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Bisimulations, Games and Logic

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Bisimulations, Games and Logic

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Abstract

In a recent paper by Joyal, Nielsen, and Winskel, bisimulation is defined in an abstract and uniform way across a wide range of different models for concurrency. In this paper, following a recent trend in theoretical computer science, we characterize their abstract definition game-theoretically and logically in a non-interleaving model. Our characterizations appear as surprisingly simple extensions of corresponding characterizations of interleaving bisimulation.

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1 Introduction

An important ingredient of the theory of concurrency is the notion of *behavioral equivalence* between processes; what does it mean for two systems to be equal with respect to their communication structures? There is no unique answer to this question, but, undoubtedly, one of the most popular and successful answers was given by Park [Par81]: Two processes (or states s and s' of two transition systems) are equivalent, or *bisimilar*, if for all actions a , every a -derivative of s is bisimilar to some a -derivative of s' , and vice versa.

One of the measures of success for a behaviour equivalence is its accompanying *theory*. And here bisimulation is particularly rich in results. Let us mention just three examples of elegant and powerful characterizations.

The first classical characterization is in terms of the existence of a bisimulation relation over states of the associated transition systems: Two transition systems are *bisimilar* iff there is a relation \mathcal{S} over states such that the initial states are related, and

- whenever $s \mathcal{S} s'$ and $s \xrightarrow{a} s_1$, there is a transition $s' \xrightarrow{a} s'_1$ such that $s_1 \mathcal{S} s'_1$, and
- whenever $s \mathcal{S} s'$ and $s' \xrightarrow{a} s'_1$, there is a transition $s \xrightarrow{a} s_1$ such that $s_1 \mathcal{S} s'_1$.

The process of exploring whether two transition systems are bisimilar or not can be viewed as a *game* between two persons, Player and Opponent, taking turns [Sti93]. This provides an operational setting in which bisimulation may be understood experimentally. Player tries to prove the systems bisimilar, whereas Opponent intends otherwise. The game is opened by Opponent who chooses a transition from the initial state of one of the systems. This transition must be matched by Player with an equally labelled transition from the initial state of the other system. The new states form the starting point for the next pair of moves, and so forth. The play continues like this *forever*, in which case Player wins, or until either Player or Opponent is unable to move, in which case the opposition wins. This game is *characteristic* for bisimulation in the sense that two transition systems are bisimilar iff Player has a *winning strategy*, i.e. iff Player is able to win every game starting from the initial states.

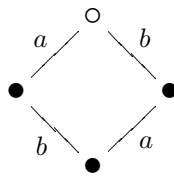


Figure 1: A transition system representing both $a \parallel b$ and $a.b + b.a$.

Another important ingredient of the theory is the associated language of logical assertions. The logic, known as *Hennesy-Milner logic* [HM85], is a modal logic in which the modalities are indexed by actions. As such, it captures precisely the discrimination power of bisimulation: Two systems are bisimilar iff they satisfy the same logical assertions. For verification and analysis, the Hennessy-Milner logic is most interesting in conjunction with *recursion*. In such logic, it is possible to express properties like deadlocks, invariants, inevitability, etc. [Sti93]. Also this very expressive logic is characteristic for bisimulation.

In the transition system model of CCS and CSP, parallelism is treated as *non-deterministic interleaving* of atomic actions. As a result, the CCS-processes $a \parallel b$, which can do the atomic actions a and b in parallel, is bisimilar to the process $a.b + b.a$, which non-deterministically chooses to do either “ a followed by b ” or “ b followed by a ”. In fact, the associated transition systems are isomorphic. Abstracting away from the names of the states, both transition systems are represented by the system of Fig. 1. Due to this identification, the transition system model is usually called an *interleaving model*, and bisimulation is traditionally called *interleaving bisimulation* when confusion is possible.

Interpreted at the machine level, non-deterministic interleaving corresponds to parallel processes sharing a single CPU. Opposed to this, Petri nets [WN94] model the *physical disjointness* of parallel processes. The processes $a \parallel b$ and $a.b + b.a$ are represented by the labelled nets of Fig. 1 [Old91].

