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BRICS RS-95-23 Cheng & Nielsen: Open Maps (at) Work

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BRICS Report Series RS-95-23
ISSN 0909-0878 April 1995

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Open Maps (at) Work*

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Abstract. The notion of bisimilarity, as defined by Park and Milner, has turned out to be one of the most fundamental notions of operational equivalences in the field of process algebras. Not only does it induce a congruence (largest bisimulation) in CCS which have nice equational properties, it has also proven itself applicable for numerous models of parallel computation and settings such as Petri Nets and semantics of functional languages. In an attempt to understand the relationships and differences between the extensive amount of research within the field, Joyal, Nielsen, and Winskel recently presented an abstract category-theoretic definition of bisimulation. They identify spans of morphisms satisfying certain "path lifting" properties, so-called open maps, as a possible abstract definition of bisimilarity. In [JNW93] they show, that they can capture Park and Milner's bisimulation. The aim of this paper is to show that the abstract definition of bisimilarity is applicable "in practice" by showing how a representative selection of *well-known* bisimulations and equivalences, such as e.g. Hennessy's testing equivalence, Milner and Sangiorgi's barbed bisimulation, and Larsen and Skou's probabilistic bisimulation, are captured in the setting of open maps and hence, that the proposed notion of open maps seems successful. Hence, we confirm that the treatment of strong bisimulation in [JNW93] is not a one-off application of open maps.

1 Introduction

As a response to some of the numerous models for concurrency proposed in the literature Winskel and Nielsen have used category theory as an attempt to understand the relationship between models like event structures, Petri nets, trace languages, and asynchronous transition systems [WN94]. From the algebraic point of view many of the operators of CCS like process algebras have been recasted using category-theoretic concepts such as products, co-products. However, a similar convincing category-theoretic way of adjoining abstract equivalences to a category of models had been missing until Joyal, Nielsen, and Winskel

* This work has been supported by The Danish Research Councils, The Danish Research Academy, and **BRICS**, Basic Research in Computer Science, Centre of the Danish National Research Foundation

proposed the notion of *span of open maps* [JNW93]. They show how these can capture Park and Milner's strong bisimulation and identify a *new* bisimulation, strong history-preserving bisimulation, on models with independence like event structures and Petri nets.

As a measure of the applicability of open maps as an abstract definition of equivalences we show that it is possible to capture not only Park and Milner's strong bisimulation but a representative selection of *well-known* bisimulations, such as e.g. Milner and Sangiorgi's barbed bisimulation and Larsen and Skou's probabilistic bisimulation. The presentation also serves as a tutorial on how open maps are applied. Although we do not identify new bisimulations, the reader should have no trouble using this setting on his or hers favourite model of computation. As an exercise, the reader is encouraged to prove the claims and proofs which are left out. Along, we make several observations clear which are either rather implicit in [JNW93, JNW94] or not mentioned at all.

The rest of the paper is structured as follows. In the next section we give a short stepwise introduction to open maps as presented in [JNW93, JNW94]. Then, in the subsequent sections, we apply the theory of open maps by instantiating the definitions with different models and notions of (simulation) morphisms and characterise the obtained abstract notion of equivalence operationally. It turns out that our choices of categories, which are guided by our intuitive understanding of what it means for a system to simulate another, yield well known notions of equivalence. More specifically, the following Sect. 3 to Sect. 7 are devoted to trace equivalence, weak bisimulation, testing equivalence, barbed bisimulation, and probabilistic bisimulation. In each of the sections we follow the steps presented in Sect. 2, the section recalling the general theory. Finally, in Sect. 8 we conclude with some remarks and hints for future research.

2 Open Maps, an Introduction

In this section we briefly recall the basic definitions from [JNW93].

As presented there, the general setting requires several steps. First, a category which represents a *model of computation* has to be identified. We denote this category $\mathcal{M}$. A morphism $m : X \longrightarrow Y$ in $\mathcal{M}$ should intuitively be thought of as a simulation of X in Y. Then, within $\mathcal{M}$ we choose a subcategory of "observation objects" and "observation extension" morphisms between these objects. We denote this *category of observations* by $\mathcal{P}$. Given an observation (object) P in $\mathcal{P}$ and a model X in $\mathcal{M}$. P is said to be an *observable behaviour* of X if there exists a morphism $p : P \longrightarrow X$ in $\mathcal{M}$.

Next, we identify morphisms $m : X \longrightarrow Y$ which have the property that whenever an observable behaviour of X can be extended via f in Y then that extension can be matched by an extension of the observable behaviour in X.

Definition 1. Open Maps
A morphism $m : X \longrightarrow Y$ in $\mathcal{M}$ is said to be $\mathcal{P}$-*open* if whenever $f : O_1 \longrightarrow O_2$ in $\mathcal{P}$, $p : O_1 \longrightarrow X$, $q : O_2 \longrightarrow Y$ in $\mathcal{M}$, and the diagram

$$
\begin{tikzcd}
O_1 \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "m"] \\
O_2 \arrow[r, "q"'] & Y
\end{tikzcd}
\tag{1}
$$

commutes, i.e. $m \circ p = q \circ f$, there exists a morphism $h : O_2 \longrightarrow X$ in $\mathcal{M}$ such that the two triangles in the diagram

$$
\begin{tikzcd}
O_1 \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "m"] \\
O_2 \arrow[r, "q"'] \arrow[ur, "h"] & Y
\end{tikzcd}
\tag{2}
$$

commute, i.e. $p = h \circ f$ and $q = m \circ h$. When no confusion is possible, we refer to $\mathcal{P}$-open morphisms as *open maps.*

The abstract definition of bisimilarity is as follows.

Definition 2. $\mathcal{P}$-bisimilarity
Two models X and Y in $\mathcal{M}$ are said to be *$\mathcal{P}$-bisimilar*, written $X \sim_{\mathcal{P}} Y$, if there exists a *span of open maps* from a common object Z:

$$
\begin{tikzcd}
& Z \arrow[dl, "m"'] \arrow[dr, "m'"] & \\
X & & Y
\end{tikzcd}
\tag{3}
$$

Notice that if $\mathcal{M}$ has pullbacks, it can be shown that $\sim_{\mathcal{P}}$ is an equivalence relation. The important observation is that pullbacks of open maps are themselves open maps. For more details, the reader is referred to [JNW93].

In the next sections, we proceed by following the above presented steps.

As a preliminary example of a category of models of computation $\mathcal{M}$ we present *labelled transition systems.*

Definition 3. A *labelled transition system* over Act is a tuple

$$(S, i, Act, \longrightarrow) \;, \tag{4}$$

where S is a set of states with *initial state* i, Act is a set of actions ranged over by α, β, ... , and $\longrightarrow \subseteq S \times Act \times S$ is the transition relation. For the sake of readability we introduce the following notation. Whenever (s_0, α_1, s_1), (s_1, α_2, s_2), ..., $(s_{n-1}, \alpha_n, s_n) \in \longrightarrow$ we denote this as $s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n$ or $s_0 \xrightarrow{v} s_n$ where $v = \alpha_1\alpha_2 \cdots \alpha_n \in Act^*$. Also, we assume that all states $s \in S$ are reachable from i, i.e. there exists a $v \in Act^*$ such that $i \xrightarrow{v} s$.

Let us briefly remind the reader about Park and Milner's definition of strong bisimulation.

Definition 4. Let $T_1 = (S_1, i_1, Act, \longrightarrow_1)$ and $T_2 = (S_2, i_2, Act, \longrightarrow_2)$. A *strong bisimulation between* T_1 *and* T_2 is a relation $R \subseteq S_1 \times S_2$ such that

$$(i_1, i_2) \in R \ , \tag{5}$$

$$((r, s) \in R \ \wedge \ r \xrightarrow{\alpha}_1 r') \ \Rightarrow \ \text{for some } s', (s \xrightarrow{\alpha}_2 s' \ \wedge \ (r', s') \in R) \ , \tag{6}$$

$$((r, s) \in R \ \wedge \ s \xrightarrow{\alpha}_2 s') \ \Rightarrow \ \text{for some } r', (r \xrightarrow{\alpha}_1 r' \ \wedge \ (r', s') \in R) \ . \tag{7}$$

T_1 and T_2 are said to be *strongly bisimilar* if there exists a strong bisimulation between them.

Henceforth, whenever no confusion is possible we drop the indexing subscripts on the transition relations and write $\longrightarrow$ instead.

By defining morphisms between labelled transition systems we can obtain a category of models of computation, $\mathcal{LTS}$, labelled transition systems.

Definition 5. Let $T_1 = (S_1, i_1, Act, \longrightarrow_1)$ and $T_2 = (S_2, i_2, Act, \longrightarrow_2)$. A morphism $m : T_1 \longrightarrow T_2$ is a function $m : S_1 \longrightarrow S_2$ such that

$$m(i_1) = i_2 \ , \tag{8}$$

$$s \xrightarrow{\alpha}_1 s' \ \Rightarrow \ m(s) \xrightarrow{\alpha}_2 m(s') \ . \tag{9}$$

Composition of morphisms is defined as the usual composition of functions. The intuition behind this specific choice of morphism is that an α labelled transition in T_1 must be simulated by an α labelled transition in T_2.

If, as done in [JNW93], one chooses $\mathcal{P}$ as the full subcategory of $\mathcal{M}$ whose objects are finite synchronisation trees with at most one maximal branch, i.e. labelled transition systems of the form

$$i \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n \ ,$$

$\mathcal{P}$-bisimilarity corresponds to Park and Milner's strong bisimulation. This follows from the following characterisation of $\mathcal{P}$-open maps [JNW93].

Lemma 6. *A morphism $m : T_1 \longrightarrow T_2$ is $\mathcal{P}$-open if and only if it satisfies the following "zig-zag" property:*

If $m(r) \xrightarrow{\alpha} s$ then there exists an r' such that $r \xrightarrow{\alpha} r'$ and $m(r') = s$.

In the following sections we shall "rediscover" well-known behavioural equivalences by varying $\mathcal{M}$ and $\mathcal{P}$. In the following, whenever we write e.g. $\mathcal{M}$, $\mathcal{P}$, or $\mathcal{P}$-bisimilarity they refer to the specific choices of categories made in the section they appear.

3 Trace Equivalence

In this section we show how trace equivalence between two labelled transition systems can be captured by open maps. Trace equivalence is perhaps the first and simplest equivalence between labelled transition systems that one can think of. The result is based on the following important fact: Two labelled transition systems are trace equivalent if and only if their underlying deterministic transition systems are bisimilar.

First, we present the category $\mathcal{LTS}_1$ of *labelled transition systems*, which will corresponds to $\mathcal{M}$. Then, we identify a subcategory, $\mathcal{P}$, of observations. Finally, we show that $\mathcal{P}$-bisimilarity corresponds to trace equivalence.

The object of $\mathcal{LTS}_1$ are the labelled transition systems (*lts*) from Definition 3. The following definition is needed in the definition of the morphisms in $\mathcal{LTS}_1$.

Definition 7. Given an *lts* $T = (S, i, Act, \longrightarrow)$. For nonempty sets $X, Y \subseteq S$, we define $X \xrightarrow{\alpha} Y$ if $Y = \{r' \in S \mid \exists\, r \in X.\, r \xrightarrow{\alpha} r'\}$. Notice that this transition relation is deterministic. As before, the transition relation can be generalised to a relation $X \xrightarrow{v} Y$, where $v \in Act^*$. Furthermore, we define $RS(T)$, *the reachability set of* T, to be the least subset of $2^S/\{\emptyset\}$, such that

$$\{i\} \in RS(T)\ , \tag{10}$$

$$X \in RS(T) \text{ and } X \xrightarrow{\alpha} Y \text{ implies } Y \in RS(T)\ . \tag{11}$$

Next, we define morphisms between two *lts*'s.

Definition 8. Given two *lts*'s, $T_j = (S_j, i_j, Act, \longrightarrow_j), j = 1, 2$. A morphism m between T_1 and T_2 is a function m from $RS(T_1)$ to $RS(T_2)$, such that

$$m(\{i_1\}) = \{i_2\}\ , \tag{12}$$

$$X \xrightarrow{\alpha} Y \text{ implies } m(X) \xrightarrow{\alpha} m(Y)\ . \tag{13}$$

Composition of morphisms is defined as the usual composition of functions. This defines the category $\mathcal{LTS}_1$.

The intuition behind this definition of (simulating) morphism is that one is only interested in what action sequences an *lts* can perform. After performing a sequence $\sigma = \alpha_1 \cdots \alpha_n$ of actions from the initial state i one may in general end up in several different states of T, i.e. a set X of states of T. These sets of states are exactly the elements of $RS(T)$. Extending the sequence σ by performing another action α then corresponds to performing an α transition from X.

