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Things that can and things that can't be done in PRA

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Abstract

It is well-known by now that large parts of (non-constructive) mathematical reasoning can be carried out in systems $\mathcal{T}$ which are conservative over primitive recursive arithmetic **PRA** (and even much weaker systems). On the other hand there are principles **S** of elementary analysis (like the Bolzano-Weierstraß principle, the existence of a limit superior for bounded sequences etc.) which are known to be equivalent to arithmetical comprehension (relative to $\mathcal{T}$) and therefore go far beyond the strength of **PRA** (when added to $\mathcal{T}$).

In this paper we determine precisely the arithmetical and computational strength (in terms of optimal conservation results and subrecursive characterizations of provably recursive functions) of weaker function parameter-free schematic versions $\mathbf{S}^-$ of **S**, thereby exhibiting different levels of strength between these principles as well as a sharp borderline between fragments of analysis which are still conservative over **PRA** and extensions which just go beyond the strength of **PRA**.

1 Introduction

It is well-known by now, mainly from work done on the program of so-called reverse mathematics (although not using the reverse direction explicitly), that substantial parts of mathematics (and in particular analysis) can be carried out in systems $\mathcal{T}$ which are conservative

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over primitive recursive arithmetic PRA (see [25] for a systematic account). This is of interest for mainly two reasons

1) If a Π^0_2-sentence A is provable in $\mathcal{T}$ and the conservation of $\mathcal{T}$ over PRA has been established proof-theoretically, then one can extract a primitive recursive program which realizes A from a given proof. Typically the resulting program will have a quite restricted complexity or rate of growth (compared to merely being primitive recursive). In fact in a series of papers we have shown that in many cases even a polynomial bound is guaranteed (see [9],[11],[14] among others).

2) One can argue that PRA formalizes what has been called finitistic reasoning (see e.g. [26]). If the conservation of $\mathcal{T}$ over PRA has been established finitistically (which is possible for mathematically strong systems $\mathcal{T}$ (see [22],[8]), then all the mathematics which can be carried out in $\mathcal{T}$ has a finitistic justification (see [24],[25] for a discussion of this).

In this paper we exhibit a sharp boundary between finistically reducible parts of analysis and extensions which provably go beyond the strength of PRA.
More precisely we study the (proof-theoretical and numerical) strength of function parameter-free schematic forms of[1]

- the convergence (with modulus of convergence) of bounded monotone sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ principle (PCM)
- the Bolzano-Weierstraß principle (BW) for $(a_n)_{n\in\mathbb{N}} \subset [0,1]^d$
- the Ascoli-Arzela principle for bounded sequences $(f_n)_{n\in\mathbb{N}} \subset C[0,1]$ of equicontinuous functions (A-A)
- the existence of the limit superior principle for $(a_n)_{n\in\mathbb{N}} \subset [0,1]$ (Limsup).

Let us discuss what we mean by 'function parameter-free schematic form' in more detail for BW:
'Schematic' means that an instance BW(t) of BW is given by a term t of the underlying system which defines a sequence in $[0,1]^d$. We allow number parameters k in t, i.e. we consider sequences $\forall k \in \mathbb{N}$ BW($t[k]$) of instances of BW, but not function parameters.
Allowing function parameters to occur in BW would make the schema equivalent to the single second-order sentence

$$(*)\ \forall (a_n) \subset [0,1]^d\ \mathrm{BW}(a_n).$$

[1] For precise formalizations of these principles in systems based on number and function variables see [12] on which the present paper partially relies. We slightly deviate from the notation used in [12] by writing (PCM),(PCM$_{ar}$) instead of (PCM2),(PCM1).

It is well-known by the work on program of reverse mathematics that (∗) is equivalent to the schema of arithmetical comprehension (relative to weak fragments of second-order arithmetic).
On the other hand, the restriction of BW to function parameter-free instances – in short: BW^- – is much weaker since the iterated use of BW is now no longer possible.

We calibrate precisely the strength of PCM^-, BW^-, $\mathrm{A\text{-}A}^-$ and Limsup^- relative second-order extensions of primitive recursive arithmetic PRA (thereby completing research started in [12]). It turns out that the results depend heavily on what type of extension of PRA we choose:
One option is straightforward: extend PRA by number and variables x^0 and quantifiers for objects f^ρ of type-level 1, i.e. $\rho = 0(0)\cdots(0)$, where $\rho(0)$ is the type of functions from $\mathbb{N}$ into objects of type ρ (note that modulo λ-abstraction objects of type $0\overbrace{(0)\dots(0)}^{n}$ are just n-ary number theoretic functions).[2] We have the axioms and rules of many-sorted classical predicate logic as well as symbols and defining equations for all primitive recursive functionals of type level ≤ 2 in the sense of Kleene [7] (i.e. ordinary primitive recursion uniformly in function parameters, for details see e.g. [6](II.1) or [21]). We also have a schema of quantifier-free induction (w.r.t. to this extended language) and λ-abstraction for number variables, i.e.

$$(\lambda \underline{y}.t[\underline{y}])\underline{x} = t[\underline{x}],\ \underline{x}, \underline{y} \text{ tuples of the same length.}$$

So PRA^2 is the second-order fragment of the (restricted) finite type system $\widehat{\mathrm{PA}}^\omega\!\restriction$ from [3]. It is clear that the resulting system PRA^2 is conservative over PRA.
We often write 1 instead of 0(0).

Another option is to impose a restriction on the type-2-functionals which are allowed. We include functionals of arbitrary Grzegorczyk level in the sense of [9][3] (including all elementary recursive functionals) but not the iteration functional

$$(It)\ \Phi_{it}(0, y, f) = y,\ \Phi_{it}(x + 1, y, f) = f(x, \Phi_{it}(x, y, f)),$$

although it is primitive recursive in the sense of Kleene (and not only in the extended sense of Gödel [5], '=' is equality between natural numbers). We call the resulting system PRA^2_-.
On easily shows that PRA^2 is a definitorial extension of $\mathrm{PRA}^2_- + (It)$.

[2]So we could have used also variables and quantifiers for n-ary functions instead and treat sequences of functions as $f_n := \lambda m.f(n, m)$. However the use of variables $f^{0(0)\cdots(0)}$ is more convenient since it avoids the use of the λ-operator in many cases.

[3]This means that we allow all the type-2-functionals Φ_n from [9] plus a bounded search operator and bounded recursion – uniformly in function parameters – on the ground type (see [9]).

EA^2 is the restriction of PRA^2_- to elementary recursive function(al)s only (see [20] for a definition of 'elementary recursive functional').

Remark 1.1 *In contrast to the class of primitive recursive functions, there exists no Grzegorzcyk hierarchy for primitive recursive functionals which would include all of them: if Φ_{it} would occur at a certain level of such a hierarchy, then this hierarchy would collapse to this level since all primitive recursive functions can be obtained from the initial functions and Φ_{it} by substitution.*

The schema of quantifier-free choice for numbers is given by

$$AC^{0,0}\text{-qf}: \ \forall x^0 \exists y^0 A_0(x,y) \rightarrow \exists f \forall x A_0(x, fx),$$

where A_0 is a quantifier-free formula.[4] We also consider the binary König's lemma as formulated in [27]:

$$\mathrm{WKL} :\equiv \forall f^1 (T(f) \wedge \forall x^0 \exists n^0 (lth(n) =_0 x \wedge f(n) =_0 0) \rightarrow \exists b \leq_1 1 \forall x^0 (f(\overline{b}x) =_0 0)),$$

where $b \leq_1 1 :\equiv \forall n(bn \leq 1)$ and

$$T(f) :\equiv \forall n^0, m^0 (f(n * m) = 0 \rightarrow f(n) = 0) \wedge \forall n^0, x^0 (f(n * \langle x \rangle) = 0 \rightarrow x \leq 1)$$

(here $lth, *, \overline{b}x, \langle \cdot \rangle$ refer to a standard elementary recursive coding of finite sequences of numbers).

One easily shows that the schema of Σ^0_1-induction is derivable in PRA^2+ $AC^{0,0}$-qf (but not in PRA^2_-+ $AC^{0,0}$-qf). The schema of recursive comprehension is already provable in PRA^2_-+ $AC^{0,0}$-qf. So PRA^2+ $AC^{0,0}$-qf (resp. PRA^2+ $AC^{0,0}$-qf + WKL) is a function variable version of the system RCA_0 (resp. WKL_0) used in reverse mathematics, which uses set variables instead of function variables.
The main results of this paper are[5]

Theorem 1.2 *1)* PRA^2_-+PCM^- *contains* PRA+Σ^0_1-IA.

2) PRA^2_-+$AC^{0,0}$-qf+WKL+PCM^-+BW^-+A-A^- *is* Π^0_3-*(but not* Π^0_4-*)conservative over* PRA+Σ^0_1-IA *and hence* Π^0_2-*conservative over* PRA.