The leftmost net consists of two independent events labelled a and b , whereas the rightmost net is a purely (nondeterministic) sequential net.

Many other closely related *non-interleaving* models have been suggested, e.g. the *asynchronous transition systems* of [Bed88, Shi85] and the *transitions systems with independence* of [WN94].

What is now the appropriate generalization of bisimulation to these

Figure 2: Labelled Petri nets representing $a \parallel b$ and $a.b + b.a$.

“independence models”? Many attempts have been made to answer this question. Unfortunately, with almost just as many different answers. Moreover, many of the proposed equivalences are incomparable. (See [GG89] and [GKP92] for definitions and comparisons of some of them.)

Apparently, the problem is that the step from interleaving models to independence models opens up for variations when trying to define an equivalence at the concrete level. [JNW93] reports on a promising attempt to define bisimulation in a *uniform* way across a wide range of different models for concurrent computation, including those described previously. However, the abstract definition is intangible. In order to obtain a better understanding of the equivalence, it is necessary to find concrete characterizations, which are indispensable for practical purposes. As a first measure of success, it is observed in [JNW93] that the abstract definition specializes to interleaving bisimulation in the case of ordinary transition systems.

In the context of an independence model, we shall denote the abstract equivalence by **Pom**_L-*bisimilarity*. The thoughts behind this choice of name will become clear later. In [JNW93], a concrete characterization of **Pom**_L-bisimilarity is given in the model of *event structures*, which may be thought of as *unfoldings* of nets or transition systems with independence. Interestingly, their characterization is not equal to any previously published equivalence; in fact, it is a strengthening of the history-preserving bisimulation [GG89, RT88].

In this paper, we give concrete characterizations of **Pom**_L-bisimilarity in the model of transition systems with independence. As a matter of fact, our choice of model is not essential, in the sense that our results could equally have been formulated and proved for nets or asynchronous

transition systems.

It turns out that surprisingly small twists of the game [Sti93] and relation [Mil89] characterizations of interleaving bisimulation lead to characterizations of \mathbf{Pom}_L -bisimilarity. On the logical side, the Hennessy-Milner logic is extended with a backwards modality. The logic is characteristic for \mathbf{Pom}_L -bisimilarity, when restricted to systems without *auto-concurrency*, i.e. to systems where no two consecutive and independent transitions are equally labelled. This restriction is necessary and has to do with the fact that our logic is based on *labels*. By strengthening the language of logical assertions, we can eliminate this restriction.

2 An Abstract Equivalence

In [JNW93], a uniform definition of bisimulation across a range of different models for parallel computation is presented. The aim of this section is to rephrase briefly parts of this work.

A model of computation is represented as a category. For a specific model, \mathbf{M} , a choice of observation is any subcategory \mathbf{P} of \mathbf{M} . Typically, a choice of observation is a selection of “observation objects” of \mathbf{M} , and \mathbf{P} is then the corresponding full subcategory.

Given a model \mathbf{M} and a choice of observation \mathbf{P} , where \mathbf{P} is a subcategory of \mathbf{M} , a morphism $f : X \rightarrow Y$ is said to be *\mathbf{P} -open* in \mathbf{M} iff whenever a square

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ m \downarrow & & \downarrow f \\ Q & \xrightarrow{q} & Y \end{array}$$

commutes, i.e. $f \circ p = q \circ m$, there is a morphism $p' : Q \rightarrow X$ such that the “triangles” in

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ m \downarrow & \nearrow p' & \downarrow f \\ Q & \xrightarrow{q} & Y \end{array}$$

commute, i.e. $p' \circ m = p$ and $f \circ p' = q$.

In the familiar example of \mathbf{M} being a category of transition systems and \mathbf{P} being sequences of labels (see [JNW93] for details), it turns out that open maps correspond to the well-known zig-zag morphisms of [Ben84].

