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Finitely Presented Heyting Algebras

Carsten Butz*

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Abstract

In this paper we study the structure of finitely presented Heyting algebras. Using algebraic techniques (as opposed to techniques from proof-theory) we show that every such Heyting algebra is in fact co-Heyting, improving on a result of Ghilardi who showed that Heyting algebras free on a finite set of generators are co-Heyting. Along the way we give a new and simple proof of the finite model property. Our main technical tool is a representation of finitely presented Heyting algebras in terms of a colimit of finite distributive lattices. As applications we construct explicitly the minimal join-irreducible elements (the atoms) and the maximal join-irreducible elements of a finitely presented Heyting algebras in terms of a given presentation. This gives as well a new proof of the disjunction property for intuitionistic propositional logic.

Unfortunately not very much is known about the structure of Heyting algebras, although it is understood that implication causes the complex structure of Heyting algebras. Just to name an example, the free Boolean algebra on one generator has four elements, the free Heyting algebra on one generator is infinite.

Our research was motivated a simple application of Pitts' uniform interpolation theorem [11]. Combining it with the old analysis of Heyting algebras free on a finite set of generators by Urquhart [13] we get a *faithful* functor

$$\mathcal{J}\colon \mathsf{HA}^{\mathrm{op}}_{\mathrm{f.p.}} \to \mathsf{PoSet},$$

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sending a finitely presented Heyting algebra to the partially ordered set of its join-irreducible elements, and a map between Heyting algebras to its left-adjoint restricted to join-irreducible elements. We will explore on the induced duality more detailed in [5].

Let us briefly browse through the contents of this paper: The first section recapitulates the basic notions, mainly that of the *implicational degree* of an element in a Heyting algebra. This is a notion relative to a given set of generators. In the next section we study finite Heyting algebras. Our contribution is a simple proof of the *finite model property* which names in particular a canonical family of finite Heyting algebras into which we can embed a given finitely presented one.

In Section 3 we recapitulate the standard duality between finite distributive lattices and finite posets. The 'new' feature here is a strict categorical formulation which helps simplifying some proofs and avoiding calculations. In the following section we recapitulate the description given by Ghilardi [8] on how to adjoin implications to a finite distributive lattice, thereby not destroying a given set of implications. This construction will be our major technical ingredient in Section 5 where we show that *every* finitely presented Heyting algebra is co-Heyting, i.e., that the operation $(-) \setminus (-)$ dual to implication is defined. This result improves on Ghilardi's [8] that this is true for Heyting algebras free on a finite set of generators.

Then we go on analysing the structure of finitely presented Heyting algebras in Section 6. We show that every element can be expressed as a finite join of join-irreducibles, and calculate explicitly the maximal join-irreducible elements in such a Heyting algebra (in terms of a given presentation). As a consequence we give a new proof of the *disjunction property* for propositional intuitionistic logic. As well, we calculate the minimal join-irreducible elements, which are nothing but the *atoms* of the Heyting algebra.

Finally, we show how all this material can be used to express the category of finitely presented Heyting algebras as a *category of fractions* of a certain category with objects morphism between finite distributive lattices.

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1 Implicational degree

The object of study in this paper are Heyting algebras. For example, the free Heyting algebra on a set of generators X is (isomorphic to) the set of equivalence classes of terms with free variables among elements of X. Here the terms are freely generated by the grammar

$$\mathrm{terms}(X) ::= x(\in X) \mid \top \mid \bot \mid t \wedge t' \mid t \vee t' \mid t \to t' ,$$

and the equivalence relation is provability in intuitionistic propositional logic, that is, t and t' are equivalent if $\vdash_i t \leftrightarrow t'$. (We refer to [6] for a possible axiomatisation.) We use the usual abbreviations $\neg t$ for $t \to \bot$ and $t \leftrightarrow t'$ for $(t \to t') \wedge (t' \to t)$, the latter we already used above. The free Heyting algebra on a set of generators X is denoted $\mathrm{HA}[X]$. In case X is a finite set (a tuple $\bar{x}$) we write as well $\mathrm{HA}[\bar{x}]$. We assume that the reader is familiar with the notions of *finitely generated* algebras, *finitely presented* algebras, and *finite algebras*. With morphisms the structure preserving maps we have the corresponding full inclusions of categories

$$\mathsf{HA}_{\mathrm{fin}} \hookrightarrow \mathsf{HA}_{\mathrm{f.p.}} \hookrightarrow \mathsf{HA}_{\mathrm{f.g.}} \hookrightarrow \mathsf{HA} .$$

We *warn the reader* that we usually confuse a term and the element denoted by it in the free Heyting algebra. Only if absolutely necessary we denote equivalence classes of terms by $[t]$.

The *implicational degree* of a term t is defined by recursion:

$$\begin{aligned}
&\deg_{\bar{x}}(\top) = \deg_{\bar{x}}(\bot) = \deg_{\bar{x}}(x_i) = 0,\\
&\deg_{\bar{x}}(t_1(\bar{x}) \wedge t_2(\bar{x})) = \deg_{\bar{x}}(t_1(\bar{x}) \vee t_2(\bar{x})) = \max\{\deg_{\bar{x}}(t_1(\bar{x})), \deg_{\bar{x}}(t_2(\bar{x}))\},\\
&\deg_{\bar{x}}(t_1(\bar{x}) \to t_2(\bar{x})) = 1 + \max\{\deg_{\bar{x}}(t_1(\bar{x})), \deg_{\bar{x}}(t_2(\bar{x}))\}.
\end{aligned}$$

If the set of variables is clear from the context we write $\deg(t)$.

The *implicational degree* of an element $h \in \mathrm{HA}[\bar{x}]$ is the minimum over all degrees of terms t that represent h. For example, in $\mathrm{HA}[\bar{x}]$ we have (identifying a term with the equivalence class it represents (!))

$$\begin{aligned}
&\deg_{\bar{x}}(x_1 \wedge (x_2 \to x_3)) = 1,\\
&\deg_{\bar{x}}(x_1 \to x_1) = 0 = \deg_{\bar{x}}(\top),\\
&\deg_{\bar{x}}(x_1 \wedge (x_1 \to x_2)) = 0 = \deg_{\bar{x}}(x_1 \wedge x_2).
\end{aligned}$$

The elements of implicational degree $\leq n$ form a finite distributive sub-lattice $D^n_{\bar{x}}$ of $\mathrm{HA}[\bar{x}]$, so that we get a sequence

$$D^0_{\bar{x}} \hookrightarrow D^1_{\bar{x}} \hookrightarrow \cdots \hookrightarrow D^n_{\bar{x}} \hookrightarrow \cdots \qquad \mathrm{HA}[\bar{x}].$$

We speak as well about the *standard filtration* of $\mathrm{HA}[\bar{x}]$ (by finite distributive lattices).

A Heyting algebra H is *finitely generated* (f.g.) if there exists a finite set $\bar{h}$ in H so that any element in H can be expressed as a term in $\bar{h}$. Equivalently, H is finitely generated if there is a surjective morphism $\chi\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow H$. Writing E^n_χ for the image of $D^n_{\bar{x}}$ under the map χ, we get a diagram of finite distributive lattices as follows:

$$\begin{tikzcd}
D^0_{\bar{x}} \arrow[r, hook] \arrow[d, "\chi^0"'] & D^1_{\bar{x}} \arrow[r, hook] \arrow[d, "\chi^1"'] & \cdots \arrow[r, hook] & D^n_{\bar{x}} \arrow[r, hook, "\alpha^{n,n+1}"] \arrow[d, "\chi^n"'] & D^{n+1}_{\bar{x}} \arrow[r, hook] \arrow[d, "\chi^{n+1}"'] & \cdots & \mathrm{HA}[\bar{x}] \arrow[d, "\chi"] \\
E^0_\chi \arrow[r, hook] & E^1_\chi \arrow[r, hook] & \cdots \arrow[r, hook] & E^n_\chi \arrow[r, hook, "\beta^{n,n+1}"'] & E^{n+1}_\chi \arrow[r, hook] & \cdots & H
\end{tikzcd} \tag{1}$$

The maps into $\mathrm{HA}[\bar{x}]$ or H are denoted $\alpha^{n,\infty}\colon D^n_{\bar{x}} \to \mathrm{HA}[\bar{x}]$, respectively $\beta^{n,\infty}\colon E^n_\chi \to H$. We call the sequence $\{E^i_\chi\}_{i\geq 0}$ the *standard filtration of* H, induced by the surjective map χ (or by the elements $\bar{h} = \chi(\bar{x})$). Elements in $E^n_\chi \setminus E^{n-1}_\chi$ are said to have degree n, and we write $\deg_{\bar{h}}(e) = \deg_\chi(e) = n$ for this degree. Diagram (1) will be our main tool for analysing the structure of H.

Next we define the *rank* of a set of generators (or of a surjection χ as above): It is the minimum of implicational degrees of terms, needed to express H as a quotient $\mathrm{HA}[\bar{x}]/(t^j(\bar{x}) = s^j(\bar{x}))_J$. Formally it becomes as follows:

Definition 1.1 *The* rank *of χ (or of the set of generators $\bar{h}$),* $\mathrm{rank}_\chi(H)$ *(or* $\mathrm{rank}_{\bar{h}}(H)$*) is defined as*

$$\min\{r \mid \exists t^j(\bar{x}), s^j(\bar{x})\colon \deg(t^j), \deg(s^j) \leq r \ \&\ H \cong \mathrm{HA}[\bar{x}]/(t^j(\bar{x}) = s^j(\bar{x}))_J\}.$$

Note that H is *finitely presented* (f.p.) if and only if $\mathrm{rank}_\chi(H)$ is finite. As we will see, the rank of a presentation is an important number related to the structure of H.[1]

[1] There is a different notion of rank available: Since any quotient just considered can be written in the form $H \cong \mathrm{HA}[\bar{x}]/(\gamma(\bar{x}) = \top)$, i.e., is obtained by forcing one element $\gamma(\bar{x})$ equal to $\top$, we could define the rank of a presentation as the degree of this element. The two ranks differ by at most one, and each of them has its advantages and disadvantages.