Corollary 1.3
The provably recursive functions of PRA^2_-+$AC^{0,0}$-qf+WKL+PCM^-+BW^-+A-A^- *are exactly the primitive recursive ones.*

[4]Throughout this paper $A_0, B_0, C_0, \ldots$ denote quantifier-free formulas.
[5]Here and in the following we denote the (conservative) extension of PRA by first-order predicate logic also by PRA.

Theorem 1.4 *1)* PRA^2_-+Limsup$^-$ *contains* PRA+Σ^0_2-IA.

2) PRA^2_-+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$+Limsup$^-$ *is* Π^0_4*-conservative over* PRA+Σ^0_2-IA.

Corollary 1.5 *The provably recursive functions of* PRA^2_-+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$+Limsup$^-$ *are exactly the* $\alpha(<\omega^{(\omega^\omega)})$*- recursive ones,*[6] *i.e. the functions definable in the fragment* T_1 *of Gödel's* T ([5]) *with recursion of level* ≤ 1 *only, which includes the Ackermann function.*
This results also holds for EA^2 *instead of* PRA^2_-.

Theorem 1.6 PRA^2+ PCM$^-$ *is closed under the function parameter-free rule* Σ^0_2-IR$^-$ *of* Σ^0_2*-induction.*

Corollary 1.7 *Every* $\alpha(<\omega^{(\omega^\omega)})$*-recursive (i.e.* T_1*-definable) function (including the Ackermann function) is provably recursive in* PRA^2+ PCM$^-$.

Together with the fact that PRA^2+AC^{0,0}-qf+WKL is Π^0_2-conservative over PRA (see [22] and for more general results [8]) this yields

Corollary 1.8 PRA^2+AC^{0,0}-qf+WKL $\not\vdash$ PCM$^-$ *(this holds a fortiori for* BW$^-$, A-A$^-$ *and* Limsup$^-$ *instead of* PCM$^-$).

Theorem 1.9 *Let* P *be* PCM$^-$, BW$^-$ *or* A-A$^-$. *Then* PRA^2+ AC^{0,0}-qf +P *contains* PRA +Π^0_2-IA (=PRA +Σ^0_2-IA).

So relative to PRA^2+ AC^{0,0}-qf, the principles PCM$^-$, BW$^-$ and A-A$^-$ are not conservative over PRA.
Relative to PRA^2_- (+AC^{0,0}-qf +WKL) these principles are conservative over PRA but the principle Limsup$^-$ is not.

2 Preliminaries

We first indicate how to represent real numbers and the basic arithmetical operations and relations on them in EA^2.
The results of this section a fortiori hold for PRA^2_- instead of EA^2.

[6]Here α-recursive is meant in the sense of [16], i.e. unnested. In contrast to this the notion of α-recursiveness as used e.g. in [2],[21] corresponds to nested recursion.

The representation of $\mathbb{R}$ presupposes a **representation of** $\mathbb{Q}$: Let j be the Cantor pairing function. Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n,m. $j(n,m)$ represents

$$\text{the rational number } \frac{\frac{n}{2}}{m+1}, \text{ if } n \text{ is even, and the negative rational } -\frac{\frac{n+1}{2}}{m+1} \text{ if } n \text{ is odd.}$$

Because of the surjectivity of j, every natural number is a code of a uniquely determined rational number. On the codes of $\mathbb{Q}$, i.e. on $\mathbb{N}$, we define an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1, j_1 n_2 \text{ both are even}$$

and analogously in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold iff $ad =_0 cb$ (for $bd > 0$).

On $\mathbb{N}$ one easily defines functions $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}} :_{\mathbb{Q}}, \max_{\mathbb{Q}}, \min_{\mathbb{Q}} \in \mathrm{EA}^2$ and (quantifier-free) relations) $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the corresponding functions and relations on $\mathbb{Q}$. In the following we sometimes omit the index $\mathbb{Q}$ if this does not cause any confusion.

Notational convention: For better readability we often write e.g. $\frac{1}{k+1}$ instead of its code $j(2,k)$ in $\mathbb{N}$. So e.g. we write $x^0 \leq_{\mathbb{Q}} \frac{1}{k+1}$ for $x \leq_{\mathbb{Q}} j(2,k)$.

Real numbers are represented as Cauchy sequences $(q_n)_{n\in\mathbb{N}}$ of rational numbers with fixed rate of convergence

$$\forall n \forall m, \tilde{m} \geq n (|q_m - q_{\tilde{m}}| \leq \frac{1}{n+1}).$$

By the coding of rational numbers as natural numbers, **sequences of rationals** are just functions f^1 (and every function f^1 can be conceived as a sequence of rational numbers in a unique way). In particular representatives of real numbers are functions f^1 modulo this coding. We now show that **every** function can be viewed of as an representative of a uniquely determined Cauchy sequence of rationals with modulus $1/(k+1)$ and therefore can be conceived as an representative of a uniquely determined real number.
To this end we need the following functional $\widehat{f}$.

Definition 2.1 *The functional $\lambda f^1.\widehat{f} \in \mathrm{EA}^2$ is defined such that*

$$\widehat{f}n = \begin{cases} fn, \ \text{if } \forall k, m, \tilde{m} \leq_0 n (m, \tilde{m} \geq_0 k \to |fm -_{\mathbb{Q}} f\tilde{m}| \leq_{\mathbb{Q}} \frac{1}{k+1}) \\ f(n_0 - 1) \text{ for } n_0 := \min l \leq_0 n [\exists k, m, \tilde{m} \leq_0 l (m, \tilde{m} \geq_0 k \wedge |fm -_{\mathbb{Q}} f\tilde{m}| >_{\mathbb{Q}} \frac{1}{k+1})], \\ \text{otherwise.} \end{cases}$$

One easily proves in EA^2 that

1) if f^1 represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$, then $\forall n^0(fn =_0 \widehat{f}n)$,

2) for every f^1 the function $\widehat{f}$ represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$.

Following the usual notation we write (x_n) instead of fn and $(\widehat{x}_n)$ instead of $\widehat{f}n$.

Definition 2.2 *1)* $(x_n) =_{\mathbb{R}} (\tilde{x}_n) :\equiv \forall k^0(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| \leq_{\mathbb{Q}} \frac{3}{k+1})$;

2) $(x_n) <_{\mathbb{R}} (\tilde{x}_n) :\equiv \exists k^0(\widehat{\tilde{x}}_k - \widehat{x}_k >_{\mathbb{Q}} \frac{3}{k+1})$;

3) $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n) :\equiv \neg(\widehat{\tilde{x}}_n) <_{\mathbb{R}} (\widehat{x}_n)$;

4) $(x_n) +_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} +_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

5) $(x_n) -_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} -_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

6) $|(x_n)|_{\mathbb{R}} := (|\widehat{x}_n|_{\mathbb{Q}})$;

7) $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2(n+1)k} \cdot_{\mathbb{Q}} \widehat{\tilde{x}}_{2(n+1)k})$, *where* $k := \lceil \max_{\mathbb{Q}}(|x_0|_{\mathbb{Q}} + 1, |\tilde{x_0}|_{\mathbb{Q}} + 1)\rceil$;

8) For (x_n) *and* l^0 *we define*

$$(x_n)^{-1} := \begin{cases} (\max_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{1}{l+1})^{-1}), & \textit{if } \widehat{x}_{2(l+1)} >_{\mathbb{Q}} 0 \\ (\min_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{-1}{l+1})^{-1}), & \textit{otherwise}; \end{cases}$$

9) $\max_{\mathbb{R}}((x_n),(\tilde{x}_n)) := (\max_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n)), \quad \min_{\mathbb{R}}((x_n),(\tilde{x}_n)) := (\min_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n))$.

Sequences of real numbers are coded as sequences $f^{1(0)}$ of codes of real numbers.

The principles PCM and PCM_{ar} of convergence for bounded monotone sequences are given by[7]

$$\mathrm{PCM}_{ar}(f^{1(0)}) :\equiv$$
$$\forall n(0 \leq_{\mathbb{R}} f(n+1) \leq_{\mathbb{R}} f(n)) \to \forall k \exists n \forall m, \tilde{m} \geq n(|fm -_{\mathbb{R}} f\tilde{m}| \leq \tfrac{1}{k+1}),$$

[7] The restriction to decreasing sequences and the special lower bound 0 is of course inessential.

$$\text{PCM}(f^{1(0)}) :\equiv$$
$$\forall n(0 \leq_{\mathbb{R}} f(n+1) \leq_{\mathbb{R}} f(n)) \to \exists g \forall k \forall m, \tilde{m} \geq gk(|fm -_{\mathbb{R}} f\tilde{m}| \leq \tfrac{1}{k+1}).$$

Relative to PRA^2_-, PCM is equivalent to the principle stating the convergence of f with a modulus of convergence (PCM_{ar} does not imply in PRA^2_- the existence of a limit since reals have to be given as Cauchy sequences with given rate of convergence). For monotone sequences the existence of a modulus of convergence can be obtained from the existence of a limit by the use of $\text{AC}^{0,0}$-qf. So relative to $\text{PRA}^2_- + \text{AC}^{0,0}$-qf we don't have to distinguish between our formulation of PCM, the existence of a limit of f and the existence of a limit together with a modulus of convergence.