Definition 2.1 Assume \mathbf{P} is a subcategory of \mathbf{M} and define two objects X and X' of \mathbf{M} to be \mathbf{P} -bisimilar, written $X \sim_{\mathbf{P}} X'$, iff there is a span of \mathbf{P} -open morphisms f and f' with common domain Y :

$$\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow f' \\ X & & X' \end{array}$$

□

Using that pullbacks of \mathbf{P} -open maps are themselves \mathbf{P} -open, it can be shown that \mathbf{P} -bisimilarity is an equivalence relation provided that \mathbf{M} has pullbacks.

The category of transition systems turns out to have pullbacks, and the notion of \mathbf{P} -bisimilarity (\mathbf{P} again being sequences of labels) turns out to coincide precisely with (strong) bisimilarity in the sense of [Mil89].

The interesting question is what you get when you lift this abstract characterization to non-interleaving models. One choice of model made in [JNW93] is transition systems with independence. Transition systems with independence are precisely what their name suggests, namely ordinary transition systems with an additional relation expressing when one transition is independent of another. The independence relation expresses which actions can happen in parallel.

Definition 2.2 A *transition system with independence* is a structure

$$X = (S, i, L, Tran, I)$$

where

- S is a set of *states* with a distinguished *initial state* i ,
- L is a set of *labels*,
- $Tran \subseteq S \times L \times S$ is a set of *transitions*¹, and
- $I \subseteq Tran^2$ is an *independence relation* which is irreflexive and symmetric.

Moreover, we require the following axioms to hold:

¹As usual, a transition $(s, a, s_1) \in Tran$ is written as $s \xrightarrow{a} s_1$.

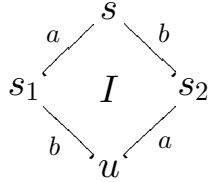
1. $s \xrightarrow{a} s_1 \sim s \xrightarrow{a} s_2 \Rightarrow s_1 = s_2$
2. $s \xrightarrow{a} s_1 \ I \ s_1 \xrightarrow{b} u \Rightarrow \exists s_2. s \xrightarrow{a} s_1 \ I \ s \xrightarrow{b} s_2 \ I \ s_2 \xrightarrow{a} u$
3. (a) $s \xrightarrow{a} s_1 \prec s_2 \xrightarrow{a} u \ I \ w \xrightarrow{b} w' \Rightarrow s \xrightarrow{a} s_1 \ I \ w \xrightarrow{b} w'$
 (b) $w \xrightarrow{b} w' \ I \ s \xrightarrow{a} s_1 \prec s_2 \xrightarrow{a} u \Rightarrow w \xrightarrow{b} w' \ I \ s_2 \xrightarrow{a} u$

where the relation \prec between transitions is defined by

$$s \xrightarrow{a} s_1 \prec s_2 \xrightarrow{a} u \Leftrightarrow \exists b. s_1 \xrightarrow{b} u \ I \ s \xrightarrow{a} s_1 \ I \ s \xrightarrow{b} s_2 \ I \ s_2 \xrightarrow{a} u,$$

and \sim is the least equivalence relation including \prec . □

The \sim -equivalence classes should be thought of as *events*. Thus, Axiom 1 asserts that the occurrence of an event at a state yields a unique state. Similarly, Axiom 3 asserts that independence respects events. Axiom 2 describes the intuitive property of independence that whenever two independent transitions occur consecutively, they can also occur in the opposite order. Hence, if $s \xrightarrow{a} s_1 \xrightarrow{b} u$ are independent transitions there is an “independence square”



Moreover, Axiom 1 implies the uniqueness of s_2 . So we are justified in saying that s_2 (or $s \xrightarrow{b} s_2 \xrightarrow{a} u$) is *the* completion of $s \xrightarrow{a} s_1 \xrightarrow{b} u$.

Notice that an ordinary labelled transition system can be viewed as a transition system with independence having empty independence relation. Furthermore, the standard labelled case graph of a labelled (safe) net, with two transitions being independent iff they represent firings of independent (in net terminology) events, is a transition system with independence [WN94]. As an example the transition system with independence above is the representation of the CCS-expression $a \parallel b$ or its corresponding net from 1 (following [WN94]).