Example 1.2 The rank of the trivial representation $\chi = \mathrm{id}\colon \mathrm{HA}[\bar{x}] \to \mathrm{HA}[\bar{x}]$ is 0 since $\mathrm{HA}[\bar{x}] \cong \mathrm{HA}[\bar{x}]/(\top = \top)$. Of more interest is the following example: Fix $D^n_{\bar{x}}$ in $\mathrm{HA}[\bar{x}]$. $D^n_{\bar{x}}$, as a finite distributive lattice, is in particular a Heyting algebra. The implication is given by $a \to_{D^n_{\bar{x}}} b = \bigvee\{c \in D^n_{\bar{x}} \mid a \wedge c \leq_{D^n_{\bar{x}}} b\}$, which is a finite suprema. The generators $\bar{x} \in \mathrm{HA}[\bar{x}]$ are of degree 0, so lie in $D^n_{\bar{x}}$. Of course, $D^n_{\bar{x}}$ is generated as a Heyting algebra by $\bar{x} \in D^n_{\bar{x}}$. Therefore, there is a surjective map of Heyting algebras, defined by

$$\pi^n\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow D^n_{\bar{x}}, \qquad \bar{x} \in \mathrm{HA}[\bar{x}] \mapsto \bar{x} \in D^n_{\bar{x}}.$$

The rank of this presentation is $n+1$: Indeed, what we have to say is what happens to a term $t(\bar{x})$ of degree $n+1$, which becomes an element in $D^n_{\bar{x}}$, to be expressed by a term of degree $\leq n$. (Strictly speaking we only showed that the rank is bounded by $n+1$. Using that the free Heyting algebra on one generator is infinite ([1], pp. 182–185, [10], p. 35) one shows that the rank equals $n+1$.)

The above example shows slightly more: $\pi^n \restriction D^n_{\bar{x}}$, which is the composite

$$D^n_{\bar{x}} \xhookrightarrow[\alpha^{n,\infty}]{} \mathrm{HA}[\bar{x}] \xrightarrow[\pi^n]{} D^n_{\bar{x}},$$

say as a map of distributive lattices, is the identity on $D^n_{\bar{x}}$. This shows that the family of maps $\{\pi^n\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow D^n_{\bar{x}}\}_n$ is jointly injective. Equivalently, we have the embedding

$$\mathrm{HA}[\bar{x}] \hookrightarrow \prod_n D^n_{\bar{x}}$$

of $\mathrm{HA}[\bar{x}]$ into the product of finite Heyting algebras.[2] This is commonly known as the *finite model property* (here of free Heyting algebras on a finite set of generators). As is well known, this holds more generally for all finitely presented Heyting algebras. The usual proof uses the completeness theorem of intuitionistic propositional logic with respect to Kripke models, and then one has to show that finite Kripke models suffice (see for example [6]). We will give a simple algebraic proof in the next section.

[2]As Mai Gehrke has pointed out we get slightly more: The embedding gives a representation of $\mathrm{HA}[\bar{x}]$ as a subdirect product of subdirectly irreducible algebras, the latter since in each $D^n_{\bar{x}}$ the top element is join-irreducible. In the general case of Proposition 2.1 we have to replace each E^n_χ by the product $\prod_{p \in \mathcal{J}(E^n_\chi)} E^n_\chi/p$ (where $\mathcal{J}(E^n_\chi)$ is the set of all join-irreducible elements of E^n_χ) to get a representation of a finitely presented Heyting algebra as a subdirect product of subdirectly irreducible algebras.

2 Finite Heyting algebras and the finite model property

In this section our fixed data is a representation $\chi\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow H$ of some finitely presented Heyting algebra H. Associated we have the sequence of finite distributive lattices E_χ^n, each of which is in particular a finite Heyting algebra.

Proposition 2.1 (The finite model property.) *Any finitely presented Heyting algebra H embeds into a product of finite ones. Even more: For $n \geq \mathrm{rank}_\chi(H)$ there are canonical morphisms of Heyting algebras $\gamma^n\colon H \twoheadrightarrow E_\chi^n$ and the family of these is jointly injective.*

Proof. Fix a presentation $H \cong \mathrm{HA}[\bar{x}]/(t^i(\bar{x}) = s^i(\bar{x}))_{i\in I}$ where the degrees of the terms s^i and t^i are bounded by $\mathrm{rank}_\chi(H)$ for all $i \in I$. Heyting algebra morphisms $\gamma^n\colon H \to E_\chi^n$ correspond to elements $\bar{e} \in E_\chi^n$ satisfying $t^i(\bar{e}) = s^i(\bar{e})$ for all $i \in I$. Obviously, $\bar{e} = \bar{x} \in E_\chi^n$ will do the job, provided $n \geq \mathrm{rank}_\chi(H)$ since $t^i(\bar{e}) = t^i[\bar{x}] = [t^i(\bar{x})] = [s^i(\bar{x})] = s^i(\bar{e})$. The family of these maps is jointly injective. $\square$

We want to show that these distributive lattices E_χ^n are related by various maps. For this we first need a definition:

Definition 2.2 *For $n \geq 0$ and a set of variables $\bar{x}$ we define the following term (in which the meet is indexed by all terms t of degree $n+1$):*

$$\Delta^n(\bar{x}) = \bigwedge \{t(\bar{x}) \leftrightarrow \delta_t(\bar{x}) \mid \deg(t) = n{+}1,\ \deg(\delta_t) \leq n \ \textit{and}\ D_{\bar{x}}^n \models t(\bar{x}) \leftrightarrow \delta_t(\bar{x})\}.$$

(Here $D_{\bar{x}}^n$ is again the set of elements of degree bounded by n in the free Heyting algebra $\mathrm{HA}[\bar{x}]$*.)*

We note that, being honest, $\Delta^n(\bar{x})$ is defined only up to provable equivalence (although we could fix a representative for this equivalence class of formulae). By definition (see Example 1.2), $D_{\bar{x}}^n \cong \mathrm{HA}[\bar{x}]/\Delta^n(\bar{x})$. Clearly, $\deg(\Delta^n(\bar{x})) \leq n+2$. So we can view $\Delta^n(\bar{x})$ as an element in $D_{\bar{x}}^m$, $m \geq n+2$, and there is thus a canonical map of Heyting algebras $D_{\bar{x}}^m/\Delta^n(\bar{x}) \to D_{\bar{x}}^n$, which is easily seen to be a bijection. We conclude that for all n and all

$m \geq n+2$ there is a map of Heyting algebras (the quotient map composed with the isomorphism above)

$$\pi^{m,n} \colon D^m_{\bar{x}} \to D^n_{\bar{x}}.$$

As well, it follows that $\Delta^n(\bar{x}) \vdash_i \Delta^m(\bar{x})$ for $m \geq n+2$.

The following proposition allows us to give a presentation of E^n_χ in terms of a presentation of H and of $D^n_{\bar{x}}$. Here the tensor stands for the usual tensor product of algebras, which is also known as push-out or as free amalgamated product.

Proposition 2.3 *For $n > \mathrm{rank}_\chi(H)$ there is an isomorphism $E^n_\chi \cong H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}$. As a consequence, for $m \geq n+2$ the square on the right is a push-out square of Heyting algebras as well:*

$$\begin{tikzcd}
\mathrm{HA}[\bar{x}] \arrow[r, two heads, "\pi^m"] \arrow[d, "\chi"'] & D^m_{\bar{x}} \arrow[r, two heads, "\pi^{m,n}"] \arrow[d, "\chi^m"] & D^n_{\bar{x}} \arrow[d, "\chi^n"] \\
H \arrow[r, two heads] & E^m_\chi \arrow[r, two heads] & E^n_\chi \cong H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}
\end{tikzcd}$$

We note that, in particular, χ^n for $n > \mathrm{rank}_\chi(H)$ is a morphism of (finite) Heyting algebras.

Proof. It is clearly enough to prove the first statement. We fix again a presentation of H of minimal degree, i.e., $H \cong \mathrm{HA}[\bar{x}]/(t^i(\bar{x}) = s^i(\bar{x}))_{i \in I}$ where $\deg(t^i), \deg(s^i) \leq \mathrm{rank}_\chi(H)$ for all $i \in I$. We write $\Psi(\bar{x})$ for $\bigwedge_I (t^i(\bar{x}) \leftrightarrow s^i(\bar{x}))$, a term of degree $\leq \mathrm{rank}_\chi(H) + 1$.

Fix now $n > \mathrm{rank}_\chi(H)$ and consider the push-out square

$$\begin{tikzcd}
& \mathrm{HA}[\bar{x}] \arrow[r, two heads, "\pi^n"] \arrow[d, "\chi"'] & D^n_{\bar{x}} \arrow[d] & \cong \quad \mathrm{HA}[\bar{x}]/\Delta^n(\bar{x}) \\
\mathrm{HA}[\bar{x}]/\Psi(\bar{x}) \quad \cong & H \arrow[r, two heads] & H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}} & \cong \quad \mathrm{HA}[\bar{x}]/\Psi(\bar{x}) \wedge \Delta^n(\bar{x})
\end{tikzcd}$$

(Being more precise, the push-out is given as the quotient of Heyting algebras $\mathrm{HA}[\bar{x}, \bar{x'}]/(\Psi(\bar{x}) = \top, \Delta^n(\bar{x'}) = \top, \bar{x} = \bar{x'})$. We already did some simplifications.) Since $D^n_{\bar{x}} \to H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}$ is surjective the composite

$$E^n_\chi \hookrightarrow H \twoheadrightarrow H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}$$

is surjective as well. Let us show that it is injective: Take $a, b \in E^n_\chi$ and suppose they become equal in the tensor product $H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}$. Choosing polynomials $a(\bar{x})$ and $b(\bar{x})$ of degree $\leq n$ for a and b, this means that

$$\Delta^n(\bar{x}), \Psi(\bar{x}) \vdash_i a(\bar{x}) \leftrightarrow b(\bar{x}).$$

Hence $\Delta^n(\bar{x}) \vdash_i (\Psi(\bar{x}) \wedge a(\bar{x})) \rightarrow b(\bar{x})$ or $[\Psi(\bar{x}) \wedge a(\bar{x})]_{D^n_{\bar{x}}} \leq [b(\bar{x})]_{D^n_{\bar{x}}}$. By our assumption on n, $\deg(\Psi(\bar{x}) \wedge a(\bar{x})) \leq n$, but $D^n_{\bar{x}}$ collapses only elements of degree $> n$, so that in fact $[\Psi(\bar{x}) \wedge a(\bar{x})]_{\mathrm{HA}[\bar{x}]} \leq [b(\bar{x})]_{\mathrm{HA}[\bar{x}]}$. Finally, since $\Psi(\bar{x})$ becomes $\top$ in H we deduce $[a(\bar{x})]_H \leq [b(\bar{x})]_H$ and thus $[a(\bar{x})]_{E^n_\chi} \leq [b(\bar{x})]_{E^n_\chi}$, since both $a(\bar{x})$ and $b(\bar{x})$ represent elements lying already in E^n_χ, a subset of H. The other inequality follows from a symmetric argument.

