Inductive Type Schemas as Functors

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Types

Definition (Types)

$$(\to)\ \frac{\rho, \tau \in \mathsf{Ty}}{\rho \to \tau \in \mathsf{Ty}} \qquad (\mu)\ \frac{\overrightarrow{\mathsf{c}} \subseteq \mathsf{Const} \qquad \overrightarrow{\rho}, \overrightarrow{\sigma} \subseteq \mathsf{Ty}}{\mu\alpha\,(\overrightarrow{\mathsf{c} : \kappa_{\overrightarrow{\rho},\overrightarrow{\sigma}}(\alpha)})}$$

Definition (Strictly positive Operator)

$$\kappa_{\overrightarrow{\rho},\overrightarrow{\sigma}}(\alpha) = \alpha \mid \rho_i \to \kappa(\alpha) \mid (\sigma_{i_1} \to \cdots \to \sigma_{i_j} \to \alpha) \to \kappa(\alpha)$$

For $\kappa_{\overrightarrow{\rho},\overrightarrow{\sigma}}(\alpha) \equiv \overrightarrow{\tau} \to \alpha$, we note $\kappa^{-}_{\overrightarrow{\rho},\overrightarrow{\sigma}}(\alpha) \equiv \overrightarrow{\tau}$

Definition (Inductive Schemas)

$$\mu\alpha\,(\overrightarrow{k : \kappa_{\overrightarrow{\pi},\overrightarrow{\theta}}(\alpha)})$$

Example

Some definable inductive types:

$\mathbf{N}$	$\hat{=}$	$\mu\alpha\,(0 : \alpha, S : \alpha \to \alpha)$	Peano's naturals
$\mathbf{L}(\rho)$	$\hat{=}$	$\mu\alpha\,(\mathsf{nil} : \alpha, \mathsf{cons} : \rho \to \alpha \to \alpha)$	lists (schema)
$\mathbf{\Sigma}(\rho, \sigma)$	$\hat{=}$	$\mu\alpha\,(\mathsf{in}_L : \rho \to \alpha, \mathsf{in}_R : \sigma \to \alpha)$	sums (schema)
$\mathbf{T}(\rho)$	$\hat{=}$	$\mu\alpha\,(\mathsf{l} : \alpha, \mathsf{b} : (\rho \to \alpha) \to \alpha)$	Tree of arity ρ

Typed Terms

Definition (Terms)

$$t ::= x \mid \lambda x^{\tau} t \mid (t\ t) \mid \mathsf{c_i}^{\mu} \mid (\!| \overrightarrow{t} |\!)^{\mu,\sigma} \quad ,$$

with $x \in \mathrm{Var}$, $i \in \mathbb{N} \setminus \{0\}$ *and* $\sigma, \mu \subseteq \mathrm{Ty}$.

Definition (Typing)

$$\frac{(x,\rho) \in \Gamma}{\Gamma \vdash x : \rho} \qquad \frac{\Gamma, x : \rho \vdash r : \sigma}{\Gamma \vdash \lambda x^{\rho}.r : \rho \to \sigma} \qquad \frac{\Gamma \vdash s : \rho \quad \Gamma \vdash r : \rho \to \sigma}{\Gamma \vdash rs : \sigma}$$

$$\frac{(\mathsf{c} : \kappa_{\vec{\rho},\vec{\sigma}}(\alpha) \in \mu) \quad \Gamma \vdash \overrightarrow{r} : \kappa_{\vec{\rho},\vec{\sigma}}^{-}(\mu)}{\Gamma \vdash \mathsf{c}\,\overrightarrow{r} : \mu} \qquad \frac{\Gamma \vdash \overrightarrow{t} : \overrightarrow{\kappa_{\vec{\rho},\vec{\sigma}}(\tau)}}{\Gamma \vdash (\!| \overrightarrow{t} |\!)^{\mu,\tau} : \mu \to \tau}$$