PCM^- and PCM^-_{ar} denote the function parameter-free schematic versions of $\text{PCM}(f)$ and $\text{PCM}_{ar}(f)$ (in the sense discussed in the introduction).

Let $\text{BW}(f)$ be the statement

$$(f^{1(0)} \text{ codes a sequence } \subset [0,1]^d \Rightarrow \text{ this sequence has a limit point in } [0,1]^d)$$

(for details see [12]). In [12] we also discuss the (relative to PRA^2_- slightly stronger) principle $\text{BW}^+(f)$ expressing that f possesses a convergent subsequence (with modulus of convergence). All the results of this paper are valid for both versions $\text{BW}(f)$ and $\text{BW}^+(f)$ and so we don't distinguish between them and denote their function parameter-free schematic forms both by BW^-. Likewise for the Arzela-Ascoli lemma (see [12] for the precise formulations of A-A(f) and A-A$^+$(f)).

We define the limit superior according to the so-called ε-definition, i.e. $a \in [-1,1]$ is the limit superior of $(x_n) \subset [-1,1]$ if[8]

$$(*)\ \forall k(\forall m \exists n > m(|a - x_n| \leq \frac{1}{k+1}) \wedge \exists l \forall j > l(x_j \leq a + \frac{1}{k+1})).$$

$(*)$ implies (relative to PRA^2_-) that a is the maximal limit point of (x_n). The reverse direction follows with the use of BW (we don't know whether it can be proved in PRA^2_-).

$\text{Limsup}(f)$ is the principle stating

$$(f \text{ codes a sequence} \subset [-1,1] \ \Rightarrow \ \text{this sequence has a } \limsup \text{ in the sense of } (*)).$$

Limsup^- is the corresponding function parameter-free schematic version.

[8] Again the restriction to the particular bound 1 is inessential.

3 Things that can be done in (a conservative extension of) PRA resp. in PRA $+\Sigma^0_2$-IA

In this section we draw some consequences of our results from [12] and [13] and formulate them in a way which fits into the present framework.

Theorem 3.1 *Every* Π^0_3*-theorem of* $\mathrm{PRA}^2_-+\mathrm{AC}^{0,0}$-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$ *is provable in* PRA+Σ^0_1-IA.

Proof: From the proofs of propositions 5.5 and 5.6 from [12] and proposition 5.5.2) below it follows that there exist instances Π^0_1-CA(ξ_j) which prove, relative to E-$\mathrm{G}_\infty\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$-qf+$F^-$ all universal closures $\tilde{G}_i$ of the instances G_i of PCM$^-$, BW$^-$ and A-A$^-$ which are used in the proof of the Π^0_3-sentence $A \equiv \forall x \exists y \forall z\, A_0(x,y,z) \in$ PRA. The instances Π^0_1-CA(ξ_j) can be coded together into a single instance Π^0_1-CA(ξ) (see again the proof of proposition 5.5 from [12]). Since furthermore $\mathrm{PRA}^2_- \subset$ E-$\mathrm{G}_\infty\mathrm{A}^\omega$ and – by [9] (section 4) – WKL can be derived in E-$\mathrm{G}_\infty\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$-qf $+F^-$,[9] we obtain

$$\text{E-G}_\infty\text{A}^\omega + \text{AC}^{1,0}\text{-qf} + F^- \vdash \Pi^0_1\text{-CA}(\xi) \to A.$$

Corollary 4.7 from [13] (combined with the elimination of extensionality procedure as used in the proof of corollary 4.5 in [13]) yields that

$$\text{G}_\infty\text{A}^\omega + \Sigma^0_1\text{-IA} \vdash A,$$

and hence (since $\mathrm{G}_\infty\mathrm{A}^\omega + \Sigma^0_1$-IA can easily be seen to be conservative over PRA+Σ^0_1-IA)[10]

$$\text{PRA} + \Sigma^0_1\text{-IA} \vdash A.$$

□

Remark 3.2 *1) In section 4 below we will show that already* PRA^2_-+PCM$^-$ *contains* PRA+Σ^0_1-IA.

2) Already PRA^2_-+ $\mathrm{AC}^{0,0}$-qf+PCM$^-$ *is not* Π^0_4*-conservative over* PRA+Σ^0_1-IA*: From proposition 5.5 below it follows that* PRA^2_-+ $\mathrm{AC}^{0,0}$-qf+PCM$^-$ *proves* Π^0_1-CA$^-$ *and therefore every function parameter-free instance of the principle of* Π^0_1*-collection principle* Π^0_1-CP. *Hence* PRA+Π^0_1-CP *is a subsystem of* PRA^2_-+ $\mathrm{AC}^{0,0}$-qf+PCM$^-$. *However from* [17] *we know that there exists an instance of* Π^0_1-CP *which cannot be proved*

[9] In the proof of theorem 4.27 from [9], $\mathrm{AC}^{0,1}$-qf is only needed to derive the strong sequential version WKL_{seq} of WKL.

[10] We work here in the variant of $\mathrm{G}_\infty\mathrm{A}^\omega$ where the universal axioms 9) are replaced by the schema of quantifier-free induction.

in PRA $+\Sigma^0_1$-IA. *The claim now follows from the fact that (the universal closure of) every instance of* Π^0_1-CP *can be shown to be equivalent to a* Π^0_4*-sentence in* PRA$+\Sigma^0_1$-IA.

Corollary 3.3 *Let* $A \equiv \forall x \exists y A_0(x,y)$ *be a* Π^0_2*-sentence in* $\mathcal{L}(\text{PRA})$. *Then the following rule holds:*

$$\begin{cases} \text{PRA}^2_- + \text{AC}^{0,0}\text{-qf} + \text{WKL} + \text{PCM}^- + \text{BW}^- + \text{A-A}^- \vdash \forall x \exists y A_0(x,y) \\ \Rightarrow \textit{ one can extract a primitive recursive function } p \textit{ such that} \\ \text{PRA} \vdash A_0(x, px). \end{cases}$$

Proof: The corollary follows from theorem 3.1 and the well-known fact that PRA$+\Sigma^0_1$-IA is Π^0_2-conservative over PRA. □

Theorem 3.4 *Every* Π^0_4*-theorem of*
$\text{PRA}^2_- + \text{AC}^{0,0}\text{-qf} + \text{WKL} + \text{PCM}^- + \text{BW}^- + \text{A-A}^- + \text{Limsup}^-$ *is provable in* PRA $+\Sigma^0_2$-IA.

Proof: One easily shows (relative to $\text{PRA}^2_- + \text{AC}^{0,0}$-qf that Limsup$^-$ follows from Π^0_2-CA$^-$: for sequences $(q_n) \subset [0,1]$ of rational numbers this is particularly straightforward (the general case can be reduced to this one): by Π^0_2-CA define f such that for $i < 2^j$

$$f(i,j) = 0 \leftrightarrow \ \infty\text{-many elements of } (q_n) \text{ are in } [\tfrac{i}{2^j}, \tfrac{i+1}{2^j}].$$

Let $g(j) :=$ maximal $i < 2^j [f(i,j) = 0]$. Then (a_n) defined by $a_n := \frac{g(j)}{2^j}$ is a Cauchy sequence which converges (with rate 2^n) to the *limsup* (in the sense of $(*)$) of (q_n).
The theorem now follows from [13](corollary 4.7) similar to the use of this corollary in the proof of theorem 3.1 above. □

Remark 3.5 *In section 5 below we will show that already* $\text{PRA}^2_- + \text{Limsup}^-$ *contains* PRA$+\Sigma^0_2$-IA.