Moreover, one checks directly that along these isomorphisms the map $D^n_{\bar{x}} \rightarrow E^n_\chi \cong H \otimes_{\mathrm{HA}[\bar{x}]} D^n_{\bar{x}}$ is really χ^n. $\square$

Lemma 2.4 *Let $\varphi\colon H \rightarrow H'$ be a morphism of Heyting algebras with H' finite, $\chi\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow H$ a presentation of the finitely presented Heyting algebra H. Then there is a minimal $n = n(\varphi)$ such that φ factors through some Heyting algebra morphism $\gamma^n\colon H \twoheadrightarrow E^n_\chi$.*

Proof. Using Proposition 2.3 it is enough to consider the case where H is the free Heyting algebra $\mathrm{HA}[\bar{x}]$ and $\chi = \mathrm{id}$. Then it is enough to look at a surjection $\varphi\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow H'$, which then gives rise to a presentation of H'. Since H' is finite the induced sequence of elements of degree $\leq n$ becomes stationary,

$$E^n_\varphi \overset{\cong}{\hookrightarrow} E^{n+1}_\varphi \overset{\cong}{\hookrightarrow} \cdots \qquad \cdots \qquad \cong H',$$

for n big enough. Then the bottom map in Lemma 2.3 becomes an isomorphism, and $\varphi^n\colon D^n_{\bar{x}} \rightarrow E^n \cong H'$ is a Heyting algebra morphism from $D^n_{\bar{x}}$ onto H'. Since φ factors through some $\gamma^n\colon \mathrm{HA}[\bar{x}] \rightarrow D^n_{\bar{x}}$ there is certainly a minimal n where this happens. $\square$

3 Duality for finite distributive lattices

For an element $p \neq \perp$ in a distributive lattice D, the following are equivalent:

— $p = a \vee b$ implies $p = a$ or $p = b$; and

— $p \leq a \vee b$ implies $p \leq a$ or $p \leq b$.

Elements with this property are called *join-irreducible*. $\mathcal{J}(D)$ denotes the set of all join-irreducible elements of D, which inherits a partial order from D. Dually, we have the notion of *meet-irreducibles* and the partially ordered set $\mathcal{M}(D)$.

The construction of $\mathcal{J}(D)$ (or $\mathcal{M}(D)$) is in no way functorial in D, unless...

Lemma 3.1 *Let $\alpha: D \to D'$ be a map of distributive lattices. If the left adjoint $\alpha_!$ of α exists, it preserves join-irreducibles of D' and yields a map*

$$\mathcal{J}(\alpha) = \alpha_! \restriction \mathcal{J}(D'): \mathcal{J}(D') \to \mathcal{J}(D).$$

Dually, if the right adjoint α_* exists, it preserves meet-irreducibles. $\square$

Thus, after restricting to appropriate sub-categories of DLat we get (contravariant) functors to PoSet. In particular, there are two functors

$$\mathcal{J}, \mathcal{M}: \mathsf{DLat}^{\mathrm{op}}_{\mathrm{fin}} \rightrightarrows \mathsf{PoSet}_{\mathrm{fin}}.$$

Proposition 3.2 *The two functors $\mathcal{J}$ and $\mathcal{M}$ are naturally isomorphic. Both induce an equivalence between the opposite of the category of finite distributive lattices and the category of finite partial orders. On the 2–categorical level both functors reverse the order: if $\alpha \leq \beta: D \rightrightarrows D'$ then $\mathcal{J}(\beta) \leq \mathcal{J}(\alpha): \mathcal{J}(D') \rightrightarrows \mathcal{J}(D)$; and similar for $\mathcal{M}$.*

Proof. The proposition just restates the well known fact that a finite distributive lattice is completely determined by its join-irreducible elements. See for example [7] for details. As well it is well known that for a finite distributive lattice D the posets $\mathcal{J}(D)$ and $\mathcal{M}(D)$ are order isomorphic, via

$$\gamma_D: \mathcal{J}(D) \to \mathcal{M}(D), \qquad p \mapsto \bigvee\{D \setminus \uparrow p\} \quad = \bigvee{}^{D}\{\mathcal{J}(D) \setminus \uparrow p\}.$$

($\gamma_D(p)$ is the unique element in D satisfying $\downarrow \gamma_D(p) = D \setminus \uparrow p$.) From this explicit description it follows easily that for $\alpha: D \to D'$ the diagrams

$$\begin{tikzcd}
\mathcal{J}(D') \arrow[r, "\gamma_{D'}"] \arrow[d, "\alpha_!"'] & \mathcal{M}(D') \arrow[r, "\gamma_{D'}^{-1}"] \arrow[d, "\alpha_*"'] & \mathcal{J}(D') \arrow[d, "\alpha_!"] \\
\mathcal{J}(D) \arrow[r, "\gamma_D"'] & \mathcal{M}(D) \arrow[r, "\gamma_D^{-1}"'] & \mathcal{J}(D)
\end{tikzcd}$$

commute. This means exactly that γ is a natural isomorphism.

Finally, suppose that $\alpha, \beta: D \rightrightarrows D'$ and $\alpha \leq \beta$. Then, by adjointness and monotonicity, $\mathrm{id} \leq \alpha\alpha_! \leq \beta\alpha_!$, so that $\beta_! \leq \alpha_!: D' \to D$, and of course this inequality holds as well if both maps are restricted to join-irreducible elements in D'. $\square$

Any finite distributive lattice is both a Heyting algebra and a co–Heyting algebra. The latter means that there is a binary operation $(-) \setminus (-)$ ('supplement') determined completely by

$$a \setminus b \leq c \qquad \text{iff} \qquad a \leq b \vee c.$$

A map $\alpha: D \to D'$ preserves implication iff $\mathcal{J}(\alpha)$ is *open*, i.e., if $p \leq \alpha_!(q)$ in $\mathcal{J}(D)$ implies there exists $q' \leq q$ in $\mathcal{J}(D')$ so that $p = \alpha_!(q')$. For dual reasons, α is a morphism of co-Heyting algebras iff $\mathcal{M}(\alpha)$ is *co-open* ($\alpha_*(m) \leq n$ in $\mathcal{M}(D)$ implies there exists an $m' \geq m$ so that $\alpha_*(m') = m$). Using the natural isomorphisms above, α preserves $(-) \setminus (-)$ iff $\mathcal{J}(\alpha)$ is co-open, that is, if $\alpha_!(q) \leq p$ in $\mathcal{J}(D)$ implies there exists $q' \in \mathcal{J}(D')$, $q \leq q'$ and $\alpha_!(q') = p$. For the record:

Lemma 3.3 *Let $\alpha: D \to D'$ be a map between* finite *distributive lattices. Then α is a morphism of Heyting algebras if and only if $\mathcal{J}(\alpha)$ is an open map; while α is a morphism of co-Heyting algebras iff $\mathcal{M}(\alpha)$ is co-open, which in turn is equivalent to $\mathcal{J}(\alpha)$ being co-open.* $\square$

4 Adjoining implications

We go back to the 'standard' filtration of a Heyting algebra by finite distributive lattices, see diagram (1). At first sight, none of the maps involved is open, i.e., is a morphism of Heyting algebras. (Of course, χ is open and we know already from Proposition 2.3 that χ^n is open if n is large enough.) But at least all the maps in this diagram come close to being open: All displayed maps are morphisms of distributive lattices, and in addition, for example $\alpha^{n,m}$ preserves implication between elements in the image of any $\alpha^{k,n}$ ($k < n$), and similarly, each χ^n preserves implications between elements in the image of $\alpha^{k,n}$. It seems worthwhile to study more detailed this situation.

Definition 4.1 *Let $g: D' \to D$ be a morphism between distributive lattices. We suppose that D has implications between elements in the image of g.*

*A morphism of distributive lattices $f\colon D \to L$ is called g–*open *if (i) L has implications between elements in the image of f; and if (ii) f preserves implications between elements in the image of g, i.e.,*

$$f(g(d_1') \to_D g(d_2')) = fg(d_1') \to_L fg(d_2'), \quad \textit{for all } d_1', d_2' \in D'.$$

At least if D' and D are finite there is no doubt that there is a minimal g-open map $r^g\colon D \to D^g$ so that any other g-open map $f\colon D \to L$ will factor as $f^g \circ r^g$ for a unique r^g–open map $f^g\colon D^g \to L$. This D^g is a quotient of the free distributive lattice $\mathrm{DLat}[x_{d_1,d_2} \mid d_1, d_2 \in D] \otimes^{\mathsf{DLat}} D$, where x_{d_1,d_2} is adjoined to represent the implication between d_1 and d_2. So D^g is the set of formal expressions

$$\bigwedge_{i=0}^{n} \bigvee_{j \in J_i} (d_1^{i,j} \to d_2^{i,j}), \qquad d_1^{i,j}, d_2^{i,j} \in D,$$

modulo some equations. We see as well that r^g is always an inclusion.

Example 4.2 Consider the standard filtration of the free Heyting algebra in a finite set of generators, (extended by one more term to the left)

$$D^{-1} \xhookrightarrow{\alpha^{-1,0}} D^0_{\bar{x}} \xhookrightarrow{\alpha^{0,1}} D^1_{\bar{x}} \xhookrightarrow{\alpha^{1,2}} \cdots \qquad D^n_{\bar{x}} \xhookrightarrow{\alpha^{n,n+1}} D^{n+1}_{\bar{x}} \hookrightarrow \cdots \qquad \mathrm{HA}[\bar{x}]$$

Here D^{-1} is the two-point lattice $\{0,1\}$, and $D^n_{\bar{x}}$ is the distributive lattice of elements of implicational degree less or equal to n in the free Heyting algebra $\mathrm{HA}[\bar{x}]$. From the sketched description above it should be clear that, for example,

$$D^{n+2}_{\bar{x}} \simeq (D^{n+1}_{\bar{x}})^{\alpha^{n,n+1}},$$

and the canonical map $r^{\alpha^{n,n+1}}$ is in this case just the inclusion $\alpha^{n+1,n+2}$.

In case that g is a map between *finite* distributive lattices, there is an explicit description of the join-irreducible elements of D^g in terms of the join-irreducibles of D and the map $\mathcal{J}(g)\colon \mathcal{J}(D) \to \mathcal{J}(D')$ due to S. Ghilardi. Since it is of vital importance for us, we will recapitulate its description, and refer the reader for a proof of its properties to [8].