Rewrite Rules and Reduction

Definition ($\beta\eta\iota$-rewrite rules)

$$
\begin{array}{llll}
(\beta) & (\lambda x^\tau \cdot t)\ u & \longmapsto_\beta & t\{{}^u/{}_x\} \\
(\iota) & (\!|\ \overrightarrow{t}\ |\!)^{\mu,\tau}(\mathsf{c_i}^\mu\ \overrightarrow{p}\ \overrightarrow{u}) & \longmapsto_\iota & t_i\ \overrightarrow{p}\ \overrightarrow{((\!|\ \overrightarrow{t}\ |\!)^{\mu,\tau} \circ u)}. \\
 & & & \text{with } g^{\sigma\to\tau} \circ f^{\overrightarrow{\rho}\to\sigma} = \lambda \overrightarrow{x}^{\overrightarrow{\rho}} g(f\ \overrightarrow{x}) \\
(\eta) & t & \longmapsto_\eta & \lambda x^\tau \cdot t\ x \\
 & & & \text{if } \begin{cases} t : \tau \to \upsilon, x \notin \mathrm{FV}(t) \\ t \text{ is not an abstraction} \\ t \text{ is not in applicative position} \end{cases}
\end{array}
$$

Definition

The one-step reduction relation $\to_R$ is obtained by taking the contextual closure of $\mapsto_R$ (and respecting the proviso of η)

Example

$$\mathrm{nil} : \mathbf{L}(\rho)$$
$$\mathrm{cons} : \rho \to \mathbf{L}(\rho) \to \mathbf{L}(\rho)$$
$$a : \tau$$
$$f : \rho \to \tau \to \tau$$
$$(\!| a, f |\!)\ \mathrm{nil} \longrightarrow_{\iota} a$$
$$(\!| a, f |\!)\ (\mathrm{cons}\ h\ t) \longrightarrow_{\iota} f\ h\ ((\!| a, f |\!)\ t)$$

We will use the infix notation $a{::}l$ for cons.

Example (List)

Given a function $f : \rho \to \rho'$, we can define:

$$\mathrm{Map}(f) ::= (\!|\mathrm{nil}, \lambda xy.(fx)::y|\!)$$

$$\begin{array}{lll} \mathrm{Map}(f)\ \mathrm{nil} & \longrightarrow_{\beta\iota} & \mathrm{nil} \\ \mathrm{Map}(f)\ a::l & \longrightarrow_{\beta\iota} & fa::\mathrm{Map}(f)\ l \end{array}$$

And given two function $f : \rho \to \rho'$ and $g : \rho' \to \rho''$, we can verify that for every list, we have:

$$\begin{array}{ll} \mathrm{Map}(g) \circ \mathrm{Map}(f) l & = \mathrm{Map}(g \circ f) l \\ \mathrm{Map}(\mathrm{id}) l & = l \end{array}$$

Functoriality of Inductive Types

Definition (The functor Cp)

Given $\mu\alpha\,(\overrightarrow{\mathsf{c} : \kappa_{\vec{\rho},\vec{\sigma}}(\alpha)})$, $\mu\alpha\,(\overrightarrow{\mathsf{c}' : \kappa_{\vec{\rho'},\vec{\sigma'}}(\alpha)})$ and $\mu\alpha\,(\overrightarrow{\mathsf{c}'' : \kappa_{\vec{\rho''},\vec{\sigma''}}(\alpha)})$, and

$$\begin{array}{ll} l : \mathsf{c} \to \mathsf{c}', & l' : \mathsf{c}' \to \mathsf{c}'', \\ f_\rho : \rho \to \rho', & g_{\rho'} : \rho' \to \rho'', \\ f_\sigma : \sigma' \to \sigma, & g_{\sigma'} : \sigma'' \to \sigma', \end{array}$$

one can define a function **Cp** such that the functorial laws are provable:

$$\begin{array}{rcl} \mathbf{Cp}(\vec{l'}, \vec{g_{\rho'}}, \vec{g_{\sigma'}}) \circ \mathbf{Cp}(\vec{l}, \vec{f_\rho}, \vec{f_\sigma}) & = & \mathbf{Cp}(\overrightarrow{l' \circ l}, \overrightarrow{g_{\rho'} \circ f_\rho}, \overrightarrow{f_\sigma \circ g_{\sigma'}}) \\ \mathbf{Cp}(\mathrm{id}, \vec{\mathrm{id}}_\rho, \vec{\mathrm{id}}_\sigma) & = & \mathrm{id}_{\mu\alpha\,(\overrightarrow{\mathsf{c}:\kappa_{\vec{\rho},\vec{\sigma}}(\alpha)})} \end{array}$$

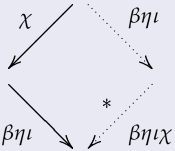
We want to add the following rules:

$$\begin{aligned} \mathbf{Cp}_{\vec{g},\vec{g'}}(\mathbf{Cp}_{\vec{f},\vec{f'}}t) &\longrightarrow_{\chi_\circ} \mathbf{Cp}_{\overrightarrow{g\circ f},\overrightarrow{f'\circ g'}}t \\ \mathbf{Cp}_{\vec{\mathrm{id}},\vec{\mathrm{id}}}t &\longrightarrow_{\chi_{\mathrm{id}}} t \end{aligned}$$