For later use, we introduce some terminology. For a transition $t = (s \xrightarrow{a} s_1)$ we shall write $src(t)$, $tgt(t)$, and $\ell(t)$ for s , s_1 , and a , respectively. The set $Seqs(X)$ consists of those transition sequences $\bar{t} = t_0 t_1 \cdots t_{n-1}$ in X beginning at the initial state ($src(t_0) = i$) which are consecutive

($\text{src}(t_{i+1}) = \text{tgt}(t_i)$). Transition sequences are always indexed from zero. We write $(\bar{t})_i$ or simply t_i for the i 'th transition in \bar{t} . The length of \bar{t} is referred to as $|\bar{t}|$. When nothing else is stated, a transition system with independence X is assumed to have components S, i, L, Tran , and I .

The category **TI** has transition systems with independence as objects. For the remaining part of this paper we fix a set L and restrict ourselves to those transition systems with independence that have labelling set L . As morphisms in the category **TI** $_L$ we choose the fiber-morphisms of [WN94]:

Definition 2.3 Let $X = (S, i, L, \text{Tran}, I)$ and $X' = (S', i', L', \text{Tran}', I')$ be transition systems with independence. A *morphism* f from X to X' is a function $f : S \rightarrow S'$ such that

- $f(i) = i'$
- for all transitions $s \xrightarrow{a} s_1$ in X , $f(s) \xrightarrow{a} f(s_1)$ in X'
- $s \xrightarrow{a} s_1 \ I \ u \xrightarrow{b} u_1$ in X implies $f(s) \xrightarrow{a} f(s_1) \ I' \ f(u) \xrightarrow{b} f(u_1)$ in X' \square

As observations it is naturally to take Pratt's pomsets [Pra86]. We identify the category **Pom** $_L$ of pomsets with its full and faithful embedding in **TI** $_L$ (for details see [JNW93]). The category **TI** $_L$ has pullbacks, so **Pom** $_L$ -bisimilarity is an equivalence relation in **TI** $_L$. The following proposition characterizes **Pom** $_L$ -open morphisms in **TI** $_L$ [JNW93].

Proposition 2.4 *A morphism $f : Y \rightarrow X$ in **TI** $_L$ is **Pom** $_L$ -open iff it is zig-zag and reflects consecutive independence, i.e. iff it has the following properties:*

- whenever r is reachable and $f(r) \xrightarrow{a} s_1$ there is a state r_1 in Y such that $r \xrightarrow{a} r_1$ & $f(r_1) = s_1$, and
- whenever r is reachable, $r \xrightarrow{a} r_1$ and $r_1 \xrightarrow{b} r_2$ are transitions in Y , and

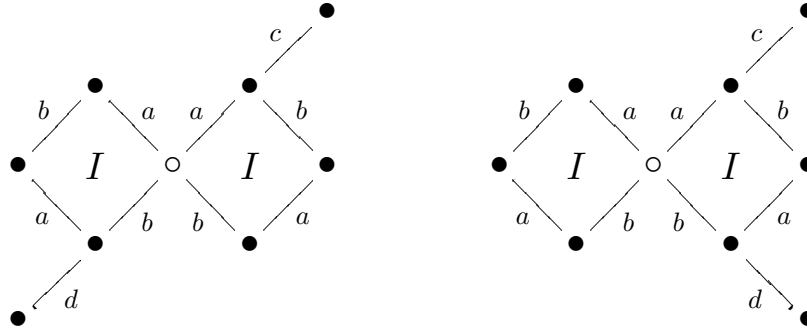
$$f(r) \xrightarrow{a} f(r_1) \ I \ f(r_1) \xrightarrow{b} f(r_2),$$

we also have $r \xrightarrow{a} r_1 \ I \ r_1 \xrightarrow{b} r_2$.