Next step is to define $\mathcal{P}$.

Definition 9. Let $\mathcal{P}$ be the full subcategory of $\mathcal{LTS}_1$ whose objects are of the form

$$i \xrightarrow{\alpha_1} r_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r_n\ , \tag{14}$$

where all states are distinct.

Apart from showing that $\mathcal{LTS}_1$ has pullbacks, the construction in the following lemma will be referred to in the main theorem of this section. Also, it follows at once from the remarks in Sect. 2 that $\mathcal{P}$-bisimilarity is an equivalence relation. We will also be able to conclude this after proving the main theorem of this section; it states that $\mathcal{P}$-bisimilarity coincides with trace equivalence, which is know to be an equivalence relation. However, in general one cannot expect that $\mathcal{P}$-bisimilarity coincides with a known equivalence relation. Lemmas as the following are sufficient for $\mathcal{P}$-bisimilarity to be an equivalence relation.

Lemma 10. *$\mathcal{LTS}_1$ has pullbacks.*

Proof. Given a diagram

$$\begin{tikzcd} T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\ & T & \end{tikzcd}$$

Define an *lts* $T' = (S', i', Act, \longrightarrow')$ as follows. $S' \subseteq RS(T_1) \times RS(T_2)$ and $\longrightarrow' \subseteq (RS(T_1) \times RS(T_2)) \times Act \times (RS(T_1) \times RS(T_2))$ are the least sets such that

- $i' = (\{i_1\}, \{i_2\}) \in S'$
- If $(X, Y) \in S'$, $X \xrightarrow{\alpha} X'$, $Y \xrightarrow{\alpha} Y'$
  then $(X', Y') \in S'$ and $((X, Y), \alpha, (X', Y')) \in \longrightarrow'$.

Notice that because the transition relations on the reachability sets are deterministic it is the case that $m_1(X) = m_2(Y)$ for any $(X, Y) \in S'$ and $RS(T')$ contains only singletons. Let $\pi_1 : T' \longrightarrow T_1$ be defined as $\pi_1(\{(X, Y)\}) = X$. It can be shown that π_1 is well defined and is a morphism from T' to T_1. We can define $\pi_2 : T' \longrightarrow T_2$ in a similar way.

Given an *lts* T'' and two morphisms $f_1 : T'' \longrightarrow T_1$ and $f_2 : T'' \longrightarrow T_2$ such that $m_1 \circ f_1 = m_2 \circ f_2$. Define $h : T'' \longrightarrow T'$ by $h(Z) = (f_1(Z), f_2(Z))$ for $Z \in RS(T'')$. From the definition of T' is should be easy to see that h is a morphism; the initial state of T'' is mapped to that of T' and transitions are preserved. Furthermore we also have $f_1 = \pi_1 \circ h$ and $f_2 = \pi_2 \circ h$. This is trivial, since there is at most one morphism between any two objects in $\mathcal{LTS}_1$. Hence, h is also unique. This gives us the desired pullback. □

The next lemma characterises the open maps in $\mathcal{LTS}_1$.

Lemma 11. *A morphism $m : T_1 \longrightarrow T_2$ is $\mathcal{P}$-open if and only if $m : RS(T_1) \longrightarrow RS(T_2)$ has the following "zig-zag" property*

If $m(X) \xrightarrow{\alpha} Y'$, then there exists an X' such that $X \xrightarrow{\alpha} X'$ and $m(X') = Y'$.

Proof. Assume m is $\mathcal{P}$-open and $m(X) \xrightarrow{\alpha} Y'$. Then it must be the case that $X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n$ for some $X_0, \ldots, X_n \in RS(T_1)$, where $X_0 = \{i_1\}$ and

$X_n = X$. Also, we have $Y_0 \xrightarrow{\alpha_1} Y_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} Y_n \xrightarrow{\alpha} Y'$, where $Y_j = m(X_j)$ for $0 \leq j \leq n$. Let O_1 be the observation

$$i \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n \quad ,$$

and O_2 be the observation

$$i' \xrightarrow{\alpha_1} s_1' \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n' \xrightarrow{\alpha} s_{n+1}' \quad .$$

Now let f denote the unique morphism from O_1 to O_2, p denote the morphism that maps $\{i\}$ to $\{i_1\}$ and $\{s_j\}$ to X_j for $1 \leq j \leq n$, and q denote the morphism that maps $\{i'\}$ to $\{i_2\}$, $\{s_j'\}$ to Y_j for $1 \leq j \leq n$, and $\{s_{n+1}'\}$ to Y'. We then have $m \circ p = q \circ f$. From our assumptions it then follows that there exists a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. We now conclude $h(\{s_n'\}) = h(f(\{s_n\})) = p(\{s_n\}) = X$, $h(\{s_n'\}) \xrightarrow{\alpha} h(\{s_{n+1}'\})$, and $m(h(\{s_{n+1}'\})) = q(\{s_{n+1}'\}) = Y'$. Now choose X' as $h(\{s_{n+1}'\})$.

Conversely, assume m has the "zig-zag" property and we are given a commuting diagram

$$\begin{tikzcd}
O_1 \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "m"] \\
O_2 \arrow[r, "q"'] & Y
\end{tikzcd}$$

where O_1 is an observations of the form

$$i \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} s_n \quad ,$$

and O_2 an observation of the form

$$i' \xrightarrow{\alpha_1} s_1' \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} s_m' \quad ,$$

and $n \leq m$. Notice f is uniquely determined (maps $\{s_j\}$ to $\{s_j'\}$ for $1 \leq j \leq n$). We will show how to define a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. We start by defining $h(\{i'\}) = \{i_1\}$ and $h(\{s_j'\}) = p(\{s_j\})$ for $1 \leq j \leq n$. Notice that we now already have $p = h \circ f$ for the partially defined h. Consequently, $q = m \circ h$ on $\{i'\}$, $\{s_1'\}, \ldots, \{s_n'\}$ because of the way f is defined and $m \circ p = q \circ f$. Now assume $n < m$. Since $m(p(\{s_n\})) = q(f(\{s_n\})) = q(\{s_n'\}) \xrightarrow{\alpha_{n+1}} q(\{s_{n+1}'\})$ we know there must exist an X' such that $p(\{s_n\}) \xrightarrow{\alpha_{n+1}} X'$ and $m(X') = q(\{s_{n+1}'\})$. Now define $h(\{s_{n+1}'\}) = X'$. Then $m(h(\{s_{n+1}'\})) = q(\{s_{n+1}'\})$. Continuing this way for the remaining $\{s_{n+2}'\}, \ldots, \{s_m'\}$ we obtain the desired morphism. □

Definition 12. Given an *lts* $T = (S, i, Act, \longrightarrow)$. The *traces/language of* T, denoted $L(T)$, is defined as

$$L(T) = \{v \in Act^* \mid i \xrightarrow{v} r \text{ for some } r \in S\} \ . \tag{15}$$

Two *lts*'s, T_1 and T_2, are said to be *trace equivalent* if $L(T_1) = L(T_2)$.

Theorem 13. *Given two* lts*'s T_1 and T_2. Then:*

T_1 and T_2 are trace equivalent if and only if they are $\mathcal{P}$-bisimilar.

Proof. The "if" direction follows from Lemma 11. For the "only if" direction, let $\mathcal{F}_1$ be the functor from $\mathcal{LTS}_1$ to $\mathcal{LTS}$ which sends an object T to $(RS(T), \{i\}, Act, \longrightarrow)$, where $\longrightarrow$ was defined in Definition 7, and an morphism $m : T \longrightarrow T'$ to the obvious morphism between $\mathcal{F}_1(T)$ and $\mathcal{F}_1(T')$ defined by $m : RS(T) \longrightarrow RS(T')$. Let $\mathcal{F}_2$ be the functor from $\mathcal{LTS}$ to $\mathcal{LTS}_1$ which maps an object to itself and a morphism $m : T \longrightarrow T'$ to the morphism determined uniquely by the induced function $m : RS(T) \longrightarrow RS(T')$. Since all observations of $\mathcal{LTS}_1$ are isomorphic to their image under $\mathcal{F}_1$ and there is at most one morphism between any two objects in $\mathcal{LTS}_1$, we conclude, using Lemma 6, that $\mathcal{F}_2$ preserves open maps. So assume that T_1 and T_2 are trace equivalent. We then know that $\mathcal{F}_1(T_1)$ and $\mathcal{F}_1(T_2)$ are strong bisimilar. From Sect. 2 this implies that there exists a span of open maps in $\mathcal{LTS}$, $m_1 : T \longrightarrow \mathcal{F}_1(T_1)$ and $m_2 : T \longrightarrow \mathcal{F}_1(T_2)$. Also, there clearly exist isomorphisms $p_1 : \mathcal{F}_2(\mathcal{F}_1(T_1)) \longrightarrow T_1$ and $p_2 : \mathcal{F}_2(\mathcal{F}_1(T_2)) \longrightarrow T_2$. Since isomorphisms are always open maps we have the following span of open maps:

$$\begin{tikzcd}
 & & \mathcal{F}_2(T) \arrow[dl, "\mathcal{F}_2(m_1)"'] \arrow[dr, "\mathcal{F}_2(m_2)"] & & \\
 & \mathcal{F}_2(\mathcal{F}_1(T_1)) \arrow[dl, "p_1"'] & & \mathcal{F}_2(\mathcal{F}_1(T_2)) \arrow[dr, "p_2"] & \\
T_1 & & & & T_2
\end{tikzcd}$$

We conclude that T_1 and T_2 are $\mathcal{P}$-bisimilar. Observe that a construction similar to the one used in Lemma 10 would also have provided a span of open maps. $\square$

Having identified trace equivalence we now continue by exploring other possibilities. In the next section we try to take "invisible" or "silent" actions into account.

4 Weak Bisimulation (Milner)

In this section we show that Milner's *weak bisimulation* [Mil89] can be characterised using the general setting of Sect. 2.

Weak bisimulation differs from strong bisimulation in at least two respects. First, a special "invisible" action, usually denoted τ, is required to be a member of the set of labels. Second, an α labelled transition in one labelled transition system is no longer required to be simulated exactly by an α labelled transition in the other system. It may be preceded and succeeded by several τ transition. We write $r \stackrel{t}{\Longrightarrow} r'$ if $r \xrightarrow{\tau^*} r_1 \xrightarrow{\alpha_1} r_1' \xrightarrow{\tau^*} \cdots \xrightarrow{\tau^*} r_n \xrightarrow{\alpha_n} r_n' \xrightarrow{\tau^*} r'$ for some $r_1, \cdots, r_n'$. Furthermore, a τ transition needn't be simulated by any transitions at all.

We start by defining a category $\mathcal{LTS}_2$, *labelled transition systems*, and a subcategory of observations, $\mathcal{P}$, in $\mathcal{LTS}_2$. Then, we show that $\mathcal{P}$-bisimilarity corresponds to Milner's weak bisimulation.

The objects of $\mathcal{LTS}_2$ are the same as those from $\mathcal{LTS}$. However, we assume that the set of actions *Act* contains a special "invisible" action τ. Guided by our intuitive understanding of how an action may be simulated, we define the morphisms between two *lts*'s as follows.

Definition 14. Given two *lts*'s, $T_j = (S_j, i_j, \mathrm{Act}, \longrightarrow_j), j = 1, 2$. A morphism between T_1 and T_2, $m : T_1 \longrightarrow T_2$, is a function m from S_1 to S_2, such that

$$m(i_1) = i_2 \ , \tag{16}$$

$$r \xrightarrow{\alpha} r' \text{ implies } m(r) \stackrel{\widehat{\alpha}}{\Longrightarrow} m(r') \ . \tag{17}$$

The function $\widehat{\ }: Act^* \longrightarrow Act^*$ removes all τ's from its argument [Mil90].

Composition of morphisms is defined as the usual composition of functions. This defines the category $\mathcal{LTS}_2$. $\mathcal{P}$, the category of observations, is defined as follows.

Definition 15. Let $\mathcal{P}$ be the subcategory of $\mathcal{LTS}_2$ whose objects are of the form

$$i \xrightarrow{\alpha_1} r_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r_n \ , \tag{18}$$

where all the states are distinct. Moreover, there will be a morphism f from an observation

$$i \xrightarrow{\alpha_1} r_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r_n \tag{19}$$

to another observation

$$i' \xrightarrow{\alpha_1} r_1' \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r_n' \xrightarrow{\alpha_{n+1}} \cdots \xrightarrow{\alpha_{n+k}} r_{n+k}' \ , \tag{20}$$

if $f(i) = i'$, $f(r_j) = r_j'$ for $1 \leq j \leq n$, and $k \geq 0$.