Adjournment

Definition (Adjournment)

Given two binary relations $\beta\eta\iota$ *and* χ, χ *is adjournable w.r.t* $\beta\eta\iota$ *if* $\chi;\beta\eta\iota \;\subseteq\; \beta\eta\iota;(\beta\eta\iota\chi)^*$ *i.e, if the following diagram can be closed:*



Lemma (Adjournment Lemma)

Given two strongly normalising relations $\beta\eta\iota$ *and* χ, $\beta\eta\iota\chi$ *is strongly normalising if* χ *is adjournable w.r.t* $\beta\eta\iota$.

Example

$$\mathrm{Map}(f) ::= (\!|\mathrm{nil}, \lambda xy.(fx)::y|\!)$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)t) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)t$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)(x)) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)(x)$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)(x))$$

Example

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$$\begin{aligned}\mathrm{Map}(g)(\mathrm{Map}(f)(x)) &\longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)(x)\\ &\longrightarrow_{\beta\eta\iota} \mathrm{Map}(h)(a::l)\\ \mathrm{Map}(g)(\mathrm{Map}(f)(x)) &\end{aligned}$$

Example

$$\mathrm{Map}(f) ::= (\!|\mathrm{nil}, \lambda xy.(fx)::y|\!)$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)t) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)t$$

$$\begin{aligned}\mathrm{Map}(g)(\mathrm{Map}(f)(x)) &\longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)(x) \\ &\longrightarrow_{\beta\eta\iota} \mathrm{Map}(h)(a::l) \\ \mathrm{Map}(g)(\mathrm{Map}(f)(x)) &\longrightarrow_{\beta\eta\iota} ???\end{aligned}$$

Definition

The set of Inductive Type together with the terms of type $\mu \to \mu'$ (where μ and μ' are inductive type) defined inductively by:

$$\mathcal{I}_1 \ni f, f' ::= \lambda x^{\mu}.x \mid (\!| \overrightarrow{t} |\!) \mid f \circ f'$$

forms a category $\mathcal{I}$.

Example

$$\mathrm{Map}(f) ::= (\!|\mathrm{nil}, \lambda xy.(fx)::y|\!)$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)t) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)t$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)(a::l)) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)(a::l)$$

$$\longrightarrow \quad g(fa)::\mathrm{Map}(g \circ f)l$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)(a::l))$$

Example

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$$\mathrm{Map}(g)(\mathrm{Map}(f)(a::l)) \longrightarrow_{\beta\iota} g(fa)::\mathrm{Map}(g)(\mathrm{Map}(f)l)$$

Example

$$\text{Map}(f) ::= (\!|\text{nil}, \lambda xy.(fx)::y|\!)$$

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$$\begin{aligned}\text{Map}(g)(\text{Map}(f)(a::l)) &\longrightarrow_{\beta\iota} g(fa)::\text{Map}(g)(\text{Map}(f)l) \\ &\longrightarrow_{\chi_\circ} g(fa)::\text{Map}(g \circ f)l\end{aligned}$$

Example

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$$\begin{aligned} \mathrm{Map}(g)(\mathrm{Map}(f)(a::l)) &\longrightarrow_{\beta\iota} g(fa)::\mathrm{Map}(g)(\mathrm{Map}(f)l) \\ &\longrightarrow_{\chi_\circ} g(fa)::\mathrm{Map}(g \circ f)l \end{aligned}$$

Example

$$\mathrm{Map}(f) ::= (\!|\mathrm{nil}, \lambda xy.(fx)::y|\!)$$

$$\mathrm{Map}(g)(\mathrm{Map}(f)t) \longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)t$$

$$\begin{aligned}
\mathrm{Map}(g)(\mathrm{Map}(f)(a::l)) &\longrightarrow_{\chi_\circ} \mathrm{Map}(g \circ f)(a::l) \\
&\longrightarrow_{\iota} ((\lambda xy.((g \circ f)x)::y)a(\mathrm{Map}(g \circ f)l)) \\
&\longrightarrow_{\beta} (g \circ f)a::\mathrm{Map}(g \circ f)l \\
&\longrightarrow g(fa)::\mathrm{Map}(g \circ f)l
\end{aligned}$$

$$\begin{aligned}
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Example

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&\longrightarrow_{\beta} (g \circ f)a::\mathrm{Map}(g \circ f)l \\
&\longrightarrow_{\beta} g(fa)::\mathrm{Map}(g \circ f)l
\end{aligned}$$

$$\begin{aligned}
\mathrm{Map}(g)(\mathrm{Map}(f)(a::l)) &\longrightarrow_{\beta\iota} g(fa)::\mathrm{Map}(g)(\mathrm{Map}(f)l) \\
&\longrightarrow_{\chi_\circ} g(fa)::\mathrm{Map}(g \circ f)l
\end{aligned}$$