Call a subset $S \subset \mathcal{J}(D)$ $\mathcal{J}(g)$-open (or simpler: $g_!$–open) if for all $s \in S$ and all $p \in \mathcal{J}(D)$

$$p \leq s \text{ implies } \exists s' \in S (s' \leq s \;\&\; g_!(p) = g_!(s'))\,.$$

Using duality, a subset $S \subset \mathcal{J}(D)$ is $\mathcal{J}(g)$–open iff the dual map of distributive lattices $D \twoheadrightarrow \downarrow S$ is g–open. Then

$$\mathcal{J}(D^g) = \{S \subset \mathcal{J}(D) \mid \ S \text{ is rooted and } \mathcal{J}(g)\text{–open}\}.$$

(Here rooted means having a *largest* element.) The canonical map $r^g\colon D \to D^g$ is induced by its dual, $r^g_! \restriction \mathcal{J}(D^g) = \mathcal{J}(r^g)\colon \mathcal{J}(D^g) \to \mathcal{J}(D)$, sending a rooted subset $S \subset \mathcal{J}(D)$ to its root. Given the dual of a g-open map $\mathcal{J}(f)\colon \mathcal{J}(L) \to \mathcal{J}(D)$ ($f\colon D \to L$ a g-open map between finite distributive lattices), the canonical map $f^g\colon D^g \to L$ is given by

$$\mathcal{J}(f^g)\colon \mathcal{J}(L) \to \mathcal{J}(D^g), \qquad q \mapsto \{\mathcal{J}(f)(q') \mid q' \leq q\}.$$

The point here is, as observed by Ghilardi, that $\mathcal{J}(r^g)$ has a right adjoint left inverse $\mathcal{J}(r^g)_*$, sending $p \in \mathcal{J}(D)$ to the downward closure of p in $\mathcal{J}(D)$, i.e., to the set $\downarrow^{\mathcal{J}(D)} p$.

On the level of distributive lattices this means that besides $r^g\colon D \to D^g$ there is another map of distributive lattices $D^g \to D$. This map has to be the right adjoint r^g_* of r^g. (This follows from Proposition 3.2: Writing s for the map of distributive lattics $D^g \to D$ we know that $r^g_! \dashv s_!$, i.e., $r^g_! s_! \leq \mathrm{id}_{\mathcal{J}(D)}$ and $\mathrm{id}_{\mathcal{J}(D')} \leq s_! r^g_!$. Using duality, which reverses the order, we deduce $sr^g \geq \mathrm{id}_D$ and $\mathrm{id}_{D^g} \geq r^g s$, which means that r^g is left-adjoint to s.[3]) We deduce that since the lattices involved are finite, r^g_* has another right adjoint r^g_{**}. Moreover, since the left adjoint of a map of distributive lattices preserves join-irreducibles, r^g preserves join-irreducibles and in fact, the map $\mathcal{J}(r^g)_*$, sending $p \in \mathcal{J}(D)$ to its downward closure, has to be the restriction of r^g to join-irreducibles in D. Finally, we see that $\mathcal{J}(r^g)$ is co-open, so that r^g is a map of co-Heyting algebras. We summarise this in the following proposition:

Proposition 4.3 (Ghilardi, see [8].) *Let $g\colon D' \to D$ be a morphism between finite distributive lattices. There exists a (unique) finite distributive lattice D^g together with a g-open map $r^g\colon D \hookrightarrow D^g$, so that any g-open map of lattices $f\colon D \to L$ (where L may be infinite) factors* uniquely *as $f^g \circ r^g$,*

[3]There is a more conceptual argument for the fact that r^g_* is a map of distributive lattices which is slightly more illuminating: Since id_D is g-open there exists an r^g-open map $s\colon D^g \to D$ satisfying $s \circ r^g = \mathrm{id}_D$. Since D^g is generated by the elements of the form $d \to_{D^g} d'$ for $d, d' \in D$ it follows that $r^g \circ s \leq \mathrm{id}_{D^g}$ (using that $r^g \circ s$ is a map of distributive lattices so that $r^g s(d \to d') \leq r^g s(d) \to r^g s(d')$) and s is right adjoint to r^g.

for $f^g\colon D^g \to L$ an r^g-open map. r^g is a map of co-Heyting algebras. Its right adjoint $r^g_*\colon D^g \to D$ is a map of distributive lattices so that r^g preserves join-irreducible elements. Moreover, $r^g_* \circ r^g = \mathrm{id}_D$. □

As mentioned before D^g is the set of formal expressions $\bigwedge_I \bigvee_J (d_1^{i,j} \to d_2^{i,j})$, $d_1^{i,j}, d_2^{i,j} \in D$, modulo some equivalence relation.

We note that r^g_* is r^g-open: Indeed, it is a map of distributive lattices, and if $d, d' \in D$ then (we supress r^g)

$$\begin{aligned} r^g_*(d \to_{D^g} d') &= \bigvee \{d \in D \mid p \leq_{D^g} d \to d'\} \\ &= \bigvee \{d \in D \mid p \wedge d \leq_{D^g} d'\} \\ &= \bigvee \{d \in D \mid p \wedge d \leq_{D} d'\} \\ &= d \to_D d' . \end{aligned}$$

Thus r^g_* is the unique lifting of the g-open map $\mathrm{id}_D\colon D \to D$. One should note, however, that r^g_* does not preserve arbitrary implications.

It remains the question which element is represented by the $g_!$-open $S \subset \mathcal{J}(D)$.

Lemma 4.4 *For a $g_!$-open subset $S \subset \mathcal{J}(D)$ we have $S \leq p \to \gamma_D(p)$ if and only if $p \notin S$ so that the $g_!$-open S represents the join-irreducible element $S = \bigwedge_{p \notin S}(p \to \gamma_D(p))$ in $\mathcal{J}(D^g) \subset D^g$. Moreover, $\gamma_{D^g}(S) = \bigvee_{p \in S}(p \to \gamma_D(p))$.*

Proof. Assuming the first part we clearly have $S \leq \bigwedge_{p \notin S}(p \to \gamma_D(p))$. For the other inequality, if $T \in \mathcal{J}(D^g)$ is an arbitrary join-irreducible element such that for all $p \notin S$ the inequality $T \leq p \to \gamma_D(p)$ holds, then $p \notin S$ implies $p \notin T$ by the first part, so that $T \subset S$ and thus $T \leq S$. (We note that this equation is already mention in [8].)

To verify the second identity of the lemma we have for $p \in S$ that $S \not\leq p \to \gamma_D(p)$, so $p \to \gamma_D(p) \leq \gamma_{D^g}(S)$ which proves one inequality. For the other if $S \not\leq T$ in $\mathcal{J}(D^g)$ then there exists some $p \in S$ so that $p \notin T$, and $T \leq p \to \gamma_D(p)$. Since $\gamma_{D^g}(S) = \bigvee^{D^g}(\mathcal{J}(D^g) \setminus \uparrow S)$ we get the other inequality.

It remains to prove that (in D^g) for all $S \in \mathcal{J}(D^g)$, $p \in \mathcal{J}(D)$,

$$S \leq p \to \gamma_D(p) \qquad \text{if and only if} \qquad p \notin S. \tag{2}$$

For this we work in $D^g = \downarrow \mathcal{J}(D^g)$.

From left to right suppose that $S \wedge p \leq \gamma_D(p)$, i.e., $\downarrow S \cap \downarrow r^g(p) \subset \bigcup_{p \not\leq p'} \downarrow r^g(p')$. If $p \in S$ then $S \cap \downarrow^{\mathcal{J}(D)} p$ is $g_!$-open, so it is in $\mathcal{J}(D^g)$, and contained in $\downarrow S \cap \downarrow r^g(p)$ so that $S \cap \downarrow^{\mathcal{J}(D)} p \subseteq \downarrow^{\mathcal{J}(D)} p'$ for some p' such that $p \not\leq p'$, a contradiction and $p \notin S$. Conversely, if $p \notin S$ then take an arbitrary T in $\downarrow S \cap \downarrow r^g(p)$, i.e., $T \subset S \cap \downarrow p$. Because p is not in S there is some p' strictly less than p so that $T \subset \downarrow p'$, and $T \in \downarrow r^g(p')$. Since $p \not\leq p'$ the inequality follows. $\square$

In the next section we will iterate the construction of adding implications infinitely many times. Here we take a look at what happens in the second step, that is, we consider the situation

$$\begin{tikzcd} D' \arrow[r, "g"] & D \arrow[r, hook, "r^g"] \arrow[d, equal] & D^g \arrow[r, hook] \arrow[d, equal] & (D^g)^{r^g} \arrow[d, equal] \\ & D^0 \arrow[r, hook, "r^{0,1}"] & D^1 \arrow[r, hook, "r^{1,2}"] & D^2 \end{tikzcd} \tag{3}$$

The following lemma is a valuable tool for calculating maps. We use the description of join-irreducibles of D^1 and D^2 as above.

Lemma 4.5 *A rooted subset $\mathfrak{S} \subset \mathcal{J}(D^1)$ is $r_!^{0,1}$-open (and therefore represents a join-irreducible element in D^2 if and only if for all $S \in \mathfrak{S}$, $S = \{\mathrm{root}(S') \mid S' \subset S$ and $S' \in \mathfrak{S}\}$.*

(We remind the reader that elements of $\mathcal{J}(D^1)$ are rooted subsets of $\mathcal{J}(D^0)$, being g-open. Thus $\mathrm{root}(S') = r_!^{0,1}(S')$.)

Proof. Suppose $\mathfrak{S} \subset \mathcal{J}(D^1)$ is $r_!^{0,1}$-open and fix $S \in \mathfrak{S}$ arbitrary. For $s \in S$ we have $S \cap \downarrow^{\mathcal{J}(D^0)} s \leq S$, and this set is g-open (i.e., in $\mathcal{J}(D^1)$) since S is. By $r_!^{0,1}$-openness of $\mathfrak{S}$ there is $S' \subset S$, S' in $\mathfrak{S}$, so that $r_!^{0,1}(S') = r_!^{0,1}(S \cap \downarrow s)$. But $\mathrm{root}(S') = r_!^{0,1}(S') = \mathrm{root}(S \cap \downarrow s) = s$, the latter because $s \in S$.

The other direction is trivial. $\square$

As an application of the lemma we show existence of a Heyting algebra morphism $\gamma^{2,0}\colon D^2 \to D^0$. On the level of join-irreducibles it is given by

$$\gamma_!^{2,0}\colon \mathcal{J}(D^0) \to \mathcal{J}(D^2), \qquad p \mapsto \{\downarrow p' \mid p' \leq p\}.$$

Clearly, $\gamma_!^{2,0}$ is monotone and well-defined by Lemma 4.5.

Lemma 4.6 *The map $\gamma^{2,0}\colon D^2 \to D^0$ is a surjective morphism of Heyting algebras satisfying $\gamma^{2,0} \circ r^{1,2} = r_*^{0,1}$. As maps of posets, $r_*^{0,2} \leq \gamma^{2,0} \leq r_!^{0,2}$.*

Proof. Obviously, $\gamma_!^{2,0}$ is injective (which implies that its dual is surjective). To show that it is open assume $\mathfrak{A} \subset \gamma_!^{2,0}(p)$, for $p \in \mathcal{J}(D^0)$ and $\mathfrak{A} \subset \mathcal{J}(D^1)$ rooted and $r_!^{0,1}$-open. Then each set in $\mathfrak{A}$ is of the form $\downarrow p'$ for some $p' \leq p$. In particular, $\mathrm{root}(\mathfrak{A}) = \downarrow a$ for some $a \leq p$. Lemma 4.5 now shows that $\mathfrak{A} = \{\downarrow a' \mid a' \leq a\} = \gamma_!^{2,0}(a)$.