Notice that morphisms between observations are required to simulate action in the "strong" sense — e.g. no additional τ's may be added. In the conclusion, Sect. 8, we will comment on this interesting choice. Allowing any morphism between two observations will in fact make $\mathcal{P}$-bisimilarity stronger than weak bisimulation; the reader should have no major difficulties in going through the proofs. Lemma 17 will no longer be true, neither will the "only if" in Theorem 19.

Having defined $\mathcal{M}$ as $\mathcal{LTS}_2$ and $\mathcal{P}$ we now show that $\mathcal{LTS}_2$ has pullbacks.

Lemma 16. *The category $\mathcal{LTS}_2$ has pullbacks.*

Proof. Given a diagram

$$\begin{tikzcd} T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\ & T & \end{tikzcd}$$

Define $T' = (S', i', Act, \longrightarrow')$ as follows. $S' \subseteq S_1 \times S_2$ and $\longrightarrow' \subseteq (S_1 \times S_2) \times Act \times (S_1 \times S_2)$ are the least sets such that

- $i' = (i_1, i_2) \in S'$
- If $(r, s) \in S'$, $r \stackrel{\widehat{\alpha}}{\Longrightarrow} r'$, $s \stackrel{\widehat{\alpha}}{\Longrightarrow} s'$, and $m_1(r') = m_2(s')$ then $(r', s') \in S'$ and $((r, s), \alpha, (r', s')) \in \longrightarrow'$.

Define $\pi_1 : T' \longrightarrow T_1$ and $\pi_2 : T' \longrightarrow T_2$ as $\pi_1((r, s)) = r$ and $\pi_2((r, s)) = s$.

We now show that this defines a pullback. Clearly, π_1 and π_2 are morphisms. Assume we have a commuting diagram

$$\begin{tikzcd} T'' \arrow[r, "g"] \arrow[d, "f"'] & T_2 \arrow[d, "m_2"] \\ T_1 \arrow[r, "m_1"'] & T \end{tikzcd}$$

Define $h : S'' \longrightarrow S_1 \times S_2$ as $h(v) = (f(v), g(v))$ for $v \in S''$. It should be easy to see, using the commutativity of the diagram and the definition of T', that $h(v) \in S'$, since v is reachable from i'', and that $v \stackrel{\alpha}{\longrightarrow} v'$ in T'' implies $h(v) \stackrel{\alpha}{\longrightarrow} h(v')$ in T'. h is then a well defined morphism from T'' to T'. Also, $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$ and moreover these equations determine h uniquely. Hence, T', π_1, and π_2 constitute a pullback. $\square$

Next, we characterise the open maps.

Lemma 17. *A morphism $m : T_1 \longrightarrow T_2$ is $\mathcal{P}$-open if and only if it satisfies the following "zig-zag" property:*

If $m(r) \stackrel{\alpha}{\longrightarrow} s$ then there exists an r' such that $r \stackrel{\widehat{\alpha}}{\Longrightarrow} r'$ and $m(r') = s$.

Proof. Assume m is open and $i_1 \stackrel{\alpha_1}{\longrightarrow} r_1 \stackrel{\alpha_2}{\longrightarrow} \cdots \stackrel{\alpha_n}{\longrightarrow} r_n = r$. Let O_1 be the observation

$$i \stackrel{\alpha_1}{\longrightarrow} r'_1 \stackrel{\alpha_2}{\longrightarrow} \cdots \stackrel{\alpha_n}{\longrightarrow} r'_n \ ,$$

O_2 be the observation

$$i' \stackrel{\alpha_1}{\longrightarrow} r''_1 \stackrel{\alpha_2}{\longrightarrow} \cdots \stackrel{\alpha_n}{\longrightarrow} r''_n \stackrel{\alpha}{\longrightarrow} s'' \ ,$$

and f the unique morphism from O_1 to O_2. Let $p : O_1 \longrightarrow T_1$ be the morphism which sends r'_j to r_j for $1 \leq j \leq n$, and $q : O_2 \longrightarrow T_2$ the morphism that sends r''_j to $m(r_j)$ for $1 \leq j \leq n$ and s'' to s. Then $m \circ p = q \circ f$ and since m is an open map there exists a morphism $h : O_2 \longrightarrow T_1$ such that the two triangles in the diagram

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "f"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "q"'] \arrow[ur, "h"] & T_2 \end{tikzcd}$$

commutes. Since $h(r''_n) \overset{\widehat{\alpha}}{\Longrightarrow} h(s'')$, $h(r''_n) = h(f(r'_n)) = p(r'_n) = r_n$, and $s = q(s'') = m(h(s''))$, we conclude that $h(r''_n) = r_n = r \overset{\widehat{\alpha}}{\Longrightarrow} h(s'')$ and $m(h(s'')) = s$. Hence, there exists a $r' = h(s'')$ such that $r \overset{\widehat{\alpha}}{\Longrightarrow} r'$ and $m(r') = s$.

For the other direction, assume m has the "zig-zag" property and we are given a commuting diagram of the form

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "f"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "q"'] & T_2 \end{tikzcd}$$

where O_1 is an observation of the form

$$i \xrightarrow{\alpha_1} r_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r_n \quad ,$$

O_2 is an observation of the form

$$i' \xrightarrow{\alpha_1} r'_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} r'_n \xrightarrow{\alpha_{n+1}} \cdots \xrightarrow{\alpha_{n+k}} r'_{n+k} \quad ,$$

and $f : O_1 \longrightarrow O_2$ is the uniquely determined morphism which sends r_j to r'_j for $1 \leq j \leq n$.

We show that there exists a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. Apart from defining $h(r'_j) = p(r_j)$ for $1 \leq j \leq n$, and of course $h(i') = i_1$, let us consider which value we should give $h(r'_{n+1})$. Assume $q(r'_n) \overset{\widehat{\alpha_{n+1}}}{\Longrightarrow} q(r'_{n+1})$ because $q(r'_n) \xrightarrow{\beta_1} v_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{l-1}} v_{l-1} \xrightarrow{\beta_l} q(r'_{n+1})$, where $l \geq 0$ and $\widehat{\alpha_{n+1}} = \widehat{\beta_1 \cdots \beta_l}$. By commutativity we have $m(p(r_n)) = q(r'_n)$ and by repeated use of the "zig-zag" property we conclude that there exist states w_j such that $p(r_n) \overset{\widehat{\beta_1}}{\Longrightarrow} w_1 \overset{\widehat{\beta_2}}{\Longrightarrow} \cdots \overset{\widehat{\beta_l}}{\Longrightarrow} w_l$, $m(w_j) = v_j$ for $1 \leq j < l$, and $m(w_l) = q(r'_{n+1})$. Define $h(r'_{n+1}) = w_l$. Then $h(r'_n) \overset{\widehat{\alpha_{n+1}}}{\Longrightarrow} h(r'_{n+1})$ and $m(h(r'_{n+1})) = m(w_l) = q(r'_{n+1})$. Continuing this process for the remaining $r'_{n+2}, \ldots, r'_{n+k}$ it is easy to see that we obtain a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. Hence, m is an open map. $\square$

For the sake of completeness we give Milner's definition of weak bisimulation [Mil89], here adapted to the case where we consider initial states of *lts*'s.

Definition 18. Given two *lts*'s T_1 and T_2. A relation $R \subseteq S_1 \times S_2$ is said to be a *weak bisimulation* over T_1 and T_2 if

$$(i_1, i_2) \in R \ , \tag{21}$$

$$((r,s) \in R \ \wedge \ r \xrightarrow{\alpha} r') \ \Rightarrow \ \text{for some } s', (s \overset{\widehat{\alpha}}{\Longrightarrow} s' \ \wedge \ (r', s') \in R) \ , \tag{22}$$

$$((r,s) \in R \ \wedge \ s \xrightarrow{\alpha} s') \ \Rightarrow \ \text{for some } r', (r \overset{\widehat{\alpha}}{\Longrightarrow} r' \ \wedge \ (r', s') \in R) \ . \tag{23}$$

T_1 and T_2 are said to be *weakly bisimilar* if there exists a weak bisimulation as defined above.

We now show that $\mathcal{P}$-bisimilarity coincides with weak bisimulation.

Theorem 19. *Given two* lts*'s T_1 and T_2. Then,*

T_1 and T_2 are weakly bisimilar if and only if T_1 and T_2 are $\mathcal{P}$-bisimilar.

Proof. Assume R is a weak bisimulation over T_1 and T_2. Define $T' = (S', i', Act, \longrightarrow')$, where $S' \subseteq S_1 \times S_2$, as follows. Let S' and $\longrightarrow'$ be the least sets such that

- $i' = (i_1, i_2) \in S'$
- If $(r,s) \in S'$, $r \overset{\widehat{\alpha}}{\Longrightarrow} r'$, $s \overset{\widehat{\alpha}}{\Longrightarrow} s'$, and $(r', s') \in R$, then $(r', s') \in S'$ and $((r,s), \alpha, (r', s')) \in \longrightarrow'$.

Notice that $S' \subseteq R$. Now define $p : T' \longrightarrow T_1$ as $p((r,s)) = r$ and $q : T' \longrightarrow T_2$ as $q((r,s)) = s$. From the definitions and the above observation it should be easy to see that p and q are open maps, i.e. T_1 and T_2 are $\mathcal{P}$-bisimilar.

Assume T_1 and T_2 are $\mathcal{P}$-bisimilar, i.e. there exist a span of open maps

$$\begin{tikzcd} & T \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & \\ T_1 & & T_2 \end{tikzcd}$$

It is enough to show that T and T_1 are weakly bisimilar since weak bisimulation is an equivalence relation. Define R to be the least relation in $S \times S_1$ such that

- $(i, i_1) \in R$,
- If $(r,s) \in R$ and $r \xrightarrow{\alpha} r'$ then $(r', m_1(r')) \in R$, and
- If $(r,s) \in R$ and $s \xrightarrow{\alpha} s'$ then $(r', s') \in R$ where r' is any state such that $r \overset{\widehat{\alpha}}{\Longrightarrow} r'$ and $m_1(r') = s'$. Such a state exists by Lemma 17.

Notice that $(r,s) \in R$ implies $m_1(r) = s$. Hence, in the last item $s = m_1(r) \overset{\widehat{\alpha}}{\Longrightarrow} m_1(r')$. It is now easy to show that R is a weak bisimulation over T and T_1. $\square$

5 Testing Equivalence (Hennessy)

In this section we modify the category from Sect. 3 (basically) only with respect to the morphisms. We then choose a new subcategory $\mathcal{P}$ of observations. This time the elements of $\mathcal{P}$ will reflect a special type of branching structure. Then we show that the obtained $\mathcal{P}$-bisimilarity coincides with Hennessy's *testing equivalence* [Hen88]. Testing equivalence is slightly stronger than trace equivalence, due to an extra requirement on the set of possible actions, so-called *acceptance sets*, from states reached by performing a sequence of actions/labels.

We continue by defining a new category $\mathcal{LTS}_3$ of transition systems. The objects are those from $\mathcal{LTS}_1$ which are finitely branching, i.e. from every state only finitely many actions can be taken. Before defining the morphisms we need some definitions, inspired by [Hen88].

Definition 20. Let $T = (S, i, Act, \longrightarrow)$ be an *lts*. Let $RS(T)$ denote the reachability set of T. For $r \in S$, $X \in RS(T)$, and $s \in Act^*$ let

$$S_T(r) = \{\alpha \in Act \mid \exists r'.\, r \xrightarrow{\alpha} r'\} \ , \tag{24}$$

$$\mathcal{A}_T(r, s) = \{S_T(r') \mid r \xrightarrow{s} r'\} \ , \tag{25}$$

$$S_T(X) = \{S_T(r) \mid r \in S\} \ . \tag{26}$$

Notice that if $\{i\} \xrightarrow{s} X$ for $s \in Act^*$ then $\{S_T(X)\} = \mathcal{A}_T(i, s)$. $S_T(X)$ is the *acceptance set of* X.

The morphisms are now defined as follows.