On event structures [JNW93] **Pom** $_L$ -bisimilarity turns out to be a slight strengthening of the history-preserving bisimilarity originally defined in [GG89, RT88]. In fact, the same strengthening has been studied in

[Bed91] in which the equivalence is denoted *hereditary* history-preserving bisimilarity. The strengthening is illustrated by the following event structures, here identified with their embeddings in \mathbf{TI}_L .

Example 2.5 Consider the following “event structures”:



The circles indicate the initial states. These “event structures” are history-preserving bisimilar but *not* hereditary history-preserving bisimilar. \square

One result on \mathbf{Pom}_L -bisimilarity for \mathbf{TI}_L mentioned in [JNW93] is the fact that two \mathbf{TI}_L -objects are \mathbf{Pom}_L -bisimilar iff their unfolded event structures are \mathbf{Pom}_L -bisimilar. The natural question is now: Does \mathbf{Pom}_L -bisimilarity for \mathbf{TI}_L have characterizations in the spirit of e.g. the relational, game-theoretical, and logical characterizations of bisimulation for standard transition systems? This question is answered positively in the next sections.

3 Game Characterizations

Following a new trend in the area of program semantics, we first present a game-theoretical characterization of \mathbf{Pom}_L -bisimilarity. The game defined can be viewed as a “backtracking” extension of Stirling’s game [Sti93]. We then show that the equivalence induced by the backtracking game can be characterized by the existence of a bisimulation relation over paths, satisfying a certain “backtracking property”.

3.1 Basic Definitions

The following definitions are inspired by [AJ92].

A *game* is a structure $\Gamma = (C, c_0, \triangleright, \lambda, W)$ where

- C is a set of *configurations* with a distinguished *initial configuration* c_0 ,
- $\triangleright \subseteq C^2$ is a set of *moves*. Formally, a *play* of Γ is a (possibly infinite) sequence of moves

$$c_0 \cdot c_1 \cdot c_2 \cdot \dots,$$

such that $c_0 \triangleright c_1 \triangleright c_2 \triangleright \dots$. The set $Pos(\Gamma)$ of *positions* consists of all finite plays. The meta-variable p ranges over positions.

- $\lambda : Pos(\Gamma) \rightarrow \{O, P\}$ is a function indicating whose *turn* it is to move in a given *position* (an element of $Pos(\Gamma)$, defined below) of a play, and
- $W \subseteq Pos(\Gamma)$ is a set of *winning positions*.

We require all plays to be *alternating*, i.e. we require \triangleright and λ together to satisfy that if $\lambda(p \cdot c) = Q$ and $c \triangleright c'$ then $\lambda(p \cdot c \cdot c') = \overline{Q}$, where $\overline{P} = O$ and $\overline{O} = P$. Furthermore, Opponent should start every play. This is expressed by demanding $\lambda(c_0) = O$.

In defining when a game is won we take Player's point of view: A *play* p is won (by P) if one of the following conditions hold:

- p is infinite,
- p is finite and $\lambda(p) = O$, or
- $p \in W$.

If p is not won, it is *lost*.

A *strategy* is a partial function $\sigma : Pos(\Gamma) \rightarrow C$ such that

$$\sigma(p \cdot c) = c' \text{ implies } c \triangleright c'.$$

We reserve the words *strategy* for Player and *counter-strategy* for Opponent and use σ and τ to range over strategies and counter-strategies, respectively. Player is said to *follow her strategy* σ in a play $c_0 \cdot c_1 \cdot \dots \cdot c_n \cdot c_{n+1} \cdot \dots$ iff $\lambda(c_0 \cdot c_1 \cdot \dots \cdot c_n) = P$ implies $c_{n+1} = \sigma(c_0 \cdot c_1 \cdot \dots \cdot c_n)$. Similarly, we can define when Opponent follows his strategy.

The intuition behind W is the following. As soon as Player can force Opponent to a position $p \in W$, she has won p and all extensions of p . We reflect this intuition by demanding that W is closed under \triangleright , i.e. that

$p \cdot c \in W$ whenever $p \in W$. To avoid a play $p \in W$ to continue forever, we require all strategies to be undefined on winning positions, i.e. $\sigma(p)$ is undefined whenever $p \in W$.