For the identity, we have to show that $r^{0,1} = r_!^{1,2} \circ \gamma_!^{2,0}\colon \mathcal{J}(D^0) \to \mathcal{J}(D^1)$. But $r_!^{1,2}\gamma_!^{2,0}(p) = \mathrm{root}(\gamma_!^{2,0}(p)) = \downarrow p = r^{0,1}(p)$. Finally note that $\gamma^{2,0} \circ r^{0,2} = \gamma^{2,0} \circ r^{1,2} \circ r^{0,1} = r_*^{0,1} \circ r^{0,1} = \mathrm{id}$, so that the inequalities follow by applying $\gamma^{2,0}$ to the inequalities $r^{0,2}r_*^{0,2} \leq \mathrm{id} \leq r^{0,2}r_!^{0,2}$. $\square$

Let us remark that we already saw these morphisms of Heyting algebras, namely as quotient maps

$$D_{\bar{x}}^{n+2} \to D_{\bar{x}}^{n}$$

(see the discussion before Lemma 2.3 and Example 4.2).

5 A representation theorem

In this section we iterate the construction of the previous one. We consider a map of finite distributive lattices $\alpha^{-1,0}\colon D^{-1} \to D^0$ to get a sequence

$$D^{-1} \xrightarrow{\alpha^{-1,0}} D^0 \xhookrightarrow{\alpha^{0,1}} D^1 \xhookrightarrow{\alpha^{1,2}} \cdots \qquad D^n \xhookrightarrow{\alpha^{n,n+1}} D^{n+1} \hookrightarrow \cdots$$

of distributive lattices, with $D^{n+1} = (D^n)^{\alpha^{n-1,n}}$ for $n \geq 0$. The colimit of this sequence (in the category of distributive lattices) is denoted $\mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \to D^0]$ or simply $\mathrm{HA}[\alpha^{-1,0}]$.

Proposition 5.1 $\mathrm{HA}[\alpha^{-1,0}]$ *is a bi-Heyting algebra. As a Heyting algebra, it has the universal property that Heyting algebra maps $\mathrm{HA}[\alpha^{-1,0}] \to H$ correspond to $\alpha^{-1,0}$-open maps (of distributive lattices) $D^0 \to H$. In particular, $\mathrm{HA}[\alpha^{-1,0}]$ is a finitely presented Heyting algebra.*

Proof. It is clear that the colimit is a bi-Heyting algebra (i.e., both a Heyting and a co-Heyting algebra). Denoting equivalence classes in the colimit by $[a]$, the supplement of $[a]$ and $[b]$, for $a \in D^n$ and $b \in D^m$, is given by

$[\alpha^{n,k}(a) \setminus_{D^k} \alpha^{m,k}(b)]$, where k is the maximum of n and m. Their implication is given by $[\alpha^{n,k+1}(a) \to_{D^{k+1}} \alpha^{m,k+1}(b)]$.

To show the universal property, denote the canonical maps $D^n \to \mathrm{HA}[\alpha^{-1,0}]$ by $\alpha^{n,\infty}$. Since $\mathrm{HA}[\alpha^{-1,0}]$ is a cone on the sequence of distributive lattices, the identities $\alpha^{n,\infty} = \alpha^{m,\infty} \circ \alpha^{n,m}$ hold.

Let now $\gamma\colon \mathrm{HA}[\alpha^{-1,0}] \to H$ be arbitrary. If $a, b \in D^n$ $(n \geq -1)$, then

$$\begin{aligned}
\gamma\alpha^{n+1,\infty}(\alpha^{n,n+1}(a) \to_{D^{n+1}} \alpha^{n,n+1}(b)) & \\
&= \gamma(a \to b) \\
&= \gamma a \to_H \gamma b \\
&= (\gamma\alpha^{n+1,\infty})\alpha^{n,n+1}(a) \to_H \\
(\gamma\alpha^{n+1,\infty})\alpha^{n,n+1}(b). &
\end{aligned}$$

Thus, $\gamma \circ \alpha^{n+1,\infty}$ is $\alpha^{n,n+1}$-open, in particular,

$$\bar{\gamma} = \gamma \circ \alpha^{0,\infty}\colon D^0 \to H$$

is $\alpha^{-1,0}$-open. By the uniqueness of the lifting in Proposition 4.3, γ is uniquely determined by $\bar{\gamma}$. By the same proposition, any such $\alpha^{-1,0}$-open map $\bar{\gamma}\colon D^0 \to H$ lifts to a sequence of maps $\gamma^i\colon D^i \to H$ and thus to a map $\gamma\colon \mathrm{HA}[\alpha^{-1,0}] \to H$. The properties of the lifting ensure that γ is a morphism of Heyting algebras.

It is now clear from the universal property that $\mathrm{HA}[\alpha^{-1,0}]$ is finitely presented. We note that, in particular, $\alpha^{0,\infty}(D^0)$ is a set of generators of $\mathrm{HA}[\alpha^{-1,0}]$. $\square$

The following proposition shows that we captured all finitely presented Heyting algebras.

Proposition 5.2 (Representation of finitely presented Heyting algebras.) *Let H be some finitely presented Heyting algebra and choose a presentation $\chi\colon \mathrm{HA}[\bar{x}] \twoheadrightarrow H$ of H. Then*

$$H \cong \mathrm{HA}[\beta^{n,n+1}\colon E^n_\chi \hookrightarrow E^{n+1}_\chi],$$

where $n \geq \mathrm{rank}_\chi(H)$, and E^n_χ (respectively E^{n+1}_χ) is the set of elements of implicational degree $\leq n$ (of degree $\leq n+1$) in H.

Proof. We will give explicitly the isomorphism. Fix the data of the proposition, and a presentation $H \cong \mathrm{HA}[\bar{x}]/(s^i(\bar{x}) = t^i(\bar{x}))_{i\in I}$ where the degrees of the occurring terms s^i and t^i are bounded by $\mathrm{rank}_\chi(H)$ for all $i \in I$.

The identity map $\mathrm{HA}[\beta^{n,n+1}] \to \mathrm{HA}[\beta^{n,n+1}]$ corresponds to a $\beta^{n,n+1}$-open map $\gamma\colon E_\chi^{n+1} \to \mathrm{HA}[\beta^{n,n+1}]$. The generators of H, the elements $\bar{x}$, are in E_χ^0, and by our assumption on n,

$$s^i(\gamma(\bar{x})) = \gamma(s^i(\bar{x})) = \gamma(t^i(\bar{x})) = t^i(\gamma(\bar{x})),$$

for all terms s^i, t^i occurring in the presentation of H, so that we get a morphism of Heyting algebras

$$H \to \mathrm{HA}[\beta^{n,n+1}], \qquad \bar{x} \mapsto \gamma(\bar{x}).$$

By the universal property of $\mathrm{HA}[\beta^{n,n+1}]$ there is a map $\mathrm{HA}[\beta^{n,n+1}] \to H$, induced by the $\beta^{n,n+1}$-open inclusion $\beta^{n+1,\infty}\colon E_\chi^{n+1} \to H$. Since by construction

$$\begin{tikzcd} \mathrm{HA}[\beta^{n,n+1}] \arrow[r] & H \\ E_\chi^{n+1} \arrow[u, hook, "\gamma"] \arrow[ur, "\beta^{n+1,\infty}"'] & \end{tikzcd}$$

commutes we get that this second map sends $\gamma(\bar{x})$ to $\beta^{n+1,\infty}(\bar{x}) = \bar{x} \in H$, and the two maps are inverse to each other. $\square$

Example 5.3 The free Heyting algebra on a finite set of generators $\mathrm{HA}[\bar{x}]$ can be described as $\mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \to D^0_{\bar{x}}]$, where D^{-1} is again the two element distributive lattice, while $D^0_{\bar{x}}$ is the lattice of elements of implicational degree 0 in $\mathrm{HA}[\bar{x}]$. Equivalently, $D^0_{\bar{x}}$ is the free distributive lattice on the set of generators $\bar{x}$, i.e., $D^0_{\bar{x}} = \mathrm{DLat}[\bar{x}]$.

Corollary 5.4 *Every finitely presented Heyting algebra is co-Heyting.* $\square$

Corollary 5.5 *A finitely presented Heyting algebra that is as well a finitely generated co-Heyting algebra is finite.*

Proof. If $H \cong \mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \hookrightarrow D^0]$ as a Heyting algebra then the co-Heyting algebra generators (finitely many) are all contained in some D^n. But the sub-co-Heyting algebra of H generated by D^n is D^n so that the inclusion $\alpha^{n,\infty}\colon D^n \to H$ is an isomorphism. $\square$

The above corollary shows as well that the dual operations $(-) \setminus (-)$ and $\sim (-) = \top \setminus (-)$ are not definable in terms of the Heyting operations in case that H is infinite.

Corollary 5.6 (Rauszer [12].) *Bi-intuitionistic logic is conservative over intuitionistic logic.*

Proof. Algebraically this just says that the canonical map from the free Heyting algebra on countably many generators into the free bi-Heyting algebra on this set of generators is an inclusion. For this it is certainly enough to verify this statement for the canonical map of Heyting algebras from the free Heyting algebra on a finite set of generators $\bar{x}$ to the free bi-Heyting algebra on this set of generators

$$\mathrm{HA}[\bar{x}] \to \mathrm{biHA}[\bar{x}].$$

But by Proposition 5.2 and Proposition 5.1 this map has a (bi-Heyting) retraction $\mathrm{biHA}[\bar{x}] \twoheadrightarrow \mathrm{HA}[\bar{x}]$. $\square$

After this representation we start analysing the structure of finitely presented Heyting algebras in the next section. (We turn this construction into a functor in Section 8.) But first we show that the maps $\beta^{n,n+1}$ are good ones, sharing all the properties of the $\alpha^{n,n+1}$.