Definition 21. Given two *lts*'s, $T_j = (S_j, i_j, Act, \longrightarrow_j), j = 1, 2$. A morphism m between T_1 and T_2 is a function m from $RS(T_1)$ to $RS(T_2)$, such that

$$m(\{i_1\}) = \{i_2\} \ , \tag{27}$$

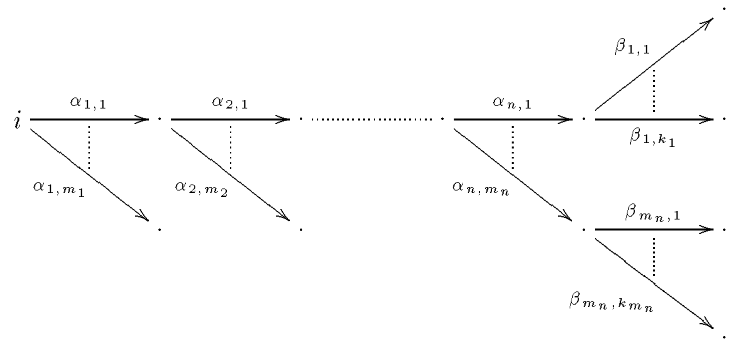
$$X \xrightarrow{\alpha} X' \text{ implies } m(X) \xrightarrow{\alpha} m(X') \ , \tag{28}$$

$$m(X) = Y \ \Rightarrow \ \forall A' \in S_{T_2}(Y).\, \exists A \in S_{T_1}(X).\, A \subseteq A' \ . \tag{29}$$

Notice how the definition of morphisms intuitively simulates Hennessy's $<\!\!<_{MAY}$ and $<\!\!<_{MUST}$ pre-orders. Being guided by the definitions in [Hen88] and our results from Sect. 3, (27) and (28) reflect that we want traces to be simulated, and (29) reflects how acceptance sets are to be matched. Composition of morphisms is defined as the usual composition of functions. This defines the category $\mathcal{LTS}_3$.

The subcategory $\mathcal{P}$ of observations will not consist of finite paths, but of trees consisting of a "trunk" and "branches" of length one, except for the "top" of the tree, where a more general branching structure is allowed.

Definition 22. Let $\mathcal{P}$ be the full subcategory of $\mathcal{LTS}_3$ whose objects are of the form



where $0 \leq m_1, \ldots, m_n, k_1, \ldots, k_{m_n}$ and all states are distinct.

Intuitively, the "trunk" corresponds to the observations in Sect. 3, i.e. it will ensure the existence of certain traces. The "top" of the the tree will ensure the existence of acceptance sets. The branches along the trunk are merely there for technical reasons. Think of a tree that has a trunk and only branches (of length one) at the top. Then allow branches of length one ("acceptance sets") to "grow" at any node. This will produce an observation in $\mathcal{P}$.

Lemma 23. *The category $\mathcal{LTS}_3$ has pullbacks.*

Proof. Assume we are given a diagram

$$\begin{tikzcd} T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\ & T_0 & \end{tikzcd}$$

where $T_j = (S_j, i_j, Act, \longrightarrow_j)$, $j = 0, 1, 2$.

We start be defining an *lts* $T = (S, i, Act, \longrightarrow)$ as follows. S will consist of triples whose first, second, and third components are elements from $RS(T_1)$, $RS(T_2)$, and subsets of Act, respectively. S and $\longrightarrow$ are defined to be the least set such that

- $i = (\{i_1\}, \{i_2\}, S_{T_1}(i_1) \cap S_{T_2}(i_2)) \in S$
- If $(X, Y, C) \in S$, $\alpha \in C$, $X \xrightarrow{\alpha} X'$, and $Y \xrightarrow{\alpha} Y'$ then $(X', Y', C') \in S$ for all $C' \in M(X', Y')$, where $M(X', Y') = \{A' \cap (\bigcup S_{T_2}(Y')) \mid A' \in S_{T_1}(X')\} \cup \{B' \cap (\bigcup S_{T_1}(X')) \mid B' \in S_{T_2}(Y')\}$. Also, $(X, Y, C) \xrightarrow{\alpha} (X', Y', C')$ for all $C' \in M(X', Y')$.

It should be easy to see that T is an *lts*. For later use we notice the following facts.

- For $(X, Y, C) \in S$ it is the case that $S_T((X, Y, C)) = C$. This follows from the definition of C.
- It is the case that $L(T) = L(T_1) \cap L(T_2)$. To see this note that $L(T) \subseteq L(T_1) \cap L(T_2)$ clearly holds. So choose any $v \in L(T_1) \cap L(T_2)$, where $v = \alpha_1 \cdots \alpha_n$, $n \geq 0$. Then there exists $X_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X_n$ in T_1 and $Y_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} Y_n$ in T_2, where $X_0 = \{i_1\}$ and $Y_0 = \{i_2\}$. Also, there must exists $A_j \in S_{T_1}(X_j)$ such that $\alpha_{j+1} \in A_j$ for $0 \leq j < n$ and clearly we have $\alpha_{j+1} \in \bigcup S_{T_2}(Y_j)$ for $0 \leq j < n$. So $(X_0, Y_0, A_0 \cap \bigcup S_{T_2}(Y_0)) \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} (X_n, Y_n, A_n \cap \bigcup S_{T_2}(Y_n))$. Hence, $v \in L(T)$.
- Given $Z \in RS(T)$. Then there exist an $X \in RS(T_1)$ and a $Y \in RS(T_2)$ such that $Z = \{(X, Y, C) \mid C \in M(X, Y)\}$. This follows from that fact that the transitions are deterministic on $RS(T_1)$ and $RS(T_2)$.

Let us define $\pi_1 : T \longrightarrow T_1$ as follows (π_2 is defined in a similar fashion). Given $Z \in RS(T)$ let $\pi_1(Z) = X$, where X is the unique first component of the elements of Z. We now show that π_1 is a morphism.

- Clearly $\pi_1(\{i\}) = \{i_1\}$.
- Assume $Z \xrightarrow{\alpha} Z'$, for $Z, Z' \in RS(T)$. Then by definition $\pi_1(Z) \xrightarrow{\alpha} \pi_1(Z')$.
- Assume $A \in S_{T_1}(X)$, where $X = \pi_1(Z)$. By definition $(X, Y, A \cap \bigcup S_{T_2}(Y)) \in Z$, where $\pi_2(Z) = Y$. Also, $S_T((X, Y, A \cap \bigcup S_{T_2}(Y)) = A \cap \bigcup S_{T_2}(Y) \subseteq A$. Hence, there exists an $C \in S_T(Z)$ such that $C \subseteq A$.

Having argued for that π_1 and π_2 are morphisms it trivially follows that $m_1 \circ \pi_1 = m_2 \circ \pi_2$.

Now assume that we are given a commuting square of the form

$$\begin{tikzcd} & T'' \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\ & T_0 & \end{tikzcd}$$

We will show that there exists a morphism $h : T'' \longrightarrow T$. We then necessarily have $f_1 = \pi_1 \circ f_1$ and $f_2 = \pi_2 \circ f_2$. Hence, we will have the desired pullback. Define $h : T'' \longrightarrow T$ by $h(V) = \{(X, Y, C) \mid C \in M(X, Y)\}$, where $X = f_1(V)$ and $Y = f_2(V)$.

- Clearly $h(\{i''\}) = i$.
- If $V \xrightarrow{\alpha} V'$ in T'' then $f_1(V) \xrightarrow{\alpha} f_1(V')$ and $f_2(V) \xrightarrow{\alpha} f_2(V')$, and hence, by previous facts we conclude that $h(V) \xrightarrow{\alpha} h(V')$.
- Assume $C \in S_T(Z)$, where $Z = h(V)$. By the previous facts we conclude that $(X, Y, C) \in Z$, where $f_1(V) = X$ and $f_2(V) = Y$. Without loss of generality we may assume C is of the form $A \cap \bigcup S_{T_2}(Y)$ for $A \in S_{T_1}(X)$. Since f_1 is a morphism we know there exists a $D \in S_{T''}(V)$ such that $D \subseteq A$. Also,

$D \subseteq \bigcup S_{T_2}(Y)$ because f_2 simulates the transitions from V by transitions from Y. Hence, $D \subseteq A \cap \bigcup S_{T_2}(Y)$.

This shows that h is a morphism and completes the proof of the lemma. $\square$

Having defined $\mathcal{M}$ and $\mathcal{P}$ we now characterise the open maps.

Lemma 24. *A morphism $m : T_1 \longrightarrow T_2$ is $\mathcal{P}$-open if and only if $m : RS(T_1) \longrightarrow RS(T_2)$ has the following "zig-zag" property*

If $m(X) \xrightarrow{\alpha} Y'$, then there exists an X' such that $X \xrightarrow{\alpha} X'$ and $m(X') = Y'$,

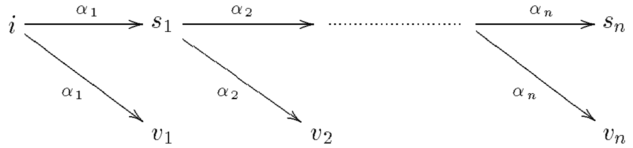
$$\textit{and } \forall A \in S_{T_1}(X).\, \exists A' \in S_{T_2}(m(X)).\, A' \subseteq A \quad .$$

Proof. Assume m is $\mathcal{P}$-open.

– If $m(X) \xrightarrow{\alpha} Y'$ then there exists

$$X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n \quad ,$$

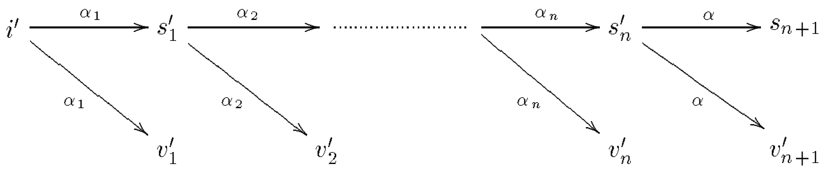
with $\{i_1\} = X_0$ and $X = X_n$ in T_1. Let $Y_j = m(X_j)$ for $0 \leq j \leq n$ and $Y_{n+1} = Y'$. Let O_1 denote the following observation. For the sake of completeness we show the underlying deterministic transition system and right below the associated acceptance sets.



$$\{i\} \xrightarrow{\alpha_1} \{s_1, v_1\} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \{s_{n-1}, v_{n-1}\} \xrightarrow{\alpha_n} \{s_n, v_n\}$$

$$\{\{\alpha_1\}\} \qquad \{\emptyset, \{\alpha_2\}\} \qquad \cdots \qquad \{\emptyset, \{\alpha_n\}\} \qquad \{\emptyset\}$$

Let O_2 denote the following observation.



$$\{i'\} \xrightarrow{\alpha_1} \{s'_1, v'_1\} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \{s'_n, v'_n\} \xrightarrow{\alpha} \{s'_{n+1}, v'_{n+1}\}$$

$$\{\{\alpha_1\}\} \qquad \{\emptyset, \{\alpha_2\}\} \qquad \cdots \qquad \{\emptyset, \{\alpha\}\} \qquad \{\emptyset\}$$

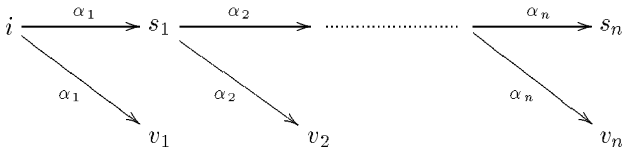
It should be easy to see that there exists a unique morphism from O_1 to O_2. Denote it $f : O_1 \longrightarrow O_2$. Also, defining $p : O_1 \longrightarrow T_1$ by $p(\{i\}) = \{i_1\}$ and $p(\{s_j, v_j\}) = X_j$ for $1 \leq j \leq n$ yields a morphism. Similarly, $q : O_2 \longrightarrow T_2$

defined by $q(\{i'\}) = \{i_2\}$ and $q(\{s'_j, v'_j\})$ for $1 \leq j \leq n+1$ is a morphism. Since there is at most one morphism between any two objects in $\mathcal{LTS}_3$ we conclude that $m \circ p = q \circ f$. Since m is $\mathcal{P}$-open there exists a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. Now $h(\{s'_n, v'_n\}) = h(f(\{s_n, v_n\})) = p(\{s_n, v_n\}) = X_n$ and since $\{s'_n, v'_n\} \xrightarrow{\alpha} \{s'_{n+1}, v'_{n+1}\}$ we conclude $h(\{s'_n, v'_n\}) \xrightarrow{\alpha} h(\{s'_{n+1}, v'_{n+1}\}). Also, m(h(\{s'_{n+1}, v'_{n+1}\})) = q(\{s'_{n+1}, v'_{n+1}\}) = Y_{n+1}$. So we conclude that there exists an X' such that $X \xrightarrow{\alpha} X'$ and $m(X') = Y'$.

