Quantum Teleportation

- Bennett, Brassard, Crepeau, Josza, Peres and Wootters, "*Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channel*", PRL, vol. 70, #13, 1993.
- Bennett and Wiesner, "Superdense Coding", PRL, vol. 69, 1992.

The Goal

- Alice has a particle ψ in an unknown state

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

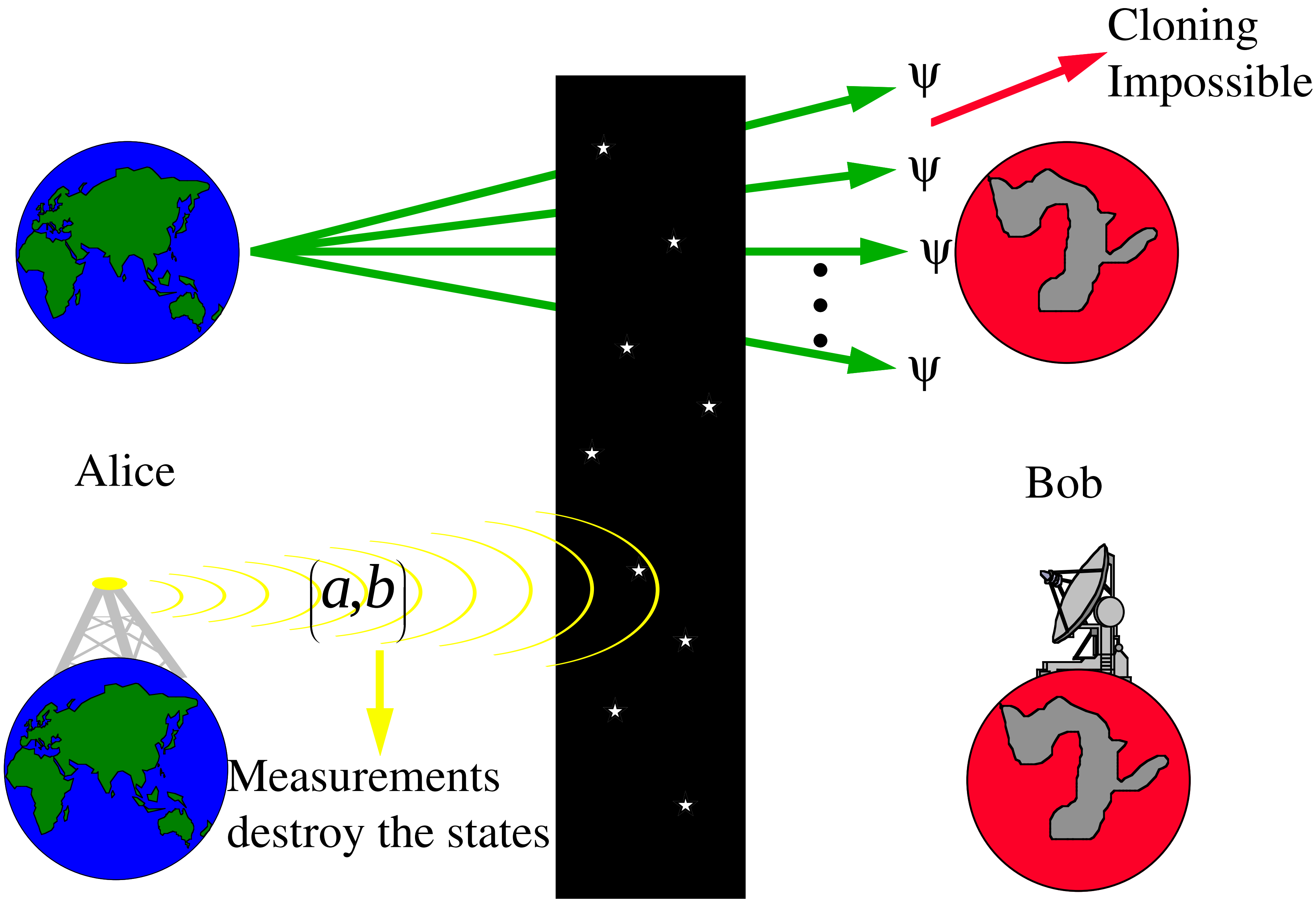
for any complex numbers a,b s.t. $|a|^2 + |b|^2 = 1$.

- Alice wants to send ψ to Bob but she does not know where Bob is in the universe.

- Alice has access to a broadcast channel that Bob can listen.

How to do that?