Lemma 5.7 *For $n > \mathrm{rank}_\chi(H)$, there is a canonical isomorphism $E_\chi^{n+2} \cong (E_\chi^{n+1})^{\beta^{n,n+1}}$, and $\beta^{n+1,n+2}$ is the canonical inclusion $E_\chi^{n+1} \hookrightarrow (E_\chi^{n+1})^{\beta^{n,n+1}}$.*

Proof. We consider the diagram

$$\begin{tikzcd}
E^n \arrow[r, hook] \arrow[d, equal] & E^{n+1} \arrow[r, hook] \arrow[d, equal] & E^{n+2} \arrow[r, hook] & \cdots & H \\
F^n \arrow[r, hook, "\gamma^{n,n+1}"'] & F^{n+1} \arrow[r, hook] & F^{n+2} \arrow[r, hook, "\gamma^{n+2,n+3}"'] \arrow[u, two heads] \arrow[d, equal] & \cdots & H \arrow[u, "\mathrm{id}"'] \\
 & & (F^{n+1})^{\gamma^{n,n+1}} & \cdots &
\end{tikzcd}$$

where all possible squares commute (the vertical surjections are induced by the universal property of the γ's). By induction one shows that $E^i \cong F^i$. For the induction step just note that in both cases, elements in $(?)^{i+1}$ are exactly those elements of H of the form $\bigwedge_J \bigvee_K (a_{jk} \to_H b_{jk})$, for a_{jk}, b_{jk} in $(?)^i$. $\square$

At the beginning of this section we defined $\mathrm{HA}[\alpha^{-1,0}]$ as the colimit of a sequence of distributive lattices. But there are more maps of distributive lattices around: Each right adjoint $\alpha_*^{n,n+1}$ is a map of distributive lattices. For each $n \geq 0$ there is a morphism of Heyting algebras $\gamma^{n+2,n}\colon D^{n+2} \to D^n$ (Lemma 4.6), which altogether gives the following picture, where all displayed maps are in DLat. An 'o' indicates that the map is open (a morphism of Heyting algebras), a 'c' that it is co-open.

$$\begin{tikzcd}
 & D^0 \arrow[dr, hook, "\alpha^{0,1}"'] & & D^2 \arrow[ll, two heads, "\gamma^{2,0}"', "o"] \arrow[dr, hook, "\alpha^{2,3}"'] \arrow[dl, "\alpha_*^{1,2}", "c"'] & & D^4 \arrow[ll, two heads, "\gamma^{4,2}"', "o"] \arrow[dl, "\alpha_*^{3,4}", "c"'] & \cdots & \mathrm{HA}[\alpha^{-1,0}] \\
D^{-1} \arrow[ur, "\alpha^{-1,0}"] & & D^1 \arrow[ul, "\alpha_*^{0,1}"', "c"] \arrow[ur, hook, "\alpha^{1,2}"'] & & D^3 \arrow[ll, two heads, "\gamma^{3,1}", "o"'] \arrow[ul, "\alpha_*^{2,3}"', "c"] \arrow[ur, hook, "\alpha^{3,4}"'] & & \cdots \arrow[ll, two heads, "\gamma^{5,3}", "o"'] &
\end{tikzcd}$$

Here the 'inner' triangles commute, i.e., $\gamma^{n+2,n}\alpha^{n+1,n} = \alpha_*^{n,n+1}$, for all $n \geq 0$. We use the notation $\gamma^{m,n}$ for the composite of the Heyting algebra morphisms $D^m \to D^n$, which is only defined if $m-n$ is even. There are as well surjective Heyting algebra morphisms ($n \geq 0$)

$$\gamma^{\infty,n}\colon \mathrm{HA}[\alpha^{-1,0}] \twoheadrightarrow D^n,$$

induced by the $\alpha^{-1,0}$-open map $\alpha^{0,n}\colon D^0 \to D^n$, which is the identity if $n = 0$. Clearly, $\gamma^{n,m}\gamma^{\infty,n} = \gamma^{\infty,m}$, since both behave the same on generators.

The set D^0 is a subset of $\mathrm{HA}[\alpha^{-1,0}]$ and as a Heyting algebra, $\mathrm{HA}[\alpha^{-1,0}]$ is generated by D^0. With respect to these generators D^n is exactly the set of elements of implicational degree $\leq n$ in $\mathrm{HA}[\alpha^{-1,0}]$. Each finite Heyting algebra D^n ($n \geq 0$) is the quotient of $\mathrm{HA}[\alpha^{-1,0}]$ by a unique element Δ^n. The same argument as in the case of free Heyting algebras (see Example 1.2) proves that Δ^n has degree $n+2$ (i.e., $\Delta^n \in D^{n+2}$), and of course

$$D^{n+2}/\Delta^n \cong D^n,$$

so that we identify (again) $\gamma^{n+2,n}\colon D^{n+2} \to D^n$ as a quotient map in the canonical way. But since $\Delta^n \in D^m$ for all $m \geq n+2$ there are Heyting algebra morphisms $D^m \to D^n$ for all $m \geq n+2$ (and not only for those m such that $m-n$ is even). The following lemma describes them explicitly:

Lemma 5.8 *For each $n \geq 0$, $\gamma^{n+2,n}\alpha_*^{n+2,n+3}\colon D^{n+3} \to D^n$ is a morphism of Heyting algebras. Moreover, $\gamma^{n+2,n}\alpha_*^{n+2,n+4} = \gamma^{n+4,n}$.*

Proof. After shifting the indices we may assume $n = 0$. The map of distributive lattices $\gamma^{2,0}\alpha_*^{2,3}$ is induced by the map of posets

$$\alpha^{2,3}\gamma_!^{2,0}: \mathcal{J}(D^0) \to \mathcal{J}(D^3), \qquad p \mapsto \downarrow^{\mathcal{J}(D^2)} \gamma_!^{2,0}(p).$$

We must show that this map is open. For this note first that $\downarrow^{\mathcal{J}(D^2)} \gamma_!^{2,0}(p) = \gamma_!^{2,0}(\downarrow p) = \{\gamma_!^{2,0}(p') \mid p' \leq p\}$, by openness of $\gamma_!^{2,0}$. So take $\mathbb{A} \subset \alpha^{2,3}\gamma_!^{2,0}(p)$ in $\mathcal{J}(D^3)$ arbitrary. By the just made remark we know that there is a subset A of $\downarrow p$ so that

$$\mathbb{A} = \{\gamma_!^{2,0}(a') \mid a' \in A\}.$$

In particular, $\mathfrak{A} = \mathrm{root}(\mathbb{A}) = \gamma_!^{2,0}(a)$ for some $a \in A$.

By Lemma 4.5, $\gamma_!^{2,0}(a) = \{\mathrm{root}(\mathfrak{B}) \mid \mathfrak{B} \subset \mathfrak{A} \text{ and } \mathfrak{B} \in \mathbb{A}\}$, i.e., for all $b \leq a$ there is some $\mathfrak{B} \in \mathbb{A}$, $\mathfrak{B} \subset \mathfrak{A}$ and

$$\downarrow b = \mathrm{root}(\mathfrak{B}).$$

Since $\mathrm{root}(\gamma_!^{2,0}(a')) = \downarrow a'$ we deduce, using the fact that $\mathfrak{B}$ is of the form $\gamma_!^{2,0}(a')$ for some a', that $\mathfrak{B} = \gamma_!^{2,0}(b) \in \mathbb{A}$ for all $b \leq a$. Therefore, $\gamma_!^{2,0}(\downarrow a) = \{\gamma_!^{2,0}(b) \mid b \leq a\} \subset \mathbb{A}$ and since $\mathrm{root}(\mathbb{A}) = \gamma_!^{2,0}(a)$ we get equality, so that $\alpha^{2,3}\gamma_!^{2,0}$ is open.

For the equality note that, by Lemma 4.6, $\gamma^{2,0}\alpha_*^{2,4} = \gamma^{2,0}\alpha_*^{2,3}\alpha_*^{3,4} = \gamma^{2,0}\gamma^{4,2}\alpha^{3,4}\alpha_*^{3,4} = \gamma^{4,0}$. □

6 The structure of finitely presented Heyting algebras

In this section we use our particular representation of finitely presented Heyting algebras to obtain information about their structure. Again, notations refer to diagram (1).

The first part of the following proposition is known for Heyting algebras, free on a finite set of generators (see for example [13]).

Proposition 6.1 *In the finitely presented Heyting algebra H, any element can be expressed as the finite join of join-irreducibles. Even more, if $h \in H$ is of degree $\leq n$, then h is the union of join-irreducibles each of which has degree $\leq n$ too, provided that $n > \mathrm{rank}_\chi(H)$.*

Proof. Since h has degree $\leq n$, it lives in E_χ^n. But E_χ^n is finite, so that h is the finite union of elements in $\mathcal{J}(E_\chi^n)$. Finally, the inclusion $\beta^{n,\infty}: E_\chi^n \to H$ preserves join-irreducibles by Lemma 5.7. □

If an element $h \in H$ admits *some* representation $h = \bigvee_I p_i$, $p_i \in \mathcal{J}(H)$, then it admits a *unique minimal* such.[4] Here minimal refers to $p_i \leq p_{i'}$ implies $p_i = p_{i'}$. Indeed, from some representation one obtains a minimal by taking the maximal elements in the set $\{p_i \mid i \in I\}$. Given two such minimal representations, say $\bigvee_I p_i = h = \bigvee_J q_j$, we deduce by join-irreducibility $p_i \leq q_{j(i)}$ and then $q_{j(i)} \leq p_{k(i)}$, for each $i \in I$. Now $p_i = q_{j(i)}$ $(= p_{k(i)})$ by minimality of the representation.

Obviously, the minimal representation of $\top_H$ yields a set of maximal elements in $\mathcal{J}(H)$. We can calculate this set even better.

Proposition 6.2 *$\mathcal{J}(H)$ has finitely many maximal and finitely many minimal elements. If $H \cong \mathrm{HA}[g: D^{-1} \to D^0]$ then the maximal elements are those of $\mathcal{J}(D^0)$ (in the colimit), the minimal ones are the minimal elements in $\mathcal{J}(D^1)$.*

Proof. We look at the beginning of the sequence which defines the Heyting algebra $H \cong \mathrm{HA}[g: D^{-1} \to D^0]$, i.e., at

$$\begin{tikzcd} D^{-1} \arrow[r, "g"] & D^0 \arrow[r, hook, "r^g"] \arrow[d, equal] & (D^0)^g \arrow[r, hook] \arrow[d, equal] & ((D^0)^g)^{r^g} \arrow[d, equal] & \cdots \\ & D \arrow[r, hook, "r^{0,1}"] & D^1 \arrow[r, hook, "r^{1,2}"] & D^2 & \cdots \end{tikzcd}$$

From the construction it is clear that each $S \in \mathcal{J}(D^1)$ is contained in $\downarrow \mathrm{root}(S)$, so only sets of the form $\downarrow p$, $p \in \mathcal{J}(D)$, can be maximal. They are maximal exactly if p was maximal in $\mathcal{J}(D)$. Since $\downarrow p = r^{0,1}(p)$ we get as well that $r^{0,1}$ (and in fact, all $r^{n,n+1}$ in the infinite iteration above) preserve and reflect maximal elements. Therefore, $\mathcal{J}(H)$ is set of equivalence classes in the colimit of the maximal elements of $\mathcal{J}(D^0)$.