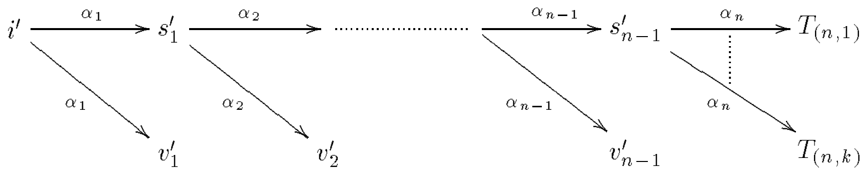
- Let $A \in S_{T_1}(X)$. We have to show that there exists an $A' \in S_{T_2}(m(X))$ such that $A' \subseteq A$. As before, assume

$$X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n \quad ,$$

with $\{i_1\} = X_0$ and $X = X_n$ in T_1. Let $Y_0, \ldots, Y_n$ denote $m(X_0), \ldots, m(X_n)$. Let O_1 denote the following observation.



Let O_2 denote



where $k = |S_{T_2}(Y)|$, and if $S_{T_2}(Y) = \{A_1, \ldots, A_k\}$ then $T_{(n,j)}$ denotes a tree of depth one, whose branches are labelled by A_j, for $1 \leq j \leq k$. For completeness have show the reachability sets and the associated acceptance sets.

$$\begin{array}{cccccc}
\{i'\} \xrightarrow{\alpha_1} & \{s'_1, v'_1\} & \cdots \{s'_{n-1}, v'_{n-1}\} \xrightarrow{\alpha_n} & \{i_{(n,1)}, \ldots, i_{(n,k)}\} & \ldots X'_1 \\
 & & & & \ddots X'_k \\
\{\{\alpha_1\}\} & \{\emptyset, \{\alpha_2\}\} \cdots & \{\emptyset, \{\alpha\}\} & S_{T_2}(Y) & \emptyset
\end{array}$$

For the sake of clarity we have used $X'_1, \ldots, X'_k$ to denote the remaining reachability sets.

As before, it should be easy to see that there exists a unique morphism $f : O_1 \longrightarrow O_2$. Define $p : O_1 \longrightarrow T_1$ as before. Finally, define $q(\{i'\}) = Y_1$, $q(\{s'_j, v'_j\}) = Y_j$ for $1 \leq j \leq n-1$, $q(\{i_{(n,1)}, \ldots, i_{(n,k)}\}) = Y_n$, and since the transitions from $\{i_{(n,1)}, \ldots, i_{(n,k)}\}$ can all be matched by those of Y_n ($S_{T_2}(Y) = S_{O_2}(\{i_{(n,1)}, \ldots, i_{(n,k)}\})$) it is possible to extend q uniquely to a morphism $q : O_2 \longrightarrow T_2$. Notice how the requirements on the acceptance sets are fulfilled. As noted before, we then necessarily have $m \circ p = q \circ f$ and therefore the existence of a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. Now the first equation tells us that $h(\{i_{(n,1)}, \ldots, i_{(n,k)}\}) = X_n$. Since $A \in S_{T_1}(X_n)$ there must exist an $A' \in S_{O_2}(\{i_{(n,1)}, \ldots, i_{(n,k)}\})$ such that $A' \subseteq A$. But since $S_{O_2}(\{i_{(n,1)}, \ldots, i_{(n,k)}\}) = S_{T_2}(Y_n) = S_{T_2}(m(X))$, we are done.

Assume m has the "zig-zag" property. Given a commuting diagram

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "m"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "q"'] & T_2 \end{tikzcd}$$

Since m has the property that $m(X) \xrightarrow{\alpha} Y'$ implies there exists $X \xrightarrow{\alpha} X'$ such that $m(X') = Y'$, transitions on the reachability sets are deterministic, and $RS(O_1)$ together with the transitions has a tree structure, we can define $h : O_2 \longrightarrow T_1$ inductively as follows.

- $h(\{i'\}) = \{i_1\}$
- If $h(Y)$ has been defined, $Y \xrightarrow{\alpha} Y'$, and h has not been defined on Y' then define $h(Y')$ as follows. By induction we may assume $q(Y) = m(h(Y))$ and we know $q(Y) \xrightarrow{\alpha} q(Y')$. Then there must exist a (necessarily unique) X' such that $h(Y) \xrightarrow{\alpha} X'$ and $m(X') = q(Y')$. Define $h(Y') = X'$. Notice also that for all $A' \in S_{T_1}(h(Y'))$ there exists an $A \in S_{O_2}(Y')$ such that $A \subseteq A'$. This follows from m being $\mathcal{P}$-open and q being a morphism; There must exist an $A'' \in S_{T_2}(m(h(Y'))) = S_{T_2}(q(Y'))$ such that $A'' \subseteq A'$ and also an $A \in S_{O_2}(Y')$ such that $A \subseteq A''$.

Checking that $h : O_2 \longrightarrow T_1$ is a morphism is now routine work. The equations $p = h \circ f$ and $q = m \circ h$ now follows trivially. □

We continue by defining Hennessy's testing equivalence.

Definition 25. Given two *lts*'s, $T_j = (S_j, i_j, Act, \longrightarrow_j), j = 1, 2$. i_1 is said to be *testing equivalent* to i_2 if

$$L(T_1) = L(T_2) \ , \tag{30}$$

and for any $s \in L(T_1)$

$$\forall A' \in \mathcal{A}(i_2, s).\, \exists A \in \mathcal{A}(i_1, s).\, A \subseteq A' \ , \tag{31}$$

$$\forall A \in \mathcal{A}(i_1, s).\, \exists A' \in \mathcal{A}(i_2, s).\, A' \subseteq A \ . \tag{32}$$

The above definition follows from Definition 2.8.8 in [Hen88]. Using the notation from [Hen88] the above definition can be rewritten to $i_1 \ll_{MAY} i_2$, $i_2 \ll_{MAY} i_1$, $i_1 \ll_{MUST} i_2$, and $i_2 \ll_{MUST} i_1$.

With the given choice of $\mathcal{M}$ and $\mathcal{P}$ it turns out that testing equivalence corresponds to $\mathcal{P}$-bisimilarity.

Theorem 26. *Given two* lts*'s T_1 and T_2. Then,*

T_1 and T_2 are testing equivalent if and only if T_1 and T_2 are $\mathcal{P}$-bisimilar.

Proof. Assume T_1 and T_2 are testing equivalent. Define $T = (S, i, Act, \longrightarrow)$, π_1 and π_2 as in the proof of Lemma 23. By symmetry it is enough to show that π_1 is $\mathcal{P}$-open.

Let $X = \pi_1(Z)$ for $Z \in RS(T)$ and assume $X \xrightarrow{\alpha} X'$. Assume $X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} X_n \xrightarrow{\alpha} X'$, where $X_0 = \{i_1\}$ and $X_n = X$. Since $L(T_1) = L(T_2)$ we know that there exist $Y_0 \xrightarrow{\alpha_1} Y_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} Y_n \xrightarrow{\alpha} Y'$ in T_2, where $Y_0 = \{i_2\}$. Then necessarily $Z \xrightarrow{\alpha} Z'$, where $Z' = \{(X', Y', C) \mid C \in M(X', Y')\}$ and $\pi_1(Z') = X'$.

Let $C \in S_T(Z)$, where $Z \in RS(T)$. We conclude that $(X, Y, C) \in Z$, where $Y = \pi_2(Z)$. There exists a $v \in L(T)$ such that $\{i_1\} \xrightarrow{v} X$ and $\{i_2\} \xrightarrow{v} Y$. Without loss of generality assume C is of the form $A \cap \bigcup S_{T_2}(Y)$, where $A \in S_{T_1}(X)$, Consider any minimal (with respect to set inclusion) $A_{min} \in S_{T_1}(X)$ such that $A_{min} \subseteq A$. Since $S_{T_1}(X) = \mathcal{A}(i_1, v) = \mathcal{A}(i_2, v) = S_{T_2}(Y)$, we conclude $A_{min} \in S_{T_2}(Y)$ and hence, $A_{min} \subseteq \bigcup S_{T_2}(U)$. Hence, $A_{min} \subseteq C$, i.e. there exists an $A' \in S_{T_1}(X)$ such that $A' \subseteq C$.

Next, assume T_1 and T_2 are $\mathcal{P}$-bisimilar, i.e. there exists a span of open maps

$$\begin{tikzcd} & T \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & \\ T_1 & & T_2 \end{tikzcd}$$

It is enough to show that T and T_1 are testing equivalent. By Lemma 24 it follows easily that $L(T) = L(T_1)$. Given an $s \in L(T)$. Assume $A \in \mathcal{A}(i, s)$. We have to show that there exists an $A_1 \in \mathcal{A}(i_1, s)$ such that $A_1 \subseteq A$. Let X be the unique element in $RS(T)$ such $\{i\} \xrightarrow{s} X$. Then $A \in S_T(X)$. Since m_1 is an open map there exists an $A_1 \in S_{T_1}(m(X))$ such that $A_1 \subseteq A$. Since $\mathcal{A}(i_1, s) = S_{T_1}(m(X))$ we are done.

Now assume $A_1 \in \mathcal{A}(i_1, s)$. We have to show that there exists an $A \in \mathcal{A}(i, s)$ such that $A \subseteq A_1$. Let X_1 be the unique element from $RS(T_1)$ such that $\{i_1\} \xrightarrow{s} X_1$. Then $A_1 \in S_{T_1}(X_1)$. Since m_1 is open there exists an $X \in RS(T)$ such that $\{i\} \xrightarrow{s} X$ and $m_1(X) = X_1$. By the definition of morphisms there exists an $A \in S_T(X)$ such that $A \subseteq A_1$. But since $S_T(X) = \mathcal{A}(i, s)$ we are done. Hence, T_1 and T_2 are testing equivalent. $\square$

6 Barbed Bisimulation (Milner & Sangiorgi)

In this section we show how we can obtain Milner and Sangiorgi's *barbed bisimulation* [MS92].

Barbed bisimulation differs from strong bisimulation in three obvious ways. First, as in the case of weak bisimulation, we distinguish between "visible" and "invisible" actions. Second, only τ transitions are required to be (bi)simulated. And third, only the *existence* of a "visible" transition has to be matched.

We start by defining the category of models $\mathcal{LTS}_4$, then the subcategory of observations $\mathcal{P}$, and finally we characterise the $\mathcal{P}$-open maps and prove that $\mathcal{P}$-bisimilarity coincides with barbed bisimulation.

Let $\mathcal{LTS}_4$ be the category of *labelled transition systems* (*lts*) whose objects are those from $\mathcal{LTS}$. Again we assume that Act contains a special "invisible" action denoted τ. Before defining the morphisms of $\mathcal{LTS}_4$ we need the following definition.

Definition 27. Given an *lts* $T = (S, i, Act, \longrightarrow)$. $R_\tau(T)$, the set of *τ-reachable states of* T, is defined to be the set $\{s \in S \mid i \xrightarrow{\tau} \cdots \xrightarrow{\tau} s\}$. We use the notation $s\downarrow$ if there exist an $\alpha \in Act - \{\tau\}$ and an $s' \in S$ such that $s \xrightarrow{\alpha} s'$.

Morphisms between two *lts* in $\mathcal{LTS}_4$ are defined as follows:

Definition 28. Given two *lts*'s, $T_1 = (S_1, i_1, Act, \longrightarrow_1)$ and $T_2 = (S_2, i_2, Act, \longrightarrow_2)$. A morphism m from T_1 to T_2 is a function $m : R_\tau(T_1) \longrightarrow R_\tau(T_2)$ such that

$$m(i_1) = i_2 \ , \tag{33}$$

$$r \xrightarrow{\tau} r' \text{ implies } m(r) \xrightarrow{\tau} m(r') \ , \tag{34}$$

$$r\downarrow \text{ implies } m(r)\downarrow \ . \tag{35}$$

This definition reflects that we only want to simulate τ's and preserve the $\downarrow$ predicate. This corresponds well to our intuitive understanding of barbed (bi)simulation. Composition of morphisms is defined as the composition of the functions between the underlying τ-reachable states.

Next we define the category of observations.

Definition 29. Let $\mathcal{P}$ be the subcategory of $\mathcal{LTS}_4$ whose objects are of the form

$$i \xrightarrow{\tau} r_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} r_{n-1} \xrightarrow{\alpha} r_n \ , \tag{36}$$

where $\alpha \in Act$ and all states are distinct.

