Normal Operators

Defn: *The* Hermitian adjoint $A^\dagger$ *(or* A^**) of* $A \in M_n(\mathcal{C})$ *is* $A^\dagger = \overline{A}^T$*.*
Hermitian adjoint for vectors is defined the same way.

- $(AB)^\dagger = B^\dagger A^\dagger$, and $\overline{AB} = \overline{A} \cdot \overline{B}$.
- Since for $x, y \in \mathcal{C}^n$, we have $y^\dagger x \in \mathcal{C}$ so $(y^\dagger x)^\dagger = \overline{y^\dagger x}$, it follows that $(y^\dagger x)^\dagger = \overline{y^\dagger x} = \overline{y^\dagger}\,\overline{x} = y^T \overline{x}$.

Notation: If ψ is a vector in $\mathcal{C}^n$ then $|\psi\rangle$ is ψ as a column vector and $\langle\psi| = |\psi\rangle^\dagger$.

$$\overline{\langle\psi_1|\psi_2\rangle} = (\langle\psi_1|\psi_2\rangle)^\dagger = |\psi_2\rangle^\dagger \langle\psi_1|^\dagger = \langle\psi_2|\psi_1\rangle.$$

Defn: *An operator* $A \in M_n(\mathcal{C})$ *is* normal *if* $A^\dagger A = AA^\dagger$ *(i.e. A commutes with its self-adjoint).*

Defn: *An operator* $A \in M_n(\mathcal{C})$ *is* Hermitian *if* $A = A^\dagger$*.*

Defn: *An operator* $U \in M_n(\mathcal{C})$ *is* unitary *if* $U^\dagger U = UU^\dagger = \mathbb{1}$*.*
Equivalently, U has orthonormal columns and orthonormal rows.

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Normal Operators (II)

- Any unitary matrix is normal $UU^\dagger = \mathbb{1} = U^\dagger U$,
- Any Hermitian matrix is normal: $A = A^\dagger \Rightarrow AA^\dagger = A^\dagger A$,
- The following operators,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

are all both Hermitian and unitary.

We define:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |+\rangle = \frac{1}{\sqrt{2}}(\ |0\rangle + \ |1\rangle), \ |-\rangle = \frac{1}{\sqrt{2}}(\ |0\rangle - \ |1\rangle).$$

Notice that,

$$|1\rangle = X\ |0\rangle,\ |0\rangle = X\ |1\rangle,\ |+\rangle = X\ |+\rangle,\ |-\rangle = -X\ |-\rangle,$$

$$|0\rangle = Z\ |0\rangle,\ |1\rangle = -Z\ |1\rangle,\ |-\rangle = Z\ |+\rangle,\ |+\rangle = Z\ |-\rangle.$$

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Eigenvalues and Eigenvectors

Defn: *Let* $A \in M_n(\mathcal{C})$ *and* $x \in \mathcal{C}^n$ *be such that*

$$A |x\rangle = \lambda |x\rangle, x \neq \mathbf{0},$$

for scalar $\lambda \in \mathcal{C}$ *then* A *is said to have an* eigenvector x *with* eigenvalue λ*.*

We have seen on the previous slide that:

$$\text{Eig}(X) = \{(+1, |+\rangle), (-1, |-\rangle)\}, \text{Eig}(Z) = \{(+1, |0\rangle), (-1, |1\rangle)\}.$$

Defn: $B \in M_n(\mathcal{C})$ *is* unitarily equivalent *to* $A \in M_n(\mathcal{C})$ *if there exists unitary* $U \in M_n(\mathcal{C})$ *such that* $B = U^\dagger A U$*.*

Defn: $B \in M_n(\mathcal{C})$ *is* unitarily diagonalizable *if* B *is unitarily equivalent to a diagonal matrix:*

$$B = U^\dagger \Sigma U \Rightarrow \Sigma = U B U^\dagger.$$

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Spectral Decomposition

Thm: *If $A \in M_n(\mathcal{C})$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ then the following are equivalent:*

- *A is normal,*
- *A is unitarily diagonalizable,*
- *A has an orthonormal set of n eigenvectors.*

Thm[Spectral Decomposition]:*If $A \in M_n(\mathcal{C})$ is normal then,*

$$A = \sum_{i=1}^{n} \lambda_i \, |f_i\rangle\langle f_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues and $\{\,|f_i\rangle\}_i$ are orthonormal eigenvectors of A.

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Hermitian Operators

Thm: *If A is Hermitian then,*

1. $\langle x | Ax \rangle \in \mathcal{R}$ *for any* $x \in \mathcal{C}^n$,
2. *all eigenvalues of A are real,*
3. $\forall S \in M_n(\mathcal{C})$, $S^\dagger A S$ *is Hermitian.*

From the spectral decomposition theorem, this means that

$$A = \sum_{i=1}^{n} \lambda_i \, |f_i\rangle\langle f_i|,$$

where $\{ |f_i\rangle\}_{i=1}^{n}$ is an orthonormal basis and $\{\lambda_i\}_{i=1}^{n}$ is the set of eigenvalues that are garanteed to be real.

The contraposite of the above theorem also holds:

Thm: *If condition 1. or 2. or 3. holds for A then A is Hermitian.*

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Positive Operators

Defn: *An operator* $A \in M_n(\mathcal{C})$ *is* positive *if for all* $x \in \mathcal{C}^n$*,* $\langle x | Ax \rangle \geq 0$.

- For $A \in M_n(\mathcal{C})$, $A^\dagger A$ is always positive.
- From last slide, any positive operator is also Hermitian.

Thm: *A is positive if all eigenvalues of A are non-negative.*

Defn: *An operator P is called a* projection *if it can be written as:*

$$P = \sum_i |f_i\rangle\langle f_i|,$$

for $\{ |f_i\rangle \}$ *an orthonormal set.*

Th eigenvalue of P being real, it follows that a projection is always Hermitian. Moreover,

Example: It is easy to verify that if P is a projection then $PP = P$.

A projection must be positive since $\langle x | Px \rangle = \langle xP | Px \rangle = \langle z | z \rangle \geq 0$.

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Trace

Defn: *Given* $A \in M_n(\mathcal{C})$*, we define* $Tr(A) = \sum_i A_{i,i}$.

The trace satisfies the following properties:

1. $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$,
2. $\text{Tr}(\lambda A) = \lambda \text{Tr}(A)$,
3. $\text{Tr}(AB) = \text{Tr}(BA)$,
4. $\text{Tr}(A) = \sum_v \langle v| Av\rangle$ where $\{ |v\rangle\}_v$ is an orthonormal basis,
5. $\text{Tr}(A) = \text{Tr}(U^\dagger AU)$ for any unitary U.

Proof of 4.:

$$\begin{aligned}\text{Tr}(A) &= \text{Tr}(\sum_i |v_i\rangle\langle v_i| A) \text{ where } \{ |v_i\rangle\}_i \text{ is an orthonormal basis,} \\ &\stackrel{1.}{=} \sum_i \text{Tr}(|v_i\rangle\langle v_i| A) \stackrel{3.}{=} \sum_i \text{Tr}(\langle v_i| A |v_i\rangle) = \sum_i \langle v_i| A |v_i\rangle.\end{aligned}$$

The following will be useful: $\text{Tr}(|v\rangle\langle v| A) = \text{Tr}(\langle v| A |v\rangle) = \langle v| A |v\rangle$.

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