What we cannot do



EPR pairs

- An EPR pair is a quantum state for two particles. The state is said to be *entangled* because measuring one affects the other:

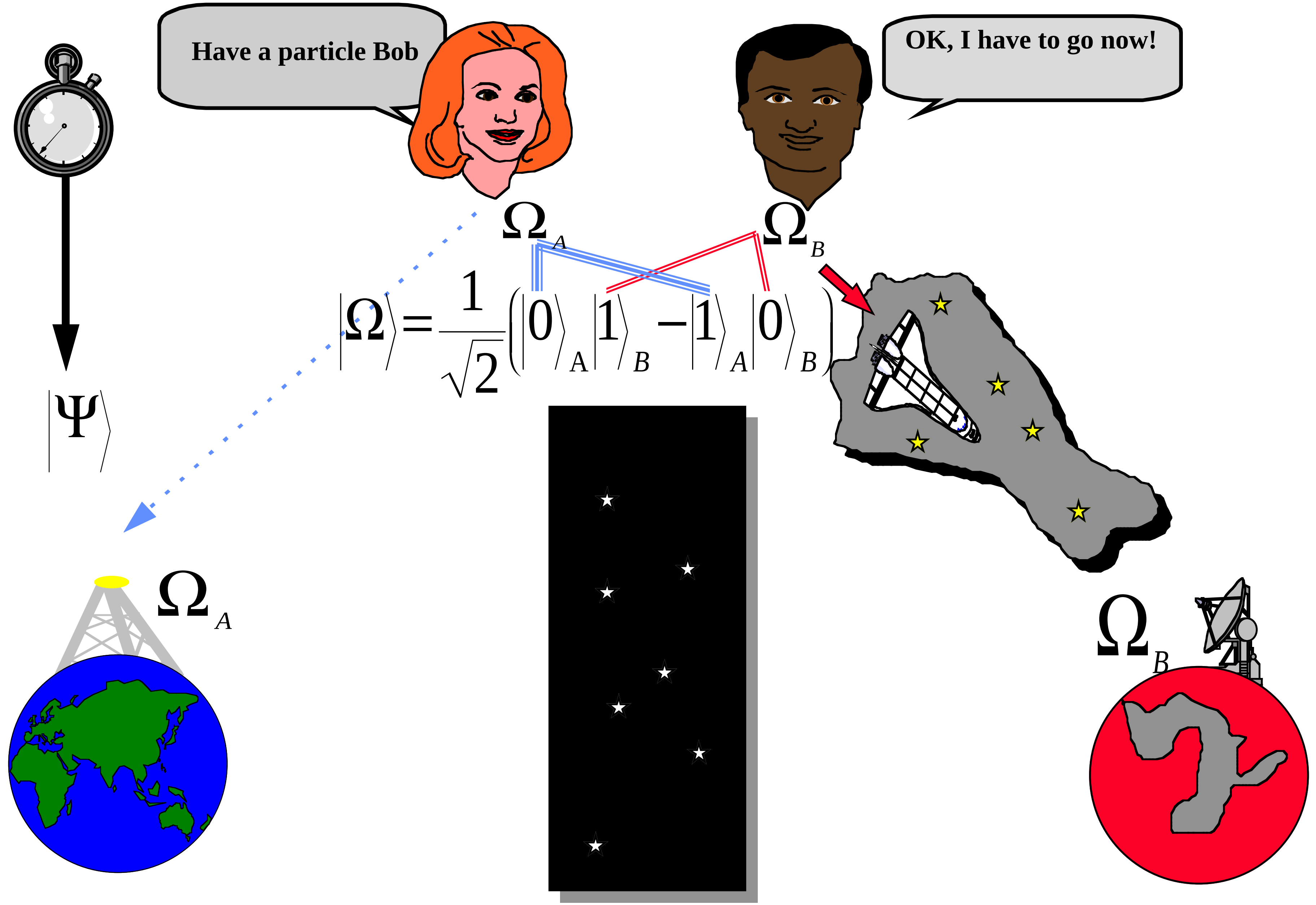
$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

- The EPR state is the *singlet* state meaning that if you represent the first and the second particle in the same basis then you get exactly the same state than above.

$$|\Omega\rangle_{xx} = \frac{1}{2\sqrt{2}}\left(\left(|0\rangle - |1\rangle\right)\left(|0\rangle + |1\rangle\right) - \left(|0\rangle + |1\rangle\right)\left(|0\rangle - |1\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(|01\rangle_{xx} - |10\rangle_{xx}\right)$$

The Setting for Teleportation



$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)$$

Bell Measurement

- Here is a basis for the Hilbert space of dimension 4. We call it the **Bell basis** and it defines a Von Neumann measurement for that space called **Bell measurement**:

$$\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right) \xrightarrow{00} |B^{+}_{00}\rangle$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right) \xrightarrow{11} |B^{-}_{00}\rangle$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle+|10\rangle\right) \xrightarrow{01} |B^{+}_{01}\rangle$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right) \xrightarrow{10} |B^{-}_{01}\rangle$$

Classical encoding

The Composite System

- The EPR pair plus the unknown system are in composite state:

Bob's particle (red)