Notice how the observations correspond to the τ-reachability of a state (r_n, when $\alpha = \tau$) and the $\downarrow$ predicate holding at it (r_{n-1}, when $\alpha \neq \tau$).

Having defined $\mathcal{M}$ as $\mathcal{LTS}_4$ and $\mathcal{P}$ we show that $\mathcal{LTS}_4$ has pullbacks.

Lemma 30. *The category $\mathcal{LTS}_4$ has pullbacks.*

Proof. Given a diagram

$$\begin{tikzcd} T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\ & T & \end{tikzcd}$$

Define $S' \subseteq R_\tau(T_1) \times R_\tau(T_2)$ and $\longrightarrow'$ as the least sets such that

- $(i_1, i_2) \in S'$
- If $(r, s) \in S'$, $r \xrightarrow{\tau} r'$, $s \xrightarrow{\tau} s'$, and $m_1(r') = m_2(s')$ then $(r', s') \in S'$ and $((r, s), \tau, (r', s')) \in \longrightarrow'$.

Intuitively, S' is the τ-reachable states of the *lts* that will constitute the pullback. Now choose any $\alpha \in Act - \{\tau\}$ and define

- $S'' = \{(r', s') | \exists (r, s) \in S'. \, r \xrightarrow{\beta} r' \ \wedge \ s \xrightarrow{\gamma} s' \ \wedge \ \beta \neq \tau \ \wedge \ \gamma \neq \tau\}$
- $\longrightarrow'' = \{((r, s), \alpha, (r', s')) \,|\, (r, s) \in S' \ \wedge \ (r', s') \in S'' \ \wedge$
  $\exists \beta \neq \tau, \gamma \neq \tau. \, r \xrightarrow{\beta} r' \ \wedge s \xrightarrow{\gamma} s'\}$

S'' and $\longrightarrow''$ ensure that the $\downarrow$ predicate has the desired value at the states in S'. Now define $T' = (S' \cup S'', (i_1, i_2), \longrightarrow' \cup \longrightarrow'', Act)$, $\pi_1 : T' \longrightarrow T_1$ as $\pi_1((r, s)) = r$, and $\pi_2 : T' \longrightarrow T_2$ as $\pi_2((r, s)) = s$. It can be shown that T', π_1, and π_2 constitute a pullback of the above diagram. $\square$

Next, we characterise the open maps.

Lemma 31. *A morphism $m : T_1 \longrightarrow T_2$ is $\mathcal{P}$-open if and only if it satisfies the following "zig-zag" property:*

If $m(r) \xrightarrow{\tau} s$ then there exists an r' such that $r \xrightarrow{\tau} r'$ and $m(r') = s$.

Also, $m(r)\downarrow$ implies $r\downarrow$.

Proof. Assume m is open, $r \in R_\tau(T_1)$, and $m(r) \xrightarrow{\tau} s$. We first prove the existence of the above mentioned r'. Assuming $i_1 \xrightarrow{\tau} r_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} r_n = r$, let O_1 be the observation

$$i \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_n \ ,$$

and O_2 the observation

$$i' \xrightarrow{\tau} s_1' \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_n' \xrightarrow{\tau} s' \ .$$

Let $f : O_1 \longrightarrow O_2$ be the unique function from O_1 to O_2 which sends s_j to s_j', $p : O_1 \longrightarrow T_1$ the morphism which sends s_j to r_j for $1 \leq j \leq n$, and $q : O_2 \longrightarrow T_2$ the morphism which sends s_j' to $m(r_j)$, for $1 \leq j \leq n$, and s' to s. Then the diagram

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "f"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "q"'] & T_2 \end{tikzcd}$$

commutes, and there exists a morphism $h : O_2 \longrightarrow T_1$ such that $p = h \circ f$ and $q = m \circ h$. Observe that $h(s'_n) = h(f(s_n)) = p(s_n) = r_n$. Hence, $r = r_n = h(s'_n) \xrightarrow{\tau} h(s')$. Also, $m(h(s')) = q(s') = s$. Now choose $r' = h(s')$. Finally, assume $m(r){\downarrow}$ because $m(r) \xrightarrow{\alpha} s$ for $\alpha \neq \tau$. Choosing O_2 as $i' \xrightarrow{\tau} s'_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s'_n \xrightarrow{\alpha} s'$ and the other components as before (adjusting the morphisms in the obvious way) we obtain another commuting diagram and a morphism $h' : O_2 \longrightarrow T_1$ such that the triangles commute. So $h'(s'_n) = h'(f(s_n)) = p(s_n) = r_n = r$ and since $s'_n{\downarrow}$ we have $r{\downarrow}$.

Assume m has the "zig-zag" property and the diagram

$$\begin{tikzcd} O_1 \arrow[r, "p"] \arrow[d, "f"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "q"'] & T_2 \end{tikzcd}$$

commutes. We proceed by a case analysis on the shape of the observations O_1 and O_2.

- Assume O_1 is of the form $i \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_n$ and O_2 of the form $i' \xrightarrow{\tau} s'_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s'_n \xrightarrow{\tau} \cdots \xrightarrow{\tau} s'_{n+k}$, where $k \geq 0$.
  Define $h(s'_j) = p(s_j)$ for $1 \leq j \leq n$. Next, we define h on s'_{n+1}. Since $q(s'_n) = q(f(s_n)) = m(p(s_n))$, $p(s_n) \in R_\tau(T_1)$, and $m(p(s_n)) \xrightarrow{\tau} q(s'_{n+1})$ there exists a r' such that $p(s_n) \xrightarrow{\tau} r'$ and $m(r') = q(s'_{n+1})$. Define $h(s'_{n+1}) = r'$. Continuing this way for the remaining state $s'_{n+2}, \ldots, s'_{n+k}$ it can be shown that the produced h is a morphism from O_2 to T_1 such that $p = h \circ f$ and $q = m \circ h$.
- Assume O_1 is of the form $i \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_{n-1} \xrightarrow{\alpha} s_n$ and O_2 of the form $i' \xrightarrow{\tau} s'_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s'_{m-1} \xrightarrow{\beta} s'_m$, where $\alpha \neq \tau$ and $\beta \neq \tau$.
  Since f is a morphism we conclude $n = m$ and $f(s_j) = s'_j$ for $1 \leq j \leq n$. It is then easy to see that $h(s'_j) = p(s_j)$ for $1 \leq j \leq n$ defines a morphism from O_2 to T_1 such that $p = h \circ f$ and $q = m \circ h$.
- Finally assume O_1 is of the form $i \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_n$ and O_2 of the form $i' \xrightarrow{\tau} s'_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s'_{m-1} \xrightarrow{\alpha} s'_m$, where $\alpha \neq \tau$.
  As before, we conclude $m > n$. Using the same procedure as in case 1, only this time observing that $m(h(s'_{m-1})) = q(s'_{m-1}){\downarrow}$ implies $h(s'_{m-1}){\downarrow}$, we obtain a morphism $h : O_2 \longrightarrow T_1$ such that the two triangles in the diagram commute.

We conclude that m is $\mathcal{P}$-open. $\square$

We now give Milner and Sangiorgi's definition of barbed bisimulation adapted to the case where the labelled transition systems have initial states.

Definition 32. Given T_1 and T_2. A *barbed bisimulation over T_1 and T_2* is a relation $R \subseteq R_\tau(T_1) \times R_\tau(T_2)$ such that

$$(i_1, i_2) \in R \ , \tag{37}$$

$$((r,s) \in R \ \wedge \ r \xrightarrow{\tau} r') \ \Rightarrow \ \text{for some } s' \ (s \xrightarrow{\tau} s' \ \wedge \ (r', s') \in R) \ , \tag{38}$$

$$((r,s) \in R \ \wedge \ s \xrightarrow{\tau} s') \ \Rightarrow \ \text{for some } r' \ (r \xrightarrow{\tau} r' \ \wedge \ (r', s') \in R) \ , \tag{39}$$

$$(r,s) \in R \ \Rightarrow \ (r\!\downarrow \ \Leftrightarrow \ s\!\downarrow) \ . \tag{40}$$

T_1 and T_2 are said to be *barbed bisimilar* if there exists such an R.

And now to the theorem relating $\mathcal{P}$-bisimilarity to barbed bisimilarity.

Theorem 33. *Given two* lts*'s, T_1 and T_2. Then:*

T_1 is barbed bisimilar to T_2 if and only if T_1 and T_2 are $\mathcal{P}$-bisimilar.

Proof. Assume R is a barbed bisimulation over T_1 and T_2 and $\gamma \in Act - \{\tau\}$. Now define

- $S' = \{(r,s) \in R \mid \exists (r_1, s_1), \ldots, (r_{n-1}, s_{n-1}) \in R.$
  $i_1 \xrightarrow{\tau} r_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} r_{n-1} \xrightarrow{\tau} r \ \wedge$
  $i_2 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_{n-1} \xrightarrow{\tau} s\}$,
- $\longrightarrow' = \{((r,s), \tau, (r', s')) \mid (r,s), (r', s') \in S' \ \wedge \ r \xrightarrow{\tau} r' \ \wedge \ s \xrightarrow{\tau} s'\}$,
- $S'' = \{(r', s') \in S_1 \times S_2 \mid \exists (r,s) \in S', \alpha, \beta \in Act - \{\tau\}. r \xrightarrow{\alpha} r' \ \wedge \ s \xrightarrow{\beta} s'\}$,
- $\longrightarrow'' = \{((r,s), \gamma, (r', s')) \mid (r,s) \in S' \ \wedge \ (r', s') \in S'' \ \wedge$
  $\exists \alpha, \beta \in Act - \{\tau\}. r \xrightarrow{\alpha} r' \ \wedge \ s \xrightarrow{\beta} s'\}$,
- $T = (S' \cup S'', (i_1, i_2), Act, \longrightarrow' \cup \longrightarrow'')$,
- $\pi_1 : T \longrightarrow T_1$ by $\pi_1((r,s)) = r$, and
- $\pi_2 : T \longrightarrow T_2$ by $\pi_1((r,s)) = s$.

It can be shown that T is a *lts* and that π_1 and π_2 are morphisms. Without loss of generality we show that π_1 is open. Choose any $(r,s) \in R_\tau(T)$.

- Assume $\pi_1((r,s)) = r \xrightarrow{\tau} r'$. Since $R_\tau(T) = S' \subseteq R$ and R was a barbed bisimulation over T_1 and T_2, we know that there exists a s' such that $s \xrightarrow{\tau} s'$ and $(r', s') \in R$. But by definition of S' we conclude $(r', s') \in S'$ and also $(r,s) \xrightarrow{\tau} (r', s')$ in T, where $\pi_1((r', s')) = r'$.
- Assume that $\pi_1((r,s)) = r\!\downarrow$, i.e. there exists an r' such that $r \xrightarrow{\alpha} r'$ and $\alpha \neq \tau$. Since $(r,s) \in R_\tau(T)$ we know that $r\!\downarrow \Rightarrow s\!\downarrow$, i.e. there exists s' such that $s \xrightarrow{\beta} s'$ and $\beta \neq \tau$. But then $(r', s') \in S''$ and $((r,s), \gamma, (r', s')) \in \longrightarrow''$, i.e. $(r,s)\!\downarrow$.

Assume T_1 and T_2 are $\mathcal{P}$-bisimilar. Then there must exist a span of open maps:

$$\begin{tikzcd} & T \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & \\ T_1 & & T_2 \end{tikzcd}$$

It is sufficient to show that if $m_1 : T \longrightarrow T_1$ ($m_2 : T \longrightarrow T_2$) is open then T and T_1 (T_2) are barbed bisimilar, since being barbed bisimilar is an equivalence relation. So define $R_1 = \{(r, m_1(r)) \mid r \in R_\tau(T)\}$. We claim that R_1 is a barbed bisimulation over T and T_1, namely;

- $(i, i_1) \in R_1$,
- $(r, s) \in R_1$ and $r \xrightarrow{\tau} r'$ implies $s = m_1(r) \xrightarrow{\tau} m_1(r')$ and $(r', m_1(r')) \in R_1$,
- if $(r, s) = (r, m_1(r)) \in R_1$ and $m_1(r) \xrightarrow{\tau} s'$ then there exists an r' such that $r \xrightarrow{\tau} r'$ and $m_1(r') = s'$, since m_1 is open, and $(r', s') = (r', m_1(r')) \in R_1$, and
- if $(r, s) \in R_1$ and $r\!\downarrow$ then $s = m_1(r)\!\downarrow$ since m_1 is a morphism and if $s\!\downarrow$ then $r\!\downarrow$ since $m_1(r) = s$, $r \in R_\tau(T)$, and m_1 is open.