Definition 3.6 *By* T_n *we denote the fragment of Gödel's calculus* T *of primitive recursive functionals in all finite types where one only has recursor constants* R_ρ *with* $deg(\rho) \le n$ *(see* [19] *for details).*

Corollary 3.7 *Let* $A \equiv \forall x \exists y A_0(x,y)$ *be a* Π^0_2*-sentence in* $\mathcal{L}(\text{PRA})$. *Then the following rule holds:*

$$\begin{cases} \text{PRA}^2_- + \text{AC}^{0,0}\text{-qf} + \text{WKL} + \text{PCM}^- + \text{BW}^- + \text{A-A}^- + \text{Limsup}^- \vdash \forall x \exists y A_0(x,y) \\ \Rightarrow \textit{ one can extract a closed term } \Phi^1 \textit{ of } T_1 \textit{ such that} \\ T_1 \vdash A_0(x, \Phi x). \end{cases}$$

Proof: The corollary follows from theorem 3.4 and Parsons' result from [19] that PRA+Σ^0_{n+1}-IA has (via negative translation) a Gödel functional interpretation in T_n. $\Box$

Remark 3.8 *Our results in* [12] *and* [13] *actually can be used to obtain stronger forms of the corollaries 3.3 and 3.7 since in* [12],[13] *we*

1) *allowed finite type extensions of the systems in the corollaries 3.3 and 3.7,*

2) *considered more general conclusions* $A \equiv \forall u^1 \forall v \leq_\rho tu \exists z^\tau A_0(x,y,z)$ *(where* ρ *is an arbitrary type and* $\tau \leq 2$*) and showed how to extract uniform bounds* $\Phi \in T_0$ *(resp.* $\in T_1$ *in the case of corollary 3.7) such that* $\forall u^1 \forall v \leq_\rho tu \exists z \leq_\tau \Phi u A_0(x,y,z)$*,*

3) *allowed the instances of* PCM, BW, A-A, Limsup *to depend on the parameters* u, v *of the conclusion and*

4) *allowed more general analytical axioms* Δ *(than only* F^-*).*

4 Some proof theory of $\mathbf{PRA}^2 + \Pi^0_1$-$\mathbf{AC}^-$

We consider the following schemata:

$$\begin{aligned}
&\Pi^0_1\text{-CA}^- &&: \exists f^1 \forall x^0 (fx = 0 \leftrightarrow \forall y A_0(x,y)),\\
&\Pi^0_1\text{-}\widehat{\text{AC}}^- &&: \exists f^1 \forall x^0, z^0 (\neg A_0(x,fx) \vee A_0(x,z)),\\
&\Pi^0_1\text{-AC}^- &&: \forall x^0 \exists y^0 \forall z^0 A_0(x,y,z) \to \exists f^1 \forall x, z A_0(x,fx,z),
\end{aligned}$$

where A_0 is quantifier–free and has **no function parameters**.
Π^0_1-CA(g) is the form of Π^0_1-CA$^-$ where $A_0(x,y)$ is replaced by $g(x,y) = 0$. Similarly for Π^0_1-$\widehat{\text{AC}}(g)$ and Π^0_1-AC(g). One easily verifies the following

Lemma 4.1

1) PRA^2 *proves the implications* Π^0_1-AC$^- \to \Pi^0_1$-$\widehat{\text{AC}}^- \to \Pi^0_1$-CA$^-$.

2) PRA^2+$AC^{0,0}$*-qf proves* Π^0_1-CA$^- \leftrightarrow \Pi^0_1$-$\widehat{\text{AC}}^- \leftrightarrow \Pi^0_1$-AC$^-$.

Proposition 4.2 1) PRA^2+Π^0_1-$\widehat{\text{AC}}^-$ *is closed under* Σ^0_2-IR$^-$ *(i.e. the induction rule for* Σ^0_2*-formulas without function parameters) and hence contains the first-order system* PRA+Σ^0_2-IR*.*

2) PRA^2 + Π^0_1-$\widehat{\text{AC}}^-$ *proves every* Π^0_3*-theorem of* PRA +Π^0_2-IA*.*

3) Every function which is definable in T_1 (i.e. every $\alpha(<\omega^{(\omega^\omega)})$-recursive function is provably recursive in $\mathrm{PRA}^2+\Pi^0_1\text{-}\widehat{\mathrm{AC}}^-$*. In particular* $\mathrm{PRA}^2+\Pi^0_1\text{-}\widehat{\mathrm{AC}}^-$ *(and a fortiori* $\mathrm{PRA}^2+\Pi^0_1\text{-AC}^-$*) proves the totality of the Ackermann function.*

Proof: 1) Let $A\equiv\exists y^0\forall z^0 A_0(a^0,x^0,y^0,z^0)$ be a Σ^0_2–formula which contains only a,x free. Suppose that PRA^2 proves:

$$\Pi^0_1\text{–}\widehat{\mathrm{AC}}^-\to\exists y\forall zA_0(a,0,y,z)\wedge\forall x(\exists y\forall zA_0(a,x,y,z)\to\exists y\forall zA_0(a,x',y,z)).$$

For notational simplicity we assume that only one instance of $\Pi^0_1\text{–}\widehat{\mathrm{AC}}^-$ without parameters is used (every instance of $\Pi^0_1\text{–}\widehat{\mathrm{AC}}^-$ with a number parameter a can be reduced to a parameter-free one by coding a and x together) and let this instance be $\exists f\forall a,b(\underbrace{\neg G_0(a,fa)\vee G_0(a,b)}_{\tilde{G}_0:\equiv})$.

Then

$$(1)\ \mathrm{PRA}^2\vdash\exists f\forall a,b\tilde{G}_0\to\exists y\forall zA_0(a,0,y,z)\text{ and}$$

$$(2)\ \mathrm{PRA}^2\vdash\exists f\forall a,b\tilde{G}_0\to\forall x(\exists y\forall zA_0(a,x,y,z)\to\exists y\forall zA_0(a,x',y,z)).$$

Since

$$\forall g(\forall a,x,y,z(\overbrace{\neg A_0(a,x,y,gaxy)\vee A_0(a,x,y,z)}^{\tilde{A}_0(a,x,y,z,g):\equiv})\to\forall a,x,y(\tilde{g}axy=0\leftrightarrow\forall zA_0(a,x,y,z))),$$

where

$$\tilde{g}axy:=\begin{cases}1,\text{if }\neg A_0(a,x,y,gaxy)\\0,\text{otherwise}\end{cases}$$

is primitive recursive in g, one has

$$(1)^*\ \mathrm{PRA}^2\vdash\forall f,g(\forall a,b\tilde{G}_0\wedge\forall a,x,y,z\tilde{A}_0\to\exists y_0(\tilde{g}(a,0,y_0)=0))$$

$$(2)^*\begin{cases}\mathrm{PRA}^2\vdash\\\forall f,g(\forall a,b\tilde{G}_0\wedge\forall a,x,y,z\tilde{A}_0\to\forall x(\exists y_1(\tilde{g}axy_1=0)\to\exists y_2(\tilde{g}ax'y_2=0))).\end{cases}$$

Using functional interpretation combined with normalization (and the fact that the finite type extension of PRA^2 obtained by adding variables and quantifiers for all finite types as well as the Π, Σ-combinators is conservative over PRA^2) or alternatively cut-elimination as in [21]) one obtains closed terms Φ_1, Φ_2 of PRA^2 such that

$$(3)\ \mathrm{PRA}^2 \vdash \begin{cases} \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \tilde{g}(a, 0, \Phi_1 fga) = 0 \\ \qquad \wedge \forall x, y_1(\tilde{g}(a, x, y_1) = 0 \to \tilde{g}(a, x', \Phi_2(fgaxy_1) = 0)). \end{cases}$$

Using ordinary (Kleene–) primitive recursion we define in PRA^2 a functional Φ by

$$\begin{cases} \Phi fga0 =_0 \Phi_1 fga \\ \Phi fgax' =_0 \Phi_2(f, g, a, x, \Phi fgax). \end{cases}$$

Using only quantifier-free induction, (3) yields

$$\mathrm{PRA}^2 \vdash \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \forall x(\tilde{g}(a, x, \Phi fgax) = 0)),$$

hence $\mathrm{PRA}^2 \vdash \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \forall x \exists y \forall z A_0(a, x, y, z)$

and therefore $\mathrm{PRA}^2 + \Pi^0_1\text{-}\widehat{\mathrm{AC}}^- \vdash \forall x \exists y \forall z A_0(a, x, y, z)$.

2) follows from 1) using the result from [19] that $\mathrm{PRA}+\Sigma^0_2$-IR proves every Π^0_3-theorem of $\mathrm{PRA}+\Pi^0_2$-IA and the fact that $\mathrm{PRA}^2 + \Sigma^0_2\text{-IR}^- \supseteq \mathrm{PRA} + \Sigma^0_2$-IR.

3) follows from 2) and the fact (see e.g. [18]) that the provably recursive functions of $\mathrm{PRA}+\Pi^0_2$–IA are just the functions definable in T_1 (i.e. the $\alpha(< \omega^{(\omega^\omega)})$-recursive functions) which includes the Ackermann function.

□

Remark 4.3 *The only part of the proof of proposition 4.2 which cannot be carried out with* PRA^2_- *instead of* PRA^2 *is the definition of* Φ.