Next, we look at the minimal elements: In general, if p is minimal in $\mathcal{J}(D)$ then $r^{0,1}(p) = \downarrow p = \{p\}$ is minimal in $\mathcal{J}(D^1)$, that is, all the maps $r^{n,n+1}$ preserve minimal elements.

Now suppose that $\mathfrak{A} \in \mathcal{J}(D^2)$ is minimal. Then $\mathfrak{A}$ can be only a singleton set $\{A\}$ (otherwise, $\mathfrak{A} \cap \downarrow A' \subsetneq \mathfrak{A}$ contradicting minimality), so that $A = \mathrm{root}(\mathfrak{A}) = r_!^{1,2}(\mathfrak{A})$. By Lemma 4.5, $A = \{\mathrm{root}(A') \mid A' \subset A,\ A' \in \mathfrak{A}\}$, so that A is a singleton set as well, $A = \{p\}$ for $p = \mathrm{root}(A)$. But any one-element set is minimal, so A is minimal in $\mathcal{J}(D^1)$, and $\mathfrak{A} = \downarrow A = r^{1,2}(A)$. $\square$

[4] A note aside: It seems that G. Birkhoff [3] was the first to observe the uniqueness of a finite irredundant representation. An observation trivial in these days.

Example 6.3 (*Maximal and minimal elements in* $\mathrm{HA}[\bar{x}]$.) We consider the free Heyting algebra $\mathrm{HA}[\bar{x}]$ on a finite set of generators, which is as mentioned several times $\mathrm{HA}[\alpha^{-1,0}\colon \{0,1\} \hookrightarrow \mathrm{DLat}[\bar{x}]]$ i.e., the defining sequence for $\mathrm{HA}[\bar{x}]$ looks like

$$\{0,1\} \xhookrightarrow[\alpha^{-1,0}]{} \mathrm{DLat}[\bar{x}] \xhookrightarrow[\alpha^{0,1}]{} D^1_{\bar{x}} \hookrightarrow D^2_{\bar{x}} \hookrightarrow \cdots$$

On the level of join irreducibles we have the sequence

$$1 \twoheadleftarrow \mathcal{J}(D^0) \rightleftarrows \mathcal{J}(D^1_{\bar{x}}) \rightleftarrows \mathcal{J}(D^2_{\bar{x}}) \rightleftarrows \cdots$$

Any one-element subset of $\mathcal{J}(D^0_{\bar{x}})$ is $\alpha_!^{-1,0}$-open, so they form exactly the minimal elements in $\mathcal{J}(D^1)$ and thus in the colimit. In particular, there are $\#\mathrm{Min}(\mathcal{J}(\mathrm{HA}[\bar{x}])) = \#\mathcal{J}(D^0_{\bar{x}})$ many of them. Using disjunctive normal form, any element in $D^0_{\bar{x}} = \mathrm{DLat}[\bar{x}]$ has the form

$$\bigvee_I \bigwedge_{X_i \subset \{x_0,\ldots,x_{n-1}\}} X_i$$

($I = \{*\}$ and $X_* = \emptyset$ gives $\top$, while $I = \emptyset$ gives $\bot$). Each of the elements $\bigwedge X$ for $X \subset \{x_0,\ldots,x_{n-1}\}$ is join-irreducible in $D^0_{\bar{x}}$, so that there are $\#\mathcal{P}\{x_0,\ldots,x_{n-1}\} = 2^n$ minimal elements in $\mathcal{J}(\mathrm{HA}[\bar{x}])$. In fact, $\mathcal{J}(D^0_{\bar{x}}) \cong \mathcal{P}\{x_0,\ldots,x_{n-1}\}$, ordered by *reverse* inclusion.

Note that the minimal elements in $\mathcal{J}(\mathrm{HA}[\bar{x}])$ are nothing but the *atoms* in $\mathrm{HA}[\bar{x}]$, and this number was as well calculated in [2], Theorem 3.0(i).

There is exactly one maximal element in $\mathcal{J}(D^0_{\bar{x}})$, namely $\top$. This means as well that in $\mathrm{HA}[\bar{x}]$ there is one maximal join-irreducible element. This is of course the algebraic version of the *disjunction property* for intuitionistic propositional calculus:

$$\vdash_i t_1(\bar{x}) \vee t_2(\bar{x}) \qquad \text{implies} \qquad \vdash_i t_1(\bar{x}) \text{ or } \vdash_i t_2(\bar{x}),$$

for all terms t_1 and t_2.

The preceeding example holds slightly more generally.

Example 6.4 (*The disjunction property.*) A finitely presented Heyting algebra H has the *disjunction property* iff $\top_H$ is join irreducible. Then $H = \mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \to D^0]$ has the disjunction property if and only if D^0 has.

Example 6.5 (*Minimal elements in Heyting algebras, free on a distributive lattice.*) The forgetful functor $\mathsf{HA} \to \mathsf{DLat}$ has a left adjoint, which sends a distributive lattice to the Heyting algebra freely generated by it.

A Heyting algebra H is free on a finite distributive lattice D iff $H \cong \mathrm{HA}[\alpha^{-1,0}\colon \{0,1\} \to D]$. As in the example above, H has $\#\mathcal{J}(D)$ many minimal join-irreducible elements. H has the disjunction property iff D has.

In Example 6.3 we calculated the number of atoms (the minimal join-irreducible elements) of $\mathrm{HA}[\bar{x}]$. The following Proposition 6.6 gives a more general way of calculating the number of atoms, and as a corollary we get an explicit description of the atoms of $\mathrm{HA}[\bar{x}]$.

Proposition 6.6 *For each finitely presented Heyting algebra H there is a canonical bijection between the set of atoms of a Heyting algebra,* $\mathrm{At}(H)$*, and the set of Heyting algebra morphisms from H to $B = \{\bot, \top\}$.*

Proof. The atoms of a distributive lattice are exactly the maximal principal prime filters. Now prime filters F of the Heyting algebra H correspond to maps of distributive lattices $\phi\colon H \to B$ under the correspondence $F = \phi^{-1}(\top)$. Then F is principal if and only if ϕ has a left-adjoint $\phi_!$ (sending the top element of B to the generator of F). Finally, one easily checks that this generator is an atom if and only if the Frobenius identity $\phi_!(a \wedge \phi(h)) = \phi_!(a) \wedge h$ holds for all $a \in B, h \in H$, which is equivalent to say that ϕ preserves implication. To sum up, we proved that the atoms correspond to those Heyting algebra morphisms $H \to B$ that have a left adjoint. But any morphism of Heyting algebras $H \to B$ is surjective and clearly B is finitely presented over H (provided that H is finitely presented), so we deduce that such a map is actually a quotient map which always has both adjoints. (Note that we do not have to appeal to Pitts' uniform interpolation theorem for this argument.) □

Corollary 6.7 *The atoms in* $\mathrm{HA}[\bar{x}]$ *are the elements*

$$a_A = \bigwedge\{x \mid x \in A\} \wedge \bigwedge\{\neg x \mid x \in \complement A\},$$

for $A \in \mathcal{P}(\bar{x})$.

Table 1: Number of join-irreducible elements

# of generators	$\#\mathcal{J}(D^0_{\bar{x}})$	$\#\mathcal{J}(D^1_{\bar{x}})$	$\#\mathcal{J}(D^2_{\bar{x}})$
0	1	1	1
1	2	3	4
2	4	13	718
3	8	159	?
4	16	33.337	?
5	32	2.147.648.859	?
6	64	9.223.372.049.740.171.909	?

Proof. Given $\varphi\colon \mathrm{HA}[\bar{x}] \to B$ induced by $x \in A \mapsto 1$ and $x \in \complement A \mapsto 0$ the filter $\varphi^{-1}(1)$ is the unique upward closed set $\uparrow a$ so that

$$\begin{tikzcd} \mathrm{HA}[\bar{x}] \arrow[r, "\varphi"] \arrow[d] & B \\ \mathrm{HA}[\bar{x}]/a \arrow[ur, "\cong", "{\substack{A\mapsto 1\\ \complement A\mapsto 0}}"'] & \end{tikzcd}$$

commutes, so that a has the form as stated in the corollary, and these are the atoms. $\square$

Finally, let us say something on the size of the posets $\mathcal{J}(D^n_{\bar{x}})$. Table 1 lists some numbers calculated using the following lemma. The values in case of two generators were independently calculated in [9].

Lemma 6.8 *(i) $\mathcal{J}(D^0_{\bar{x}})$ has 2^n elements.*

(ii) If $j(k)$ denotes the number of elements in $\mathcal{J}(D^1_{x_1,\dots,x_k})$ (or in $\mathcal{J}(D^1_{\emptyset})$ if $k=0$) then $j(n)$ can be calculated using the recursion

$$j(n) = 2^{(2^n-1)} + \sum_{k=0}^{n-1} (-1)^{(n-1)-k} \binom{n}{k} j(k).$$

Or in closed form, $j(n) = \sum_{k=0}^{n} \binom{n}{k} 2^{(2^k-1)}$.

Proof. The first part holds since $\mathcal{J}(D^0_{\bar{x}})$ is isomorphic to $\mathcal{P}(\bar{x})$. For the second, $\mathrm{HA}[\emptyset] = \{0,1\}$ and $\mathcal{J}(D^1_{\emptyset}) = \{1\}$. The recursion just counts rooted subsets in the poset $\mathcal{P}(\bar{x})^{\mathrm{op}}$. $\square$

7 Prime formulae

We start with a well known observation:

Lemma 7.1 *The following are equivalent for a finitely presented Heyting algebra $H = \mathrm{HA}[\alpha^{-1,0}]$:*

(i) H has the disjunction property.

(ii) All D^n have the disjunction property.

(iii) The unique elements $\Delta^n \in D^{n+2}$ that satisfy $H/\Delta^n \cong D^n$ are join-irreducible.

Proof. The first two equivalences are immediate, the others follow from the following straight forward lemma. □

Lemma 7.2 *Consider a quotient map of Heyting algebras $H \twoheadrightarrow H/h$. Then $h \in \mathcal{J}(H)$ iff H/h has the disjunction property.* □

Example 7.3 In Definition 2.2 we defined the terms $\Delta^n(\bar{x})$ which represent the unique elements in the free Heyting algebra $\mathrm{HA}[\bar{x}]$ such that $\mathrm{HA}[\bar{x}]/\Delta^n(\bar{x})$ is isomorphic to the algebra of terms of implicational degree bounded by n, i.e., to $D^n_{\bar{x}}$. Since $\mathrm{HA}[\bar{x}]$ has the disjunction property we deduce that $\Delta^n(\bar{x})$ is join-irreducible in H.

We want to say more about the elements $\Delta^n(\bar{x}) \in \mathrm{HA}[\bar{x}]$. In fact, we look at the more general situation of a Heyting algebra $H \cong \mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \to D^0]$ and $\Delta^n \in D^{n+2}$ the elements satisfying $H/\Delta^n \cong D^n$. In the sequel we use the order isomorphism $\gamma_D\colon \mathcal{J}(D) \overset{\cong}{\to} \mathcal{M}(D)$ of a finite distributive lattice D.