The set $Plays(\sigma, \tau)$ of plays in which both Player and Opponent follow their strategies is easily seen to be prefix-closed. This leads to the following definition of *the play of a strategy σ against a counter-strategy τ* :

$$\langle \sigma | \tau \rangle = \sqcup \{p \mid p \in Plays(\sigma, \tau)\},$$

where the least upper bound refers to the prefix-ordering. Finally, σ is said to be a *winning strategy* iff $\langle \sigma | \tau \rangle$ is won for any counter-strategy τ . Similarly, we define τ to be a *winning counter-strategy* iff $\langle \sigma | \tau \rangle$ is lost for any σ .

Any game of the above kind possesses the nice property that there can be no ties, and hence there is either a winning strategy or a winning counter-strategy.

Proposition 3.1 *For any game, there is a winning strategy iff there is no winning counter-strategy.*

Proof See Appendix A. □

3.2 A Characteristic Game for Interleaving Bisimulation

The first game considered is a “sequence variant” of the game defined by Stirling [Sti93]. Given two ordinary transition systems X and X' we take as configurations (ordered) pairs of *transition sequences* with the pair consisting of empty sequences as initial configuration. Informally, a play progresses as follows. Opponent starts out by first choosing either X or X' and then a transition from the initial state of the system chosen. If Player can't match the move with an equally labelled transition from the initial state of the other system, she loses. Otherwise, she chooses such a matching transition, and it's again Opponent's turn to move. He chooses a system, not necessarily the same as before, and a transition of that system leading out of the state arrived at in the previous pairs of moves. Again, Player is required to match with an equally labelled transition in the other system. The play continues like this *forever*, in which case

Player wins, or until either Player or Opponent is *stuck* (unable to move), in which case the other participant wins.

The above description is now formalized to fit the basic definitions of games. We define the *interleaving game* between transition systems X and X' to be $\Gamma(X, X') = (C, c_0, \triangleright, \lambda, W)$ where

- $C = Seqs(X) \times Seqs(X')$. As a convention, writing configurations (\bar{t}, \bar{t}') , $(\bar{t}\bar{t}, \bar{t}')$, and $(\bar{t}, \bar{t}'t')$ implicitly means that $|\bar{t}| = |\bar{t}'|$.
- $c_0 = (\varepsilon, \varepsilon)$.
- $\lambda : Pos(\Gamma(X, X')) \rightarrow \{O, P\}$ is defined by taking

$$\lambda(\bar{t}, \bar{t}') = O \quad \text{and} \quad \lambda(\bar{t}\bar{t}, \bar{t}') = \lambda(\bar{t}, \bar{t}'t') = P.$$

- $W = \emptyset$.
- $\triangleright \subseteq C^2$ is defined by the rules

$$\begin{array}{lll} (\bar{t}, \bar{t}') \triangleright (\bar{t}\bar{t}, \bar{t}') & \text{if } \bar{t}\bar{t} \in Seqs(X) \\ (\bar{t}, \bar{t}') \triangleright (\bar{t}, \bar{t}'t') & \text{if } \bar{t}'t' \in Seqs(X') \\ (\bar{t}\bar{t}, \bar{t}') \triangleright (\bar{t}\bar{t}, \bar{t}'t') & \text{if } \bar{t}'t' \in Seqs(X') \ \& \ \ell(t') = \ell(t) \\ (\bar{t}, \bar{t}'t') \triangleright (\bar{t}\bar{t}, \bar{t}'t') & \text{if } \bar{t}\bar{t} \in Seqs(X) \ \& \ \ell(t) = \ell(t') \end{array}$$

Just like Stirling's game, the game $\Gamma(X, X')$ is *characteristic* for interleaving bisimulation in the following sense.