Proposition 4.4 $\mathrm{PRA}^2 +\ \mathrm{AC}^{0,0}\text{-qf} + \Pi^0_1\text{-CA}^-$ *contains* $\mathrm{PRA} + \Pi^0_2$-IA (=$\mathrm{PRA} + \Sigma^0_2$-IA).

Proof: One easily shows that $\mathrm{PRA}^2 +\ \mathrm{AC}^{0,0}$-qf proves the second-order axiom of Σ^0_1-induction

$$\forall f(\exists y(f(0, y) = 0 \wedge \forall x(\exists y(f(x, y) = 0) \to \exists y(f(x', y) = 0)) \to \forall x \exists y(f(x, y) = 0)).$$

Together with $\Pi^0_1\text{-CA}^-$ this yields every function parameter-free instance of Σ^0_2-induction.
□

5 Where the convergence of bounded monotone sequences of real numbers goes beyond PRA

We now determine the pointwise relationship of PCM_{ar} and PCM to Σ^0_1-IA and $\Pi^0_1\text{-}\widehat{\mathrm{AC}}$ and use this to calibrate the strength of PCM^- relative to PRA^2.

We first show a result which in particular implies that, relatively to EA^2, the principle (PCM_{ar}) is equivalent to the axiom of Σ^0_1-induction

$$\Sigma^0_1\text{-IA} : \forall g^{000}(\exists y^0(g0y =_0 0) \wedge \forall x^0(\exists y^0(gxy =_0 0) \to \exists y^0(gx'y =_0 0)) \to \forall x^0 \exists y^0(gxy =_0 0)).$$

Remark 5.1 *This axiom is (relative to* EA^2*) equivalent to the schema of induction for all* Σ^0_1*-formulas in* $\mathcal{L}(\mathrm{EA}^2)$ *: Let* $\exists y^0 A_0(\underline{f}, \underline{x}, y)$ *be a* Σ^0_1*–formula (containing only* $\underline{f}, \underline{x}$ *as free function and number variables). There exists a term* $t_{A_0} \in \mathrm{EA}^2$ *such that*

$$\mathrm{EA}^2 \vdash \forall \underline{x}(\exists y^0 A_0(\underline{f}, \underline{x}, y) \leftrightarrow \exists y^0(t_{A_0}\underline{f}\,\underline{x}y =_0 0)).$$

Proposition 5.2 *One can construct functionals* $\Psi_1, \Psi_2 \in \mathrm{EA}^2$ *such that:*
1) EA^2 *proves*

$$\begin{aligned} \forall a^{1(0)} \Big(&\forall k^0[\exists y^0(\Psi_1 ak0y =_0 0) \wedge \forall x^0(\exists y^0(\Psi_1 akxy =_0 0) \to \exists y^0(\Psi_1 akx'y =_0 0)) \to \\ &\forall x^0 \exists y^0(\Psi_1 akxy =_0 0)] \to [\forall n^0(0 \le_{\mathbb{R}} a(n+1) \le_{\mathbb{R}} an) \\ &\to \forall k^0 \exists n^0 \forall m, \tilde{m} \ge_0 n(|am -_{\mathbb{R}} a\tilde{m}| \le_{\mathbb{R}} \tfrac{1}{k+1})]\Big). \end{aligned}$$

2) EA^2 *proves*

$$\begin{aligned} \forall g^{000}\Big(&[\forall n^0(0 \le_{\mathbb{Q}} \Psi_2 g(n+1) \le_{\mathbb{Q}} \Psi_2 gn \le_{\mathbb{Q}} 1) \to \\ &\forall k^0 \exists n^0 \forall m, \tilde{m} \ge_0 n(|\Psi_2 gm -_{\mathbb{Q}} \Psi_2 g\tilde{m}| \le_{\mathbb{Q}} \tfrac{1}{k+1})] \\ &\to [\exists y^0(g0y =_0 0) \wedge \forall x^0(\exists y^0(gxy =_0 0) \to \exists y^0(gx'y =_0 0)) \to \forall x^0 \exists y^0(gxy =_0 0)]\Big). \end{aligned}$$

Proof: 1) Assume that $\forall n^0(0 \le_{\mathbb{R}} a(n+1) \le_{\mathbb{R}} an)$ and
$\exists k \forall n \exists m > n(|am -_{\mathbb{R}} an| >_{\mathbb{R}} \frac{1}{k+1})$. By Σ^0_1–IA one proves that

$$(+)\ \forall n^0 \exists i^0(lth(i) = n \wedge \forall j <_0 n((i)_j < (i)_{j+1} \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)})).$$

Let $C \in \mathbb{N}$ be such that $C \ge a_0$. For $n := 3C(k+1)$, (+) yields an $i \in \mathbb{N}$ such that

$$\forall j < 3C(k+1)((i)_j < (i)_{j+1}) \text{ and}$$
$$\forall j < 3C(k+1)(a((i)_j) -_{\mathbb{R}} a((i)_{j+1}) >_{\mathbb{R}} \tfrac{1}{3(k+1)}).$$

Hence $a((i)_0) -_{\mathbb{R}} a((i)_{3C(k+1)}) > C$ which contradicts the assumption $\forall n(0 \leq_{\mathbb{R}} a_n \leq_{\mathbb{R}} C)$.
Define

$$\Psi_1 akni :=_0 \begin{cases} 0, \text{ if } lth(i) = n \wedge \forall j <_0 n((i)_j < (i)_{j+1} \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)}) \\ 1, \text{otherwise.} \end{cases}$$

2) Define $\Psi_2 \in \mathrm{EA}^2$ such that $\Psi_2 gn =_{\mathbb{Q}} 1 -_{\mathbb{Q}} \sum_{i=1}^{n} \frac{\chi gni}{i(i+1)}$, where $\chi \in \mathrm{EA}^2$ such that

$$\chi gni =_0 \begin{cases} 1, \text{ if } \exists l \leq_0 n(gil =_0 0) \\ 0, \text{ otherwise.} \end{cases}$$

Using $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$ (which is provable in EA^2) it follows that

$$\forall n^0(0 \leq_{\mathbb{Q}} \Psi_2 g(n+1) \leq_{\mathbb{Q}} \Psi_2 gn \leq_{\mathbb{Q}} 1).$$

By the assumption there exists an n_x for every $\mathbb{N} \ni x > 0$ such that

$$\forall m, \tilde{m} \geq n_x(|\Psi_2 gm -_{\mathbb{Q}} \Psi_2 g\tilde{m}| < \frac{1}{x(x+1)}).$$

Claim: $\forall \tilde{x}(0 < \tilde{x} \leq_0 x \to (\exists y(g\tilde{x}y = 0) \leftrightarrow \exists y \leq n_x(g\tilde{x}y = 0)))$:
Assume that $\exists l^0(g\tilde{x}l = 0) \wedge \forall l \leq n_x(g\tilde{x}l \neq 0)$ for some $\tilde{x} > 0$ with $\tilde{x} \leq x$.
Subclaim: Let l_0 be minimal such that $g\tilde{x}l_0 = 0$. Then $l_0 > n_x$ and

$$\Psi_2 g(\max(l_0, \tilde{x})) \leq_{\mathbb{Q}} \Psi_2 g(\max(l_0, \tilde{x}) - 1) -_{\mathbb{Q}} \frac{1}{\tilde{x}(\tilde{x}+1)}.$$

Proof of the subclaim: i) $\sum_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0,\tilde{x}))i}{i(i+1)}$ contains $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum, since $g\tilde{x}l_0 = 0$ and therefore $\chi g(\max(l_0, \tilde{x}))\tilde{x} = 1$.
ii) $\sum_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0,\tilde{x})-1)i}{i(i+1)}$ does not contain $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum:
Case 1. $\tilde{x} \geq l_0$: Then $\max(l_0, \tilde{x}) - 1 = \tilde{x} - 1 < \tilde{x}$.
Case 2. $l_0 > \tilde{x}$: Then $\max(l_0, \tilde{x}) - 1 = l_0 - 1$. Since l_0 is the minimal l such that $g\tilde{x}l = 0$, it follows that

$$\forall i \leq \max(l_0, \tilde{x}) - 1(g\tilde{x}i \neq 0) \text{ and thus } \chi g(\max(l_0, \tilde{x}) - 1)\tilde{x} = 0,$$

which finishes case 2.
Because of

$$\chi g(\max(l_0,\tilde{x})-1)i \neq 0 \to \chi g(\max(l_0,\tilde{x}))i \neq 0,$$

i) and ii) yield

$$\sum_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0,\tilde{x}))i}{i(i+1)} \geq \sum_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0,\tilde{x})-1)i}{i(i+1)} + \frac{1}{\tilde{x}(\tilde{x}+1)},$$

which concludes the proof of the subclaim.
The subclaim implies

$$\max(l_0,\tilde{x})-1 \geq n_x \wedge |\Psi_2 g(\max(l_0,\tilde{x})) -_{\mathbb{Q}} \Psi_2 g(\max(l_0,\tilde{x})-1)| \geq \frac{1}{x(x+1)}.$$