Modified ι-reductions

$$(\!|\ \vec{t}\ |\!)^{\mu,\tau}(c_i{}^{\mu}\,\vec{p}\,\vec{u}) \quad \longmapsto_\iota \quad t_i\ \vec{p}\ \overrightarrow{((\!|\ \vec{t}\ |\!)^{\mu,\tau} \circ u)}$$

Definition

$$\overrightarrow{(\!|\lambda\vec{x}\,\vec{y}.t|\!)}c_i\,\vec{p}\,\vec{r} \quad \longmapsto_{\iota 2} \quad t_i\{\vec{p}\,/\,\vec{x}\}\langle\overrightarrow{(\!|\lambda\vec{x}\,\vec{y}.t|\!)\bullet r}\,/\,\vec{y}\rangle$$

$$\mathcal{I}t(\vec{y}) \ni t ::= y_i\ \vec{t}\ \mid x \mid \lambda z.t \mid tt \mid (\!|\ \vec{t}\ |\!) \mid c_i$$

$$\begin{aligned} y_i\ \vec{t}\ \langle\overrightarrow{u\bullet r}\,/\,\vec{y}\rangle &::= u(r\ \vec{t}\ ) \\ x\langle\overrightarrow{u\bullet r}\,/\,\vec{y}\rangle &::= x \end{aligned}$$

Example

$$\text{Map}(g \circ f)(a{::}l) \longrightarrow_\iota (g \circ f)a{::}\text{Map}(g \circ f)l$$

Convergence

Theorem (Convergence of modified ι)

The system equipped with the modififed ι-reduction is convergent and generate the same conversion relation as with the standard ι-conversion.

Proof.

$\beta\eta\iota 2$ is embeddable in $\beta\eta\iota$, i.e. for each reduction in $\beta\eta\iota 2$ there exist a sequence of reduction in $\beta\eta\iota$, hence $\beta\eta\iota 2$ is strongly normalizing.

The set of normal form for $\beta\eta\iota$ an $\beta\eta\iota 2$ are the same. Moreover $\xrightarrow{+}_{\beta\eta\iota 2}$ is a subrelation of $\xrightarrow{+}_{\beta\eta\iota}$, hence the set of normal form of a term t are the same in the two systems, hence $\beta\eta\iota 2$ is confluent. □

Definition (The Copy function on the category of functions defined by iterators)

$\mathbf{Cp}(I, \overrightarrow{f}, \overrightarrow{f'}) := (\!|\overrightarrow{t}|\!)$ *with* $\overrightarrow{t} = t_1, \ldots, t_n$, $I(\mathrm{c_k}) = \mathrm{c}'_k$, *and*

$$t_k = \lambda \overrightarrow{x}\, \overrightarrow{y_0}\, \overrightarrow{y_1} \cdot c'_k \; \overrightarrow{\circ_x(f)} \; \overrightarrow{y_0} \; \overrightarrow{\lambda \overrightarrow{z}.y_1 \overrightarrow{\circ_z(f')}} \; .$$

where the function $\circ_x(f_k)$ *is defined by induction on the structure of* f_k*(resp.* f'_k*):*

- $\circ_x(\lambda x^{\mu}.x) := x$
- $\circ_x((\!|\overrightarrow{t}|\!)) := (\!|\overrightarrow{t}|\!)x$
- $\circ_x(f \circ f') := \circ_x(f)\{^{\circ_x(f')}/x\}$

Example

$$\mathrm{Map}(g \circ f) := (\!|\mathrm{nil}, \lambda xy.(g(fx))::y|\!) \neq (\!|\mathrm{nil}, \lambda xy.((g \circ f)x))::y|\!)$$

Theorem

The χ-reductions restricted to the category of iterator $\mathcal{I}$ is convergent.

Proof.

- SN by adjournment.
- Local confluence by case analysis.

□

Conclusion

- Other subcategory of the system.
- Generalize to the whole underlying category.
- Other method to prove strong normalization.

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