Lemma 7.4 *(i) $\Delta^n = \bigwedge\{(p \to m) \to m \mid p \in \mathcal{J}(D^n), m \in \mathcal{M}(D^n), p \not\leq m\}$.*

(ii) $\Delta^n = \bigwedge\{(p \to \gamma_{D^n}(p)) \to \gamma_{D^n}(p) \mid p \in \mathcal{J}(D^n)\}$.

Proof. Write π^n for the quotient $H \to D^n$. As said before, all information contained in Δ^n is what happens to elements in D^{n+1}, i.e.,

$$\Delta^n = \bigwedge\{d \leftrightarrow \pi^n d \mid d \in D^{n+1}\}.$$

As a lattice D^{n+1} is generated by elements of the form $a \to b$ $(a, b \in D^n)$, and a is the join of join-irreducibles in D^n, and b is the meet of meet-irreducibles, so that

$$\Delta^n = \bigwedge\{(p \to m) \leftrightarrow \pi^n(p \to m) \mid p \in \mathcal{J}(D^n), m \in \mathcal{M}(D^n), p \not\leq m\}$$

(the pairs where $p \leq m$ do not contribute). But $\pi^n(p \to m) = p \to_{D^n} m$, which in case $p \not\leq m$ equals m since $p \to_{D^n} m \wedge p \leq m$. Finally, since $(p \to m) \leftrightarrow m = (p \to m) \to m$ the first equation follows.

For the second we note that by Lemma 4.4 the lattice D^{n+1} is generated by the elements $p \to \gamma_{D^n}(p)$ for $p \in \mathcal{J}(D^n)$. Since $\gamma_{D^n}(p)$ is meet-irreducible and $p \not\leq \gamma_{D^n}(p)$ the same calculation as above proves the second equality. $\Box$

Example 7.5 From the explicit description of the free Heyting algebra on one generator (see for example [10], p. 35) one easily calculates that

$$\Delta^0(x) = \neg\neg x;$$

$$\Delta^1(x) = \neg\neg x \to x;$$

$$\Delta^2(x) = (\neg\neg x \to x) \to (x \vee \neg x)$$

and for example $\Delta^0(x, y) = \neg\neg(x \wedge y) \wedge (x \to y) \to y \wedge (y \to x) \to x$. While these terms are easy to calculate we note that complexity increases dramatically. To calculate $\Delta^2(x, y)$ one has to know the poset $\mathcal{J}(D^2_{\bar{x}})$, a poset with 718 elements (see Table 1).

Note that the inclusion $D^0_{\bar{x}} \hookrightarrow D^0_{\bar{x},\bar{y}}$ is induced on the level of join-irreducible elements by the projection $\mathcal{P}(\bar{x}, \bar{y}) \twoheadrightarrow \mathcal{P}(\bar{x})$, which is both open and co-open. Thus

$$D^0_{\bar{x}} \hookrightarrow D^0_{\bar{x},\bar{y}}, \qquad \bar{x} \mapsto \bar{x} \in D^0_{\bar{x},\bar{y}}$$

is a morphism of Heyting algebras (and co-Heyting) and $\Delta^0(\bar{x}, \bar{y}) \vdash \Delta^0(\bar{x})$.

In contrast, the canonical map $D^1_{x,y} \hookrightarrow D^1_{x,y,z}$ is *not* a morphism of Heyting algebras.

8 Functoriality

In this section, we turn the construction $\mathrm{HA}[\alpha^{-1,0}\colon D^{-1} \to D^0]$ into a functor. Let $\mathcal{D}$ denote the following sub-category of the arrow category $\mathsf{DLat}^{\to}$:

(i) *Objects* are morphisms $\alpha^{-1,0}: D^{-1} \to D^0$ between distributive lattices (not necessarily finite) such that D^0 has implications between elements in the image of $\alpha^{-1,0}$.

(ii) An *arrow* $f: (\alpha^{-1,0}: D^{-1} \to D^0) \to (\beta^{-1,0}: E^{-1} \to E^0)$ is a pair of morphisms (f^{-1}, f^0) of distributive lattices such that

$$\begin{tikzcd} D^{-1} \arrow[r, "\alpha^{-1,0}"] \arrow[d, "f^{-1}"'] & D^0 \arrow[d, "f^0"] \\ E^{-1} \arrow[r, "\beta^{-1,0}"'] & E^0 \end{tikzcd}$$

commutes, and where f^0 is $\alpha^{-1,0}$-open. (Note that this requires implicitly that E^0 has implications between elements in the image of f^0.)

Given $\alpha^{-1,0}: D^{-1} \to D^0$, we consider the Heyting algebra $\mathrm{HA}[\alpha^{-1,0}: D^{-1} \to D^0]$, the Heyting algebra with the universal property that Heyting algebra morphisms $\mathrm{HA}[\alpha^{-1,0}] \to H$ correspond naturally to $\alpha^{-1,0}$-open distributive lattice maps $D^0 \to H$. There is no doubt that $\mathrm{HA}[\alpha^{-1,0}]$ exists: it is a quotient of the Heyting algebra freely generated by the lattice D^0, which in turn is a quotient of the free Heyting algebra on the *set* D^0. This shows as well that given $\alpha^{-1,0}$, there is a *canonical* way to represent $\mathrm{HA}[\alpha^{-1,0}]$, namely as the set of equivalence classes of an equivalence relation on the set of terms in the set of free variables D^0. This last observation is needed to turn our construction into a functor, and not just into a pseudo-functor.

Next we define $\mathrm{HA}[-]$ on arrows in $\mathcal{D}$: Suppose we are given $f = (f^{-1}, f^0): (\alpha^{-1,0}: D^{-1} \to D^0) \to (\beta^{-1,0}: E^{-1} \to E^0)$. By the universal property of $\mathrm{HA}[\beta^{-1,0}]$ there is a canonical $\beta^{-1,0}$-open map $\beta^{0,\infty}: E^0 \to \mathrm{HA}[\beta^{-1,0}]$. The composite $\beta^{0,\infty} \circ f^0: D^0 \to \mathrm{HA}[\beta^{-1,0}]$ is $\alpha^{-1,0}$-open, so induces a map of Heyting algebras $\mathrm{HA}[f]: \mathrm{HA}[\alpha^{-1,0}] \to \mathrm{HA}[\beta^{-1,0}]$, and of course, the following diagram commutes:

$$\begin{tikzcd} D^0 \arrow[r, "\alpha^{0,\infty}"] \arrow[d, "f^0"'] & \mathrm{HA}[\alpha^{-1,0}] \arrow[d, "\mathrm{HA}[f]"] \\ E^0 \arrow[r, "\beta^{0,\infty}"'] & \mathrm{HA}[\beta^{-1,0}] \end{tikzcd}$$

The universal property of the Heyting algebras $\mathrm{HA}[-]$ guarantees that we obtain this way a functor.

Lemma 8.1 *The functor* $\mathrm{HA}[-]: \mathcal{D} \to \mathsf{HA}$ *is left adjoint to the inclusion functor* $i_{(-)}: \mathsf{HA} \to \mathcal{D}$, *($i_{(-)}$ sends a Heyting algebra H to* $(\mathrm{id}_H: H \to H)$*). Moreover,* $\mathrm{HA}[-] \circ i_{(-)} \cong \mathrm{id}_{\mathsf{HA}}$.

Proof. There are natural bijections between the following data:

$$
\begin{array}{cl}
\mathrm{HA}[\alpha^{-1,0}] \to H & \text{in } \mathsf{HA} \\
\hline
\alpha^{-1,0}\text{-open maps } D^0 \to H & \text{in } \mathsf{DLat} \\
\hline
\text{maps} \quad \begin{array}{ccc} D^{-1} & \longrightarrow & H \\ {\scriptstyle \alpha^{-1,0}}\big\downarrow & & \big\| \\ D^0 & \longrightarrow & H \end{array} & \text{in } \mathcal{D}
\end{array}
$$

which shows adjointness. Finally, given H in HA, id_H-open maps $H \to H'$ are nothing but Heyting algebra morphisms $H \to H'$, so that $H \cong \mathrm{HA}[\mathrm{id}_H]$. $\square$

We note that it follows from general category theory that the category HA can be obtained from $\mathcal{D}$ as a category of fractions (see for example [4], Section I.5.9).

After these general remarks we restrict ourselves to the full sub-category $\mathcal{D}_{\mathrm{fin}}$ of $\mathcal{D}$ containing those objects $\alpha^{-1,0}: D^{-1} \to D^0$ where D^{-1} and D^0 are finite. Using Proposition 5.1 we get a functor

$$\mathrm{HA}[-]: \mathcal{D}_{\mathrm{fin}} \to \mathsf{HA}_{\mathrm{f.p.}},$$

up to isomorphism full on objects (Proposition 5.2). Note that in this case the right adjoint does not exist. Still, $\mathrm{HA}[-]$ so restricted is 'full' on arrows in the following liberate sense: Given $\varphi: H \to H'$ in $\mathsf{HA}_{\mathrm{f.p.}}$ there exists $f: (\alpha^{-1,0}) \to (\beta^{-1,0})$ in $\mathcal{D}_{\mathrm{fin}}$ so that $\mathrm{HA}[\alpha^{-1,0}] \cong H$, $\mathrm{HA}[\beta^{-1,0}] \cong H'$, and

$$
\begin{array}{ccc}
\mathrm{HA}[\alpha^{-1,0}] & \cong & H \\
{\scriptstyle \mathrm{HA}[f]}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\
\mathrm{HA}[\beta^{-1,0}] & \cong & H'
\end{array}
$$

commutes.

Write Σ for the class of arrows in $\mathcal{D}_{\mathrm{fin}}$ which are send by $\mathrm{HA}[-]$ to isomorphisms in $\mathsf{HA}_{\mathrm{f.p.}}$. It is clear that $\mathsf{HA}_{\mathrm{f.p.}}$ is just the category of fractions of $\mathcal{D}_{\mathrm{fin}}$, inverting the arrows in Σ, that is, $\mathsf{HA}_{\mathrm{f.p.}} \cong \mathcal{D}_{\mathrm{fin}}[\Sigma^{-1}]$. We leave it to the reader to show that Σ in fact admits a calculus of fractions:

Proposition 8.2 *The class Σ of those arrows in $\mathcal{D}_{\text{fin}}$ which are sent to isomorphisms in $\mathsf{HA}_{\text{f.p.}}$ admits a calculus of fractions, and $\mathcal{D}_{\text{fin}}[\Sigma^{-1}] \cong \mathsf{HA}_{\text{f.p.}}$.* $\Box$

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