However this contradicts the construction of n_x and therefore concludes the proof of the claim.
Assume

$$(a)\ \exists y_0(g0y_0=0).$$

Define $\Phi \in \mathrm{EA}^2$ such that

$$\Phi g\tilde{x}y = \begin{cases} \min \tilde{y} \leq_0 y[g\tilde{x}\tilde{y} =_0 0], \text{ if } \exists \tilde{y} \leq_0 y(g\tilde{x}\tilde{y} =_0 0) \\ 0^0, \text{otherwise.} \end{cases}$$

By the claim above and (a) we obtain for $y := \max(n_x, y_0)$:

$$(b)\ \forall \tilde{x} \leq_0 x(\exists \tilde{y}(g\tilde{x}\tilde{y} =_0 0) \leftrightarrow g\tilde{x}(\Phi g\tilde{x}y) =_0 0).$$

QF–IA applied to $A_0(x) :\equiv (gx(\Phi gxy) =_0 0)$ yields

$$g0(\Phi g0y)=0 \wedge \forall \tilde{x} < x(g\tilde{x}(\Phi g\tilde{x}y)=0 \to g\tilde{x}'(\Phi g\tilde{x}'y)=0) \to gx(\Phi gxy)=0.$$

From this and $(a),(b)$ we obtain

$$\exists y_0(g0y_0=0) \wedge \forall \tilde{x} < x(\exists \tilde{y}(g\tilde{x}\tilde{y}=0) \to \exists \tilde{y}(g\tilde{x}'\tilde{y}=0)) \to \exists \tilde{y}(gx\tilde{y}=0).$$

Corollary 5.3

$$\mathrm{EA}^2 \vdash \Sigma^0_1\text{-IA} \leftrightarrow \mathrm{PCM}_{ar}.$$

Remark 5.4 *1) From the proof of proposition 5.2 it follows that 2) is already provable in the intuitionistic variant* EA^2_i *of* EA^2*. In particular*

$$\mathrm{EA}^2_i \vdash \mathrm{PCM}_{ar} \to \Sigma^0_1\text{-IA}.$$

The other implication Σ^0_1-IA $\to$ (PCM_{ar}) *cannot be proved intuitionistically since* (PCM_{ar}) *implies the non–constructive so–called 'limited principle of omniscience' (see* [15] *for a discussion on this).*

2) Proposition 5.2 provides much more information than corollary 5.3. In particular one can compute (in EA^2*) uniformly in g a decreasing sequence of positive rational numbers such that the Cauchy property of this sequence implies induction for the* Σ^0_1*–formula* $A(x) :\equiv \exists y(gxy = 0)$*. The converse is not that explicit but* Ψ_1 *provides an* **arithmetical family** $A_k(x) :\equiv \exists y(\Psi_1 akxy = 0)$ *of* Σ^0_1*–formulas such that the induction principle for these formulas implies the Cauchy property of the decreasing sequence of positive reals a.*

3) The proof of proposition 5.2.2) could be simplified a bit by using $\sum_{i=0}^{\infty} 2^{-i}$ *instead of* $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$*. However* $a_n :=_{\mathbb{R}} \sum_{i=1}^{n} \frac{1}{i(i+1)}$ *as a sequence of real numbers (but not as rational numbers) can be defined already at the second level of the Grzegorczyk hierarchy so that the implication* $\mathrm{PCM}_{ar} \to \Sigma^0_1$-IA *holds even in* G_2A^ω *(see* [14]*).*

We now show that Π^0_1-$\widehat{\mathrm{AC}}(a)$ can be reduced to $\mathrm{PCM}(\xi a)$ (for a suitable $\xi \in \mathrm{EA}^2$) relative to EA^2 and that $\mathrm{PCM}(a)$ can be reduced to Π^0_1-$\mathrm{AC}(\zeta a)$.

Proposition 5.5 *1)* $\mathrm{EA}^2 \vdash \forall f^{1(0)}(\mathrm{PCM}(\lambda n^0.\Psi_2 f'n) \to \Pi^0_1\text{-}\widehat{\mathrm{AC}}(f))$,[11]
where $\Psi_2 \in \mathrm{EA}^2$ *is the functional from prop. 5.2.2) such that* $\Psi_2 fn =_{\mathbb{Q}} 1 -_{\mathbb{Q}} \sum\limits_{i=1}^{n} \frac{\chi f n i}{i(i+1)}$
and $\chi \in \mathrm{EA}^2$ *such that*

$$\chi f n i =_0 \begin{cases} 1^0, \text{if } \exists l \leq_0 n(fil =_0 0) \\ 0^0, \text{otherwise}, \quad \text{and} \end{cases}$$

$$f' := \lambda x, y.\overline{sg}(fxy).$$

2) For a suitable closed term Φ *of* EA^2 *we have*

$$\mathrm{EA}^2 \vdash \forall f^1(\Pi^0_1\text{-AC}(\Phi f) \to \mathrm{PCM}(f)).$$

[11]Strictly speaking we refer here to the embedding $\lambda k.\Psi_2 f'n$ of $\mathbb{Q}$ into $\mathbb{R}$ instead of $\Psi_2 f'n$.

Proof: 1) From the proof of prop.5.2.2) we know

$$(1)\ \forall n^0(0 \leq_{\mathbb{Q}} \Psi_2 f'(n+1) \leq_{\mathbb{Q}} \Psi_2 f'n)$$

and

$$(2) \begin{cases} \forall x >_0 0 \forall n\Big((\forall m, \tilde{m} \geq n \to |\Psi_2 f'm -_{\mathbb{Q}} \Psi_2 f'\tilde{m}| <_{\mathbb{Q}} \frac{1}{x(x+1)}) \to \\ \forall \tilde{x}(0 <_0 \tilde{x} \leq_0 x \to (\exists y(f'\tilde{x}y = 0) \leftrightarrow \exists y \leq_0 n(f'\tilde{x}y = 0)))\Big) \end{cases}$$

By (1) and (PCM)($\lambda n^0.\Psi_2 f'n$) there exists a function h^1 such that

$$\forall x >_0 0 \forall m, \tilde{m} \geq_0 hx(|\Psi_2 f'm -_{\mathbb{Q}} \Psi_2 f'\tilde{m}| <_{\mathbb{Q}} \frac{1}{x(x+1)}).$$

Hence by (2)

$$\forall x >_0 0(\exists y(f'xy = 0) \leftrightarrow \exists y \leq_0 hx(f'xy = 0)).$$

Furthermore, classical logic yields $\exists z_0(f0z_0 \neq_0 0 \vee \forall y(f0y = 0))$. One now easily concludes that Π^0_1-$\widehat{\mathrm{AC}}(f)$.
2) Let $\Psi_1 \in \mathrm{EA}^2$ be as in proposition 5.2.1. By Π^0_1-CA($\tilde{\Psi}_1 f$), where $\tilde{\Psi}_1 fxy = \Psi_1(f, j_1 x, j_2 x, y)$, there exists a function g such that

$$\forall k^0, x^0(gkx = 0 \leftrightarrow \exists y(\Psi_1(f, k, x, y) = 0)).$$

Hence (by applying QF-IA to '$gkx = 0$') one gets

$$\forall k^0(\exists y^0(\Psi_1 fk0y =_0 0) \wedge \forall x^0(\exists y^0(\Psi_1 fkxy =_0 0) \to \exists y^0(\Psi_1 fkx'y =_0 0)) \\ \to \forall x^0 \exists y^0(\Psi_1 fkxy =_0 0))$$

and therefore (by proposition 5.2.1) PCM$_{ar}(f)$. For a suitable $\tilde{\Phi} \in \mathrm{EA}^2$, Π^0_1-AC($\tilde{\Phi} f$) allows to derive PCM(f) from PCM$_{ar}(f)$. Π^0_1-CA($\tilde{\Psi}_1 f$) follows from Π^0_1-AC($\hat{\Psi}_1 f$) for a suitable $\hat{\Psi}_1 \in \mathrm{EA}^2$. Finally both instances Π^0_1-AC($\tilde{\Phi} f$) and Π^0_1-AC($\hat{\Psi}_1 f$) can be coded together into a single instance Π^0_1-AC(Φf) for a suitable $\Phi \in \mathrm{EA}^2$ (using that the universal closure w.r.t. arithmetical parameters is incorporated into the definition of Π^0_1-AC(f)). Hence

$$\mathrm{EA}^2 \vdash \forall f^1(\Pi^0_1\text{-AC}(\Phi f) \to \mathrm{PCM}(f)).$$

□

Lemma 4.1.2) and proposition 5.5 imply

Corollary 5.6 $\mathrm{EA}^2+AC^{0,0}$-*qf* $\vdash \Pi^0_1$-$\mathrm{CA}^- \leftrightarrow \mathrm{PCM}^-$ *and* $\mathrm{EA}^2 \vdash \mathrm{PCM}^- \to \Pi^0_1$-$\widehat{\mathrm{AC}}^-$. *Analogously for* PRA^2_-, PRA^2 *instead of* EA^2.