This concludes the proof. $\square$

7 Probabilistic Transition Systems (Larsen & Skou)

In this section we show that the *probabilistic bisimulation* of Larsen and Skou [LS91] can be characterised using the general setting in Sect. 2. We will however apply the theory in a slightly different way. Until now, we have tried to characterise $\mathcal{P}$-bisimilarity between objects of $\mathcal{M}$, for the specific choices of $\mathcal{P}$ and $\mathcal{M}$. In this section we will focus on $\mathcal{P}$-bisimilarity between objects of a subcategory of $\mathcal{M}$. This application of the theory of open maps still turns out "successful".

Intuitively, Larsen and Skou's *probabilistic bisimulation* differs from strong bisimulation in at least two respects. First, to each labelled transition there is associated a real number from the interval $[0; 1]$ which is to be understood as the probability with which the transition can be performed. Second, it is no longer single labelled transitions between two states that have to be matched but a set of identically labelled transitions into an equivalence class of probabilistic bisimilar states.

Based on [LS91] we start by defining a category $\mathcal{PPTS}$, *partial probabilistic transition systems*, corresponding to $\mathcal{M}$, and a subcategory of observations, $\mathcal{P}$, in $\mathcal{PPTS}$. Then, we show that $\mathcal{P}$-bisimilarity in the full subcategory of *probabilistic transition systems*, $\mathcal{PTS}$, in $\mathcal{PPTS}$, corresponds to Larsen and Skou's probabilistic bisimulation. Contrary to Larsen and Skou we do not assume lower limit on the probability of transitions. Because we wish to allow arbitrary small probabilities and for technical reasons, we consider $\mathbb{R}^*$, the field of hyperreal numbers, instead of $I\!R$, the field of real numbers. $I\!R^*$ is the proper ordered extension of $I\!R$ containing infinitesimals. An element $\epsilon \in I\!R^*$ is infinitesimal if $0 < |\epsilon| < r$ for all positive real numbers r. We reserve the symbol ϵ to denote infinitesimals. For a thorough presentation the reader is referred to [Kei76].

Definition 34. A *partial probabilistic transition system* (*ppts*) is a tuple

$$T = (Pr, i, Act, Can, \mu) \ , \tag{41}$$

where Pr is a set of *processes* (or states), i is the *initial state*, Act is the set of *observable actions* that processes may perform, Can is an Act-indexed family of sets of processes, and μ is a family of *partial probability distributions* indexed by states and actions. For any action $a \in Act$ and any process $r \in Pr$, $r \in Can_a$ indicates that the process r can perform an a-action, in which case $\mu_{r,a} : Pr \rightarrow [0;1] \subseteq I\!R^*$ is a function such that $\sum_{r'} \mu_{r,a}(r') \leq 1$.

In general, we do not require the sum to be equal to 1; hence the name *partial* probability distribution. If all $\mu_{r,a}$ are probability distributions, i.e. $\mu_{r,a}$ maps into the real numbers and $\sum_{r'} \mu_{r,a}(r') = 1$, we leave out the term *partial* and refer to T as a *probabilistic transition system* (*pts*). $\mu_{r,a}(r') = \mu$ can intuitively be read as "r can perform the action a and with probability μ become r'".

Given a *ppts* T. We shall use the following notations:

$r \xrightarrow{a}_{\mu} r'$ whenever $r \in Can_a$ and $\mu_{r,a}(r') = \mu$
$r \xrightarrow{a} r'$ whenever $r \xrightarrow{a}_{\mu} r'$ for some $\mu > 0$
$r \xrightarrow{a}$ whenever $r \in Can_a$
$r \not\xrightarrow{a}$ whenever $r \notin Can_a$
$r \xrightarrow{a}_{\mu} S$ whenever S is *any* set of processes,
$r \in Can_a$ and $\mu = \sum_{r' \in S} \mu_{r,a}(r')$.

We assume the set Act to be fixed and that all processes in Pr are reachable from the initial state via transitions having non-zero probabilities. Finally, two *ppts* will be said to be *distinct* if their sets of processes are disjoint.

Next, we define morphisms between *ppts*'s.

Definition 35. A *ppts-morphism* f between two *ptts*'s, $T_j = (Pr_j, i_j, Act, Can_j, \mu_j)$, $j = 1, 2$, is a function between Pr_1 and Pr_2 such that

$$f(i_1) = i_2 \ , \tag{42}$$

$$f(r) \xrightarrow{a} f(r') \text{ whenever } r \xrightarrow{a} r' \ , \tag{43}$$

$$\text{If } r \xrightarrow{a} r' \text{ and } f(r) \xrightarrow{a}_{\mu'} f(r') \text{ then } \sum_{r \xrightarrow{a}_{\mu} r'' f(r'') = f(r')} \mu \leq \mu' \ . \tag{44}$$

The intuition behind (44) is that all transitions from r in T_1 which are simulated by a transition from $f(r)$ can occur with a probability which is no higher than the probability of the simulating transition from $f(r)$.

Let $\mathcal{PPTS}$ denote the category of partial probabilistic transition systems, whose objects are *ppts*'s and morphisms are *ppts-morphisms*, with composition of morphisms defined as the usual composition of functions. Let $\mathcal{PTS}$ denote the full subcategory of $\mathcal{PPTS}$ whose objects are *pts*'s.

In our model of computation, $\mathcal{PPTS}$, we identify the following subcategory $\mathcal{P}$ of observations.

Definition 36. Let $\mathcal{P}$ be the full subcategory of $\mathcal{PPTS}$ whose objects are *ppts*'s of the following form

$$i \xrightarrow{a_1}_{\epsilon_1} r_1 \xrightarrow{a_2}_{\epsilon_2} \cdots \xrightarrow{a_n}_{\epsilon_n} r_n \quad , \tag{45}$$

for some natural number n, distinct states, and actions $a_1, \ldots, a_n \in Act$. Notice that all the probabilities are infinitesimals.

The intuition behind using only infinitesimals on the transitions is that we will only be interested in whether or not a transition can occur rather the probability with which is occurs. This is only true because $\mathcal{PTS}$ are the models which we consider.

This time, we postpone the investigation of the existence of pullbacks in $\mathcal{PPTS}$. Instead, we now try to characterise the $\mathcal{P}$-open maps in $\mathcal{PPTS}$ between any two *pts*'s.

Lemma 37. *A morphism $m : T_1 \longrightarrow T_2$ between two* pts*'s is $\mathcal{P}$-open if and only if it is "zig-zag" in the following sense:*

If $m(r) \xrightarrow{a} s$ then there exists an r' such that $r \xrightarrow{a} r'$ and $m(r') = s$.

Proof. Assume m is $\mathcal{P}$-open and $m(r) \xrightarrow{a} s$. Since r is reachable from i_1 there exists

$$i_1 = r_0 \xrightarrow{a_1} r_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} r_n = r$$

in T_1. Let O_1 be any observation of the form

$$i \xrightarrow{a_1}_{\epsilon_1} s_1 \xrightarrow{a_2}_{\epsilon_2} \cdots \xrightarrow{a_n}_{\epsilon_n} s_n \quad ,$$

and O_2 any observation of the form

$$i' \xrightarrow{a_1}_{\epsilon'_1} s'_1 \xrightarrow{a_2}_{\epsilon'_2} \cdots \xrightarrow{a_n}_{\epsilon'_n} s'_n \xrightarrow{a}_{\epsilon'} s' \quad ,$$

where $\epsilon_j \leq \epsilon'_j$ for $1 \leq j \leq n$. Let h denote the unique morphism between O_1 and O_2, f denote the morphism from O_1 to T_1 which maps s_j into r_j for $1 \leq j \leq n$, and g denote the morphism from O_2 to T_2 which maps O_2 into

$$i_2 = m(r_0) \xrightarrow{a_1} m(r_1) \xrightarrow{a_2} \cdots \xrightarrow{a_n} m(r_n) = m(r) \xrightarrow{a} s \quad .$$

It is easy to see, that the square

$$\begin{tikzcd} O_1 \arrow[r, "f"] \arrow[d, "h"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "g"'] & T_2 \end{tikzcd}$$

commutes. But since m is $\mathcal{P}$-open there exists a morphism $m' : O_2 \longrightarrow T_1$ such that $m' \circ h = f$ and $m \circ m' = g$. This implies the existence of an r' such that $r \xrightarrow{a} r'$ and $m(r') = s$.

Conversely, assume that m is "zig-zag". Let O_1 and O_2 be any two observations and f, g, and h any morphisms such that the square

$$\begin{tikzcd} O_1 \arrow[r, "f"] \arrow[d, "h"'] & T_1 \arrow[d, "m"] \\ O_2 \arrow[r, "g"'] & T_2 \end{tikzcd}$$

commutes. We now define a morphism m' from O_2 to T_1 such that $m' \circ h = f$ and $m \circ m' = g$.

Assume O_1 has the form

$$i \xrightarrow{a_1}_{\epsilon_1} r_1 \xrightarrow{a_2}_{\epsilon_2} \cdots \xrightarrow{a_n}_{\epsilon_n} r_n \quad ,$$

and O_2 has the form

$$i' \xrightarrow{a_1}_{\epsilon'_1} r'_1 \xrightarrow{a_2}_{\epsilon'_2} \cdots \xrightarrow{a_n}_{\epsilon'_n} r'_n \xrightarrow{a_{n+1}}_{\epsilon'_{n+1}} \cdots \xrightarrow{a_{n+k}}_{\epsilon'_{n+k}} r'_{n+k} \quad ,$$

where $\epsilon_j \leq \epsilon'_j$ for $1 \leq j \leq n$. Apart from $m'(i') = i_1$, we must define $m'(r'_j) = f(r_j)$ for $1 \leq j \leq n$. Since $g(r'_n) = m(f(r_n)) \xrightarrow{a_{n+1}} g(r'_{n+1})$ and m is "zig-zag", there exists an s such that $f(r_n) \xrightarrow{a_{n+1}} s$ and $m(f(s)) = g(r'_{n+1})$. Define $m'(r_{n+1}) = s$. Continuing this process for the remaining $r'_{n+2}, \ldots, r'_{n+k}$ we obtain the map $m' : O_2 \longrightarrow T_1$. It is now easy to show that m' is indeed a morphism, since the transitions in the observations have infinitesimal probabilities, and that the "triangles" in the diagram commute. Hence, m must be $\mathcal{P}$-open $\square$

Going through the proof the reader should be able to realise why only infinitesimal probabilities are allowed on the observations from $\mathcal{P}$. Allowing arbitrary probabilities would imply that two *pts*'s which are related by an open map m would be locally isomorphic in the following sense: if $m(r) \xrightarrow{a}_{\mu} s'$, then there exists an $r \xrightarrow{a}_{\mu} r'$ such that $m(r') = s'$.

From the definition of the morphisms in $\mathcal{PPTS}$ one observes the following facts:

- If $m : T_1 \longrightarrow T_2$ and T_1 is a *pts* then T_2 must also be a *pts*.

– $\mathcal{PPTS}$ does not have pullbacks, neither does $\mathcal{PTS}$. Consider the following example which illustrates three *pts*'s.

$$
\begin{tikzcd}
i \arrow[d, "1"', "a"] \\
r
\end{tikzcd}
\qquad
\begin{tikzcd}
 & i' \arrow[dl, "0.55"', "a"] \arrow[dr, "a", "0.45"'] & \\
s & & s'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & i'' \arrow[dl, "0.5"', "a"] \arrow[dr, "a", "0.5"'] & \\
s'' & & s'''
\end{tikzcd}
$$

Let T, T_1, and T_2 denote the *pts*'s from left to right. Clearly there are uniquely determined morphisms from T_1 to T and from T_2 to T. Together, they form a diagram which does not have a pullback.

However, for $\mathcal{P}$-bisimilarity to be an equivalence relation (a transitive relation, to be more precise) it is in general not necessary for the category $\mathcal{M}$ to have pullbacks. The following weaker result suffices.

Theorem 38. *Given two $\mathcal{P}$-open morphisms between* pts*'s, $m_1 : T_1 \longrightarrow T_0$ and $m_2 : T_2 \longrightarrow T_0$. There exists a pts T and $\mathcal{P}$-open morphisms π_1 and π_2 such that*

$$
\begin{tikzcd}
 & T \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
T_1 \arrow[dr, "m_1"'] & & T_2 \arrow[dl, "m_2"] \\
 & T_0 &
\end{tikzcd}
$$

is a commuting square.