Alice's particles (blue)

$$|\Delta\rangle = |\Psi\rangle \otimes |\Omega\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\left(a|0\rangle + b|1\rangle\right) \otimes \left(|01\rangle - |10\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[a|0\rangle \otimes |01\rangle - a|0\rangle \otimes |10\rangle + b|1\rangle \otimes |01\rangle - b|1\rangle \otimes |10\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle\right]$$

Alice's Measurement

- We now express Alice's part of the composite system Λ in Bell basis:

$$|\Lambda\rangle = \frac{1}{\sqrt{2}}\left[a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle\right]$$

$$\frac{1}{\sqrt{2}}\left(|B^{+}_{00}\rangle + |B^{-}_{00}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|B^{+}_{01}\rangle + |B^{-}_{01}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|B^{+}_{01}\rangle - |B^{-}_{01}\rangle\right) \quad \frac{1}{\sqrt{2}}\left(|B^{+}_{00}\rangle - |B^{-}_{00}\rangle\right)$$

Once simplified gives:

$$|\Lambda\rangle = \frac{1}{2}\left[|B^{+}_{00}\rangle\left(a|1\rangle - b|0\rangle\right) + |B^{-}_{00}\rangle\left(a|1\rangle + b|0\rangle\right) + |B^{+}_{01}\rangle\left(-a|0\rangle + b|1\rangle\right) + |B^{-}_{01}\rangle\left(-a|0\rangle - b|1\rangle\right)\right]$$

Outcomes for Bell Measurement

- Assume Alice measures her two particles in Bell Basis.

$$|\Lambda\rangle = \frac{1}{2}\left[\left|B^{+}_{00}\right\rangle\left(a|1\rangle - b|0\rangle\right) + \left|B^{-}_{00}\right\rangle\left(a|1\rangle + b|0\rangle\right) + \left|B^{+}_{01}\right\rangle\left(-a|0\rangle + b|1\rangle\right) + \left|B^{-}_{01}\right\rangle\left(-a|0\rangle - b|1\rangle\right)\right]$$

with prob. 1/4	with prob. 1/4	with prob. 1/4	with prob. 1/4
outcome:00	outcome:11	outcome:01	outcome:10

New state for Bob's particle

$$a|1\rangle - b|0\rangle = \begin{pmatrix} -b \\ a \end{pmatrix} \quad a|1\rangle + b|0\rangle = \begin{pmatrix} b \\ a \end{pmatrix} \quad -a|0\rangle + b|1\rangle = \begin{pmatrix} -a \\ b \end{pmatrix} \quad -a|0\rangle - b|1\rangle = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

- By announcing the outcome, Alice can tell Bob what to do to his particle in order to recover the original state:

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

How to Recover ψ

- Alice announces the outcome of the Bell measurement by broadcasting the two classical bits (x,y).

- Bob receives (x,y) and applies the unitary transformation σ_{xy} on his particle defined as:

$$\sigma_{00} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_{01} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \sigma_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{10} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

- As expected Bob finally gets:

$(0,0) \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix}$

$(0,1) \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} -a \\ b \end{pmatrix}$

$(1,0) \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \begin{pmatrix} -a \\ -b \end{pmatrix}$

$(1,1) \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$

Conclusion

- Alice and Bob shares an EPR pair.
- Alice receives an unknown particle ψ.
- Alice measures her EPR particle together with ψ in the Bell basis.
- Alice broadcasts the two classical bits describing the outcome.
- Bob, given the two bits, applies an unitary transformation on his part of the EPR pair and gets the original particle ψ.

Superdense coding

- It seems natural to think that a quantum state in a 2 dimensional Hilbert space can only be used to send 1 classical bit of information.*(in general lg(Dim(Hilbert Space)) bits)*
- It is FALSE!
- Superdense coding allows to transmit 2 classical bits of information by sending one q-bit!
- However, Alice and Bob must share an EPR pair before the communication takes place.

The Trick

- Assume Alice and Bob shares an EPR pair:

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

NOTE: $|\Omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$

- Assume Alice applies σ_{xy} on her part of the system:

$$(\sigma_{00}\otimes I)|\Omega\rangle = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}|\Omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\-1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\-1\end{pmatrix} = -|B^{+}_{00}\rangle$$

Similarly we have:

$$(\sigma_{01}\otimes I)|\Omega\rangle = -|B^{+}_{01}\rangle \quad \star \quad (\sigma_{10}\otimes I)|\Omega\rangle = -|B^{-}_{01}\rangle \quad \star \quad (\sigma_{11}\otimes I)|\Omega\rangle = -|B^{-}_{00}\rangle$$

The Procedure

- Alice and Bob shares an EPR pair
- If Alice wants to send the classical pair of bits *(x,y)* then she applies σ_{xy} to her particle
- Alice send her particle to Bob
- Bob performs a Bell measurement on both particles and get the classical outcome *(x,y)*.