Theorem 3.1, remark 3.2.2) and corollary 5.6 yield theorem 1.2 from the introduction.

Remark 5.7 *Proposition 5.5 in particular yields that relatively to* EA^2 *the principle* $\mathrm{PCM} :\equiv \forall f\ \mathrm{PCM}(f)$ *implies* CA_{ar}. *For* RCA_0 *instead of* EA^2 *this implication is stated in* [4]. *A proof (which is different from our proof) can be found in* [23].

Proposition 4.2 and proposition 5.5 together yield (using the fact that finitely many instances of Π^0_1-$\widehat{\mathrm{AC}}^-$ can be coded into a single function **and** number parameter-free instance)

Theorem 5.8 *Let* $A \in \Pi^0_3$*-theorem of* $\mathrm{PRA} + \Pi^0_2$-IA. *Then one can construct a primitive recursive sequence* $(q_n)^1$ *of (codes of) rational numbers such that*

$$\mathrm{PRA} \vdash \forall n, m (n \geq_0 m \to 0 \leq_{\mathbb{Q}} q_n \leq_{\mathbb{Q}} q_m \leq_{\mathbb{Q}} 1)$$

and

$$\mathrm{PRA}^2 + \mathrm{PCM}(q_n) \vdash A.$$

Corollary 5.9 $\mathrm{PRA}^2 + \mathrm{PCM}^-$ *proves every* Π^0_3*-theorem of* $\mathrm{PRA}+\Pi^0_2$-IA. *In particular:* $\mathrm{PRA}^2 + \mathrm{PCM}^-$ *proves the totality of the Ackermann function (and more general of every* $\alpha(< \omega^{(\omega^\omega)})$*-recursive function, i.e. of every function definable in* T_1*).*

Theorem 5.10 *Let* P *be* PCM^-, BW^- *or* $\mathrm{A\text{-}A}^-$. *Then* $\mathrm{PRA}^2 + \mathrm{AC}^{0,0}$-qf $+P$ *contains* $\mathrm{PRA} + \Pi^0_2$-IA ($=\mathrm{PRA} + \Sigma^0_2$-IA).

Proof: For PCM^- this follows from proposition 4.4, lemma 4.1 and proposition 5.5. BW^- and $\mathrm{A\text{-}A}^-$ imply PCM^- relative to $\mathrm{PRA}^2 + \mathrm{AC}^{0,0}$-qf. $\Box$

6 Where the existence of the limit superior of bounded sequences goes beyond PRA

Theorem 6.1 $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$ *contains* $\mathrm{PRA} + \Sigma^0_2$-IA.

Proof: Let f be a function $\mathbb{N} \to \mathbb{N}$ and define $q^f_n := \frac{1}{f(n)+1}$. Then obviously $(q_n)_{\mathbb{N}} \subset [0,1] \cap \mathbb{Q}$. Let $a := \limsup_{n\to\infty} q^f_n$.

Claim 1: For $k \in \mathbb{N}, k > 0$ we have

$$a \geq_{\mathbb{R}} \frac{1}{k} \leftrightarrow a >_{\mathbb{R}} \frac{1}{k+1} \leftrightarrow \forall n \exists m \geq n (f(m) < k).$$

Proof of claim 1: $\overset{1}{\to}$ is trivial.
$\overset{2}{\to}$: Assume $\exists n \forall m \geq n(f(m) \geq k)$. Then $\exists n \forall m \geq n(q^f_m \leq_{\mathbb{Q}} \frac{1}{k+1})$ and hence (since a is a limit point of (q^f_m)) $a \leq_{\mathbb{R}} \frac{1}{k+1}$.
$\overset{2}{\leftarrow}$: $\forall n \exists m \geq n(f(m) < k)$ implies $\forall n \exists m \geq n(f(m) \leq k-1)$ and therefore

$$(1)\ \forall n \exists m \geq n(q^f_m \geq_{\mathbb{Q}} \frac{1}{k} =_{\mathbb{Q}} \frac{1}{k+1} + \frac{1}{k(k+1)}).$$

Since a is the maximal limit point of $(q^f_n)_{n\mathbb{N}}$, we have

$$(2)\ \exists n \forall m \geq n(q^f_m <_{\mathbb{R}} a + \frac{1}{k(k+1)}).$$

(1) and (2) yield that $a >_{\mathbb{R}} \frac{1}{k+1}$.
$\overset{1}{\leftarrow}$: We have already shown that $a >_{\mathbb{R}} \frac{1}{k+1}$ implies $\forall n \exists m \geq n(f(m) \leq k-1)$ and so $\forall n \exists m \geq n(q^f_m \geq \frac{1}{k})$ and hence $a \geq_{\mathbb{R}} \frac{1}{k}$.
Claim 2: Relative to PRA^2_- we have

$$\begin{cases} \forall a^1, k^0 (a =_{\mathbb{R}} \limsup_{n\to\infty} q^f_n \wedge \forall n \exists m \geq n(f(m) < k) \\ \qquad \to \exists k_0 \leq k(k_0 \text{ minimal such that } \forall n \exists m \geq n(f(m) < k_0)). \end{cases}$$

Proof of claim 2: Assume $a =_{\mathbb{R}} \limsup_{n\to\infty} q^f_n$ and $\forall n \exists m \geq n(f(m) < k)$. Then, by claim 1, $a \geq_{\mathbb{R}} \frac{1}{k}$. We now show that there exists a k_0 such that $0 < k_0 \leq k$ and $a =_{\mathbb{R}} \frac{1}{k_0}$ (it is clear that k_0 is minimal such that $\forall n \exists m \geq n(f(m) < k_0)$ since otherwise (by claim 1) $a \geq_{\mathbb{R}} \frac{1}{k_0-1}$). Let k_0, $0 < k_0 \leq k$, be such that $|\frac{1}{k_0} -_{\mathbb{Q}} a(2k(k+1))|$ is minimal. Then $\frac{1}{k_0+1} <_{\mathbb{R}} a$ and, if $k_0 - 1 > 0$, $a <_{\mathbb{R}} \frac{1}{k_0-1}$, since

$$\frac{1}{2k(k+1)} \leq \frac{1}{2}\left(\frac{1}{k_0} - \frac{1}{k_0+1}\right) \overset{\text{if } k_0-1>0}{<} \frac{1}{2}\left(\frac{1}{k_0-1} - \frac{1}{k_0}\right)$$

and $|a - a(2k(k+1))| < \frac{1}{2k(k+1)}$.
Claim 1 now implies that $a =_{\mathbb{R}} \frac{1}{k_0}$.
Claim 3: Relative to PRA^2_- we have

$$\begin{cases} \forall a^1, k^0 (a =_{\mathbb{R}} \limsup_{n\to\infty} q^f_n \wedge \forall n \exists m \geq n(f(m) = k) \\ \qquad \to \exists k_0 \leq k(k_0 \text{ minimal such that } \forall n \exists m \geq n(f(m) = k_0)). \end{cases}$$

Proof of claim 3: Assume that $\exists a^1(a =_{\mathbb{R}} \limsup_{n\to\infty} q_n^f)$. Then

$$\begin{aligned}
&\exists k\forall n\exists m \geq n(fm = k) \;\Rightarrow \\
&\exists k\forall n\exists m \geq n(fm < k+1) \;\stackrel{\text{Claim2}}{\Rightarrow} \\
&\exists k(\text{ } k \text{ least such that } \forall n\exists m \geq n(fm < k+1)) \;\Rightarrow \\
&\exists k(\text{ } k \text{ least such that } \forall n\exists m \geq n(fm = k)).
\end{aligned}$$

Claim 4: Let $R(l^0, k^0, m^0)$ be a primitive recursive predicate. Then there exists a primitive recursive function f such that

$$\text{PRA } \vdash \forall l, k\forall \tilde{k} \leq k(\forall n\exists m \geq n\, R(l, \tilde{k}, m) \leftrightarrow \forall n\exists m \geq n(flkm = \tilde{k})).$$

Proof of Claim 4: Define (using the Cantor pairing function j and its projections j_i)

$$\tilde{t}lkm := \begin{cases} j_1 m, \text{ if } R(l, j_1 m, j_2 m) \\ k+1, \text{otherwise.} \end{cases}$$