Proof. We define a *pts*, $T = (Pr, i, Act, Can, \mu)$, and two maps $\pi_1 : T \longrightarrow T_1$, $\pi_2 : T \longrightarrow T_2$ with the desired properties.

– Define $S = \{m_1^{-1}(r) \times m_2^{-1}(r) \mid r \in Pr_0\}$.
– Let $Pr \subseteq S$ and $\rightsquigarrow \subseteq S \times Act \times [0; 1] \times S$ be the least sets such that
  - $i = (i_1, i_2) \in Pr$
  - If $(r, s) \in Pr$, $r \xrightarrow{a}_{\mu_1} r'$, $\mu_1 > 0$, $s \xrightarrow{a}_{\mu_2} s'$, $\mu_2 > 0$, $m_1(r') = m_2(s')$, and $\mu' = \frac{\mu_1\mu_2}{\mu}$, where $\mu > 0$ is uniquely determined by $m_1(r) = m_2(s) \xrightarrow{a}_{\mu} m_2(s') = m_1(r')$, then $(r', s') \in Pr$ and $((r, s), a, \mu', (r', s')) \in \rightsquigarrow$, written $(r, s) \overset{a}{\rightsquigarrow}_{\mu'} (r', s')$.

– Now for $(r, s) \in Pr$, $a \in Act$ define $(r, s) \in Can_a$ if $(r, s) \overset{a}{\rightsquigarrow}_{\mu} (r', s')$ for some $(r', s') \in Pr$. Also, define $\mu_{(r,s),a}((r', s')) = \mu' > 0$ if $(r, s) \overset{a}{\rightsquigarrow}_{\mu'} (r', s')$ and otherwise 0.
– Define $\pi_1 : T \longrightarrow T_1$ by $\pi_1((r, s)) = r$ and $\pi_2 : T \longrightarrow T_2$ by $\pi_2((r, s)) = s$.

It can be shown that π_1 and π_2 are morphisms. Here we merely show that T is a *pts*.

Clearly, all states in Pr are reachable from the initial state. Since m_1 and m_2 form a co-span of open maps between *pts*'s we know e.g. that if $m_1(r) \xrightarrow{a}_\mu v$ then

$$\mu = \sum_{r' m_1(r')=v} \mu_{r,a}(r') = \sum_{r \xrightarrow{a}_\mu r' m_1(r')=v} \mu \ .$$

Also, if $(r,s) \in Can_a$ then $(r,s) \overset{a}{\leadsto}_{\mu'} (r',s')$ for some $\mu' > 0$ and we have to show

$$\sum_{(r',s')} \mu_{(r,s),a}((r's')) = 1 \ .$$

It follows from the definition that $r \in Can_a$ and $s \in Can_a$. We then have

$$\begin{aligned}
&\sum_{(r',s')} \mu_{(r,s),a}((r's')) \\
&= \sum_{I_1} \sum_{I_2} \sum_{I_3} \frac{\mu_1 \mu_2}{\mu} \\
&= \sum_{I_1} \sum_{I_2} \frac{\mu_1}{\mu} \sum_{I_3} \mu_2 \\
&= \sum_{I_1} \sum_{I_2} \frac{\mu_1}{\mu} \mu \\
&= \sum_{I_1} \sum_{I_2} \mu_1 \\
&= \sum_{I_1} \mu \\
&= 1 \ .
\end{aligned}$$

where the index sets are

$$I_1 = \{v \mid m_1(r) \xrightarrow{a}_\mu v \ \wedge \ \mu > 0\} \ ,$$

$$I_2 = \{r' \mid r \xrightarrow{a}_{\mu_1} r' \ \wedge \ m_1(r') = v\} \ ,$$

and

$$I_3 = \{s' \mid s \xrightarrow{a}_{\mu_2} s' \ \wedge \ m_2(s') = v\} \ .$$

□

Noticing that the composition of two open maps is itself an open map [JNW93], we obtain the following corollary.

Corollary 39. *$\mathcal{P}$-bisimilarity between* pts*'s is an equivalence relation.*

If, as done by Larsen and Skou in [LS91], we had assumed a lower limit γ on the probability of transitions (*minimal probability assumption*) and only considered the field of real numbers it would have been hard to obtain a result as the above, at least for us. The problem is that "pullback like" constructions of T involves expressions of the type $\frac{\mu_1\mu_2}{\mu}$ which may denote values smaller than γ.

Next, we recall the definition of probabilistic bisimulation from [LS91]. We have adapted it to the case where the probabilistic transition systems have initial states.

Definition 40. Let $T_j = (Pr_j, i_j, Act, Can_j, \mu_j)$, where $j = 1, 2$, be two distinct *pts*'s. A *probabilistic bisimulation* between T_1 and T_2 is an equivalence $\equiv$ on $Pr = Pr_1 \cup Pr_2$ such that $i_1 \equiv i_2$ and whenever $r \equiv s$, then the following holds:

$$\forall a \in Act. \forall S \in Pr/_{\equiv}.\, r \xrightarrow{a}_{\mu} S \Leftrightarrow s \xrightarrow{a}_{\mu} S \ , \tag{46}$$

where the notation $r \xrightarrow{a}_{\mu} S$ was defined after Definition 34.

Now to the main result of this section.

Theorem 41. *Given two* pts*'s, T_1 and T_2. Then :*

T_1 is probabilistic bisimilar to T_2 if and only if T_1 is $\mathcal{P}$-bisimilar to T_2.

Proof. Assume $\equiv$ is a probabilistic bisimulation between T_1 and T_2. We define a *pts* $T = (Pr, i, Act, Can, \mu)$ and two $\mathcal{P}$-open morphisms, $m_1 : T \longrightarrow T_1$ and $m_2 : T \longrightarrow T_2$, which constitute a span of open maps, showing that T_1 and T_2 are $\mathcal{P}$-bisimilar.

- Let $S = \{(r, s) \in Pr_1 \times Pr_2 \mid r \equiv s\}$.
- Let $Pr \subseteq S$ and $\rightsquigarrow \subseteq S \times Act \times [0; 1] \times S$ be the least sets such that
  - $i = (i_1, i_2) \in Pr$
  - If $(r, s) \in Pr$, $r \xrightarrow{a}_{\mu_1} r'$, $\mu_1 > 0$, $s \xrightarrow{a}_{\mu_2} s'$, $\mu_2 > 0$, $r' \equiv s'$, and $\mu' = \frac{\mu_1\mu_2}{\mu}$, where $\mu > 0$ is uniquely determined by $r \xrightarrow{a}_{\mu} [r']$ and $[r']$ is the equivalence class of r' under $\equiv$, then $(r', s') \in Pr$ and $((r, s), a, \mu', (r', s')) \in \rightsquigarrow$, written $(r, s) \overset{a}{\rightsquigarrow}_{\mu'} (r', s')$.
- Now for $(r, s) \in Pr$, $a \in Act$ define $(r, s) \in Can_a$ if $(r, s) \overset{a}{\rightsquigarrow}_{\mu} (r', s')$ for some $(r', s') \in Pr$. Also, define $\mu_{(r,s),a}((r', s')) = \mu' > 0$ if $(r, s) \overset{a}{\rightsquigarrow}_{\mu'} (r', s')$ and otherwise 0.
- Define $m_1 : T \longrightarrow T_1$ by $m_1((r, s)) = r$ and $m_2 : T \longrightarrow T_2$ by $m_2((r, s)) = s$.

It can be shown that T is a *pts*, that m_1 and m_2 are open maps, and that together they constitute the desired span of open maps.

Now assume there is a span of open maps:

$$\begin{tikzcd} & T \arrow[dl, "m_1"'] \arrow[dr, "m_2"] & \\ T_1 & & T_2 \end{tikzcd}$$

Define the relation $\sim \subseteq Pr \times Pr$ as

$$r \sim r' \text{ if and only if } m_1(r) = m_1(r') \text{ or } m_2(r) = m_2(r') \enspace ,$$

and let $\approx = \sim^*$. Now define an equivalence relation $\equiv$ on $Pr_1 \cup Pr_2$ whose set of equivalence classes are

$$\{m_1(S') \cup m_2(S') \mid S' \in Pr/_{\approx}\} \enspace .$$

That $\equiv$ is indeed an equivalence relation on $Pr_1 \cup Pr_2$ follows from the definition of $\approx$ and that m_1 and m_2 are $\mathcal{P}$-open morphisms, which implies that they are surjective functions.

We now claim that $\equiv$ is a probabilistic bisimulation between T_1 and T_2. This follows from the following observations (without loss of generality stated for T_1).

- $i_1 \equiv i_2$.
- If $s \in Pr_1$, $a \in Act$, and S is an equivalence class of $\equiv$ then $s \xrightarrow{a}_{\mu} S$ if and only if $s \xrightarrow{a}_{\mu} (S \cap Pr_1)$. Also, for any $r \in m_1^{-1}(s)$, $s \xrightarrow{a}_{\mu} S$ implies $r \xrightarrow{a}_{\mu} S'$ for the unique equivalence class $S' = m_1^{-1}(S \cap Pr_1) \in Pr/_{\approx}$, and $r \xrightarrow{a}_{\mu} S'$ for $S' \in Pr/_{\approx}$ implies $m_1(r) \xrightarrow{a}_{\mu} m_1(S')$.
- For $r_1 \approx r_2$ in T and any $S' \in Pr/_{\approx}$ we have $r_1 \xrightarrow{a}_{\mu} S'$ if and only if $r_2 \xrightarrow{a}_{\mu} S'$.

$\square$

8 Conclusion

In this paper we have tried in practice to investigate the applicability of Joyal, Nielsen, and Winskel's theory of open maps which was proposed as an abstract definition of equivalence in categories of models of computations.

Guided by our intuitive understanding of what it means for a system X to be simulated by a system Y we defined different categories of models of computation. Our choices of (sub)categories of observations were also guided by which behaviours ought to be observable.

It turned out that we could identify well-know notions of behavioural equivalences. We started by the most fundamental (or coarsest) namely, trace equivalence. Then, we considered "invisible" actions and identified weak bisimulation, testing equivalence, and barbed bisimulation.

Finally, in a more technical section, we showed how the theory of open maps could be relaxed and we identified Larsen and Skou's probabilistic bisimulation. For technical reasons we applied the theory to a category in which the subcategory of observations was disjoint from the subcategory of models we were interested in.

Our results have shown that the theory of open maps does give meaningful equivalences when applied to well know models of computation.

We also noticed that the equivalences we identified could be captured by several different choices of observations. E.g., from the proofs in Sect. 5 it is clear that we could choose a smaller subcategory of observations; binary branching along the "trunks" of the trees is sufficient. A similar observation has been made in [JNW94] in connection with strong history preserving bisimulation. On the other hand, consider the characterisation of strong bisimulation from [JNW93] (see also Sect. 2). Had $\mathcal{P}$ been chosen to be of the form

$$\begin{tikzcd}
& & & & s_{n,1} \arrow[dd, dash, dotted] \\
i \arrow[r, "\alpha_1"] & s_1 \arrow[r, "\alpha_2"] & \cdots\cdots \arrow[r, "\alpha_{n-1}"] & s_{n-1} \arrow[ur, "\alpha_{n,1}"] \arrow[dr, "\alpha_{n,m}"'] & \\
& & & & s_{n,m}
\end{tikzcd}$$

then an $\mathcal{P}$-open map m would have been characterised by the usual "zig-zag" property

$$\text{if } m(r) \xrightarrow{\alpha} s', \text{ then there exists an } r' \text{ such that } r \xrightarrow{\alpha} r' \text{ and } m(r') = s',$$

and the additional "local bijection" property

$$\text{for any } \alpha \in \mathit{Act},\ \{r' \mid r \xrightarrow{\alpha} r'\} \text{ is in bijective correspondence with}$$
$$\{s' \mid m(r) \xrightarrow{\alpha} s'\} \text{ under } m.$$

So apart from helping identifying "characterising observations" for behavioural equivalences, the theory of open maps also allows us to test how "robust" an equivalence is against different choices of observations.

Also, the choice of simulating morphisms turned out to be important. In the category of labelled transition systems $\mathcal{LTS}$ we didn't expect to be able to capture weak bisimulation just by changing the choice of observations. We defined new morphisms which intuitively corresponded to a "weak simulation".

As for future work, the section on weak bisimulation can be seen as starting point to understand how weak bisimulations can be obtained abstractly from (strong) bisimulations using category theory. Glynn Winskel has observed that one can motivate the choice of morphism between the observations (which were the same as those used for characterising strong bisimulation in [JNW93]) from a category theoretical point of view using monads.

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