We show (for all l and all $\tilde{k} \leq k$)

$$\forall n\exists m \geq n(\tilde{t}lkm = \tilde{k}) \leftrightarrow \forall n\exists m \geq n\, R(l, \tilde{k}, m).$$

'$\to$': Let $n_0 := \max_{i\leq n} j(\tilde{k}, i)$ and $m > n_0$ such that $\tilde{t}lkm = \tilde{k}$. Then $j_1 m = \tilde{k}$, $R(l, \tilde{k}, j_2 m)$ and $j_2 m > n$, since $m = j(\tilde{k}, j_2 m) > n_0$. Hence $\exists m \geq n\, R(l, \tilde{k}, m)$.
'$\leftarrow$': Let $m \geq n$ be such that $R(l, \tilde{k}, m)$. Then $\tilde{t}(l, k, j(\tilde{k}, m)) = \tilde{k}$. Since $j(\tilde{k}, m) \geq m \geq n$, we get $\exists m \geq n(\tilde{t}lkm = \tilde{k})$.
Claim 5: Let $R(k, n, m)$ be primitive recursive and
$\tilde{R}(k, n, m) :\equiv R(k, n, m) \wedge \forall \tilde{m} < m \neg R(k, n, \tilde{m})$. Then PRA $+\Sigma_1^0$-IA proves

$$\forall k(\forall n\exists m\, R(k, n, m) \leftrightarrow \forall n\exists m \geq n(lth(j_2 m) = j_1 m + 1 \wedge \forall \tilde{n} \leq j_1 m\, \tilde{R}(k, \tilde{n}, (j_2 m)_{\tilde{n}}))).$$

Proof of Claim 5:
'$\to$': Assume $\forall n\exists m\, R(k, n, m)$ and hence $\forall n\exists m\, \tilde{R}(k, n, m)$. By the principle of finite choice for Σ_1^0-formulas (which follows from Σ_1^0-IA, see [17]) we obtain
$\exists \tilde{m}(lth(\tilde{m}) = n+1 \wedge \forall \tilde{n} \leq n\, \tilde{R}(k, \tilde{n}, (\tilde{m})_{\tilde{n}}))$. So $m := j(n, \tilde{m})$ satisfies the right-hand side of the equivalence.
'$\leftarrow$': Assume

$$(+)\ \forall n\exists m \geq n(lth(j_2 m) = j_1 m + 1 \wedge \forall \tilde{n} \leq j_1 m\, \tilde{R}(k, \tilde{n}, (j_2 m)_{\tilde{n}}))$$

and suppose that $\exists n \forall m \neg R(k,n,m)$ and hence $\exists n \forall m \neg \tilde{R}(k,n,m)$. By the least number principle for Π^0_1-formulas (which easily follows from Σ^0_1-IA) we get a least such n, call it n_0. Hence

$$\forall n < n_0 \exists m \, \tilde{R}(k,n,m).$$

Again by finite Σ^0_1-choice we obtain

$$(++) \; \exists m_0 (lth(m_0) = n_0 \wedge \forall n < n_0 \tilde{R}(k,n,(m_0)_n)).$$

By $(+)$ there exists an $m > j(n_0 \dot{-} 1, m_0)$ such that

$$(+++) \; lth(j_2 m) = j_1 m + 1 \wedge \forall \tilde{n} \le j_1 m \, \tilde{R}(k,\tilde{n},(j_2 m)_{\tilde{n}}).$$

Then either $j_1 m \ge n_0$ or $j_1 m < n_0 \wedge j_2 m > m_0$. The first case yields a contradiction to $\forall m \neg \, \tilde{R}(k,n_0,m)$ and the second case contradicts the fact that (by $\tilde{R}$-definition) $(++)$ and $(+++)$ imply

$$\forall \tilde{n} < lth(j_2 m)((j_2 m)_{\tilde{n}} = (m_0)_{\tilde{n}}).$$

We now finish the proof of the theorem. By the claims 3-5 and the fact that $\mathrm{PRA}^2_- +$ $\mathrm{Limsup}^- \vdash \mathrm{PCM}^-_{ar}$ (which in turn yields Σ^0_1-IA^- by proposition 5.2.2, so that PRA $+\Sigma^0_1$-IA is a subsystem of $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$), we obtain in $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$ the least number principle instance

$$\exists k \forall n \exists m \, R(l,k,n,m) \rightarrow \exists k(\; k \text{ minimal such that } \forall n \exists m \, R(l,k,n,m)).$$

Hence $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$ proves every function parameter-free Π^0_2-instance of the least number principle, i.e. Π^0_2-LNP^-. It is an easy exercise to show that this in turn implies Σ^0_2-IA^- which concludes the proof of the theorem since PRA$+\Sigma^0_2$-IA is a pure first-order theory. □

As an immediate corollary of the theorems 3.4 and 6.1 we get theorem 1.4 from the introduction. Corollary 1.5 follows from theorem 1.4 using the fact that PRA$+\Sigma^0_2$-IA has via negative translation a Gödel functional interpretation in T_1 (see [19]) and that the functions definable in T_1 are exactly the $\alpha(< \omega^{(\omega^\omega)})$-recursive ones (see [18]).

References

[1] Buss, S.R. (editor), Handbook of Proof Theory. Studies in Logic and the Foundations of Mathematics Vol 137, Elsevier, vii+811 pp. (1998).

[2] Fairlough, M., Wainer, S., Hierarchies of provably recursive functions. In: [1] pp. 149-207.

[3] Feferman, S., Theories of finite type related to mathematical practice. In: Barwise, J. (ed.), Handbook of Mathematical Logic, pp. 913-972, North-Holland, Amsterdam (1977).

[4] Friedman, H., Systems of second-order arithmetic with restricted induction (abstract). J. Symbolic Logic **41**, pp. 558-559 (1976)

[5] Gödel, K., Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280-287 (1958).

[6] Hinman P.G., Recursion-Theoretic Hierarchies. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, Heidelberg, New York 1978.

[7] Kleene, S.C., Introduction to Metamathematics. North-Holland (Amsterdam), Noordhoff (Groningen), Van Nostrand (New-York) 1952.

[8] Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

[9] Kohlenbach, U., Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. Arch. Math. Logic **36**, pp. 31-71 (1996).

[10] Kohlenbach, U., Elimination of Skolem functions for monotone formulas in analysis. Arch. Math. Logic **37**, pp. 363-390 (1998).

[11] Kohlenbach, U., The use of a logical principle of uniform boundedness in analysis. To appear in: Proc. 'Logic in Florence 1995'.

[12] Kohlenbach, U., Arithmetizing proofs in analysis. In: Larrazabal, J.M. et al. (eds.), Proceedings Logic Colloquium 96 (San Sebastian), Springer Lecture Notes in Logic **12**, pp. 115–158 (1998).

[13] Kohlenbach, U., On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness. To appear in: Ann. Pure and Applied Logic.

[14] Kohlenbach, U., Proof theory and computational analysis.
Electronic Notes in Theoretical Computer Science **13**, Elsevier
(http://www.elsevier.nl/locate/entcs/volume13.html), 1998.

[15] Mandelkern, M., Limited omniscience and the Bolzano–Weierstraß principle. Bull. London Math. Soc. **20**, pp. 319-320 (1988).

[16] Parsons, C., Ordinal recursion in partial systems of number theory (abstract). Notices AMS **13**, pp. 857-858 (1966).

[17] Parsons, C., On a number theoretic choice schema and its relation to induction. In: Intuitionism and proof theory, pp. 459-473. North-Holland, Amsterdam (1970).

[18] Parsons, C., Proof-theoretic analysis of restricted induction schemata (abstract). J. Symbolic Logic **36**, p.361 (1971).

[19] Parsons, C., On n-quantifier induction. J. Symbolic Logic **37**, pp. 466-482 (1972).

[20] Rose, H.E., Subrecursion: Functions and hierarchies. Oxford Logic Guides **9**, Clarendon Press Oxford 1984.

[21] Schwichtenberg, H., Proof theory: some applications of cut-elimination. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 867-895 (1977).

[22] Sieg, W., Fragments of arithmetic. Ann. Pure Appl. Logic **28**,pp. 33–71 (1985).

[23] Simpson, S.G., Reverse Mathematics. Proc. Symposia Pure Math. **42**, pp. 461-471, AMS, Providence (1985).

[24] Simpson, S.G., Partial realizations of Hilbert's Program. J. Symbolic Logic **53**, pp. 349-363 (1988).

[25] Simpson, S.G., Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic, Springer-Verlag. To appear.

[26] Tait, W.W., Finitism. Journal of Philosophy **78**, pp. 524-546 (1981).

[27] Troelstra, A.S., Note on the fan theorem. J. Symbolic Logic **39**, pp. 584–596 (1974).

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