Theorem 3.2 *Two transition systems X and X' are bisimilar iff Player has a winning strategy in $\Gamma(X, X')$.*

Proof Small modifications of the reasoning of Stirling [Sti93]. □

The game presented above has the property that if X and X' exhibit infinite behaviour, then there exist infinite plays, even if both systems are finite state. For some purposes this property is undesirable, and can indeed be eliminated by choosing the set W of winning positions appropriately. To be concrete, we might in $\Gamma(X, X')$ have chosen W to consist of those positions p which are duplicate-free in the sense that no two configurations (\bar{t}, \bar{t}') , (\bar{r}, \bar{r}') in p have $tgt(\bar{t}) = tgt(\bar{r})$ and $tgt(\bar{t}') = tgt(\bar{r}')$. By the pigeonhole principle this modification of $\Gamma(X, X')$ would bound the length of any play by $2 \cdot |S| \cdot |S'| + 1$ where $|S|$ and $|S'|$ are the number of states in X and X' , respectively. Furthermore, it is quite easy to see that the characterization result also holds for the modified game.

3.3 Allowing Opponent to Backtrack

Throughout this paper we shall take “backtrack” to mean *trace backwards* within *the present observation*.

With this interpretation, backtracking in an ordinary transition system means to trace backwards *along* the transition sequence observed. In terms of games, we can express this by allowing Opponent to do backwards moves like

$$(\bar{t}t, \bar{t}'t') \triangleright (B, \bar{t}, \bar{t}'),$$

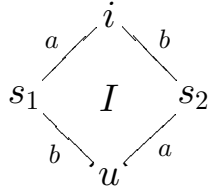
where the B is a directive to Player to play backwards on the longer of the sequences. Player must match with the move

$$(B, \bar{t}, \bar{t}') \triangleright (\bar{t}, \bar{t}').$$

It is easy to see that these additional rules do *not* give Opponent more opportunities to beat Player, nor the other way around.

Proposition 3.3 *Two transition systems are bisimilar iff Player has a winning strategy in their associated game with backtracking.*

Backtracking in an independence model is much more interesting. Consider a simple transition system with independence X



consisting of a single independence square. Since $i \xrightarrow{a} s_1$ and $s_1 \xrightarrow{b} u$ are independent, the sequence $\bar{t} = i \xrightarrow{a} s_1 \xrightarrow{b} u$ in X represents the observation “ a and b in parallel”. Another representative of this observation is the sequence $i \xrightarrow{b} s_2 \xrightarrow{a} u$, so this gives us two ways to backtrack within \bar{t} : Either along $s_1 \xrightarrow{b} u$, leaving behind the sequence $i \xrightarrow{a} s_1$, *or* along $s_2 \xrightarrow{a} u$ leaving behind the sequence $i \xrightarrow{b} s_2$.

In terms of the net representation of $a \parallel b$ from Section 1 this amounts to the following: After firing the a -transition followed by the b -transition you may naturally backtrack on the a -transition, since the firing of the b -transition has in no way affected the post-conditions of the a -transition.

In the above example the event represented by $i \xrightarrow{a} s_1$ has an occurrence – namely $s_2 \xrightarrow{a} u$ – ending in u . We say that $i \xrightarrow{a} s_1$ is *backwards enabled* in

the sequence $i \xrightarrow{a} s_1 \xrightarrow{b} u$. In general, a transition t_i of a sequence \bar{t} is said to be backwards enabled iff it by repeated use of Axiom 2 of Definition 2.2 can be “pushed to last position in \bar{t} .” By Axiom 3, this is equivalent to requiring t_i to be independent of all transitions t_j in \bar{t} with $j > i$. This leads to the following formal definition.

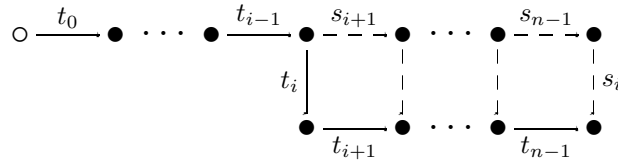
Definition 3.4 For $\bar{t} = t_0 \cdots t_{n-1}$, a sequence in a transition system with independence X , and $i \in \{0, \dots, n-1\}$, we define

$$t_i \in BEn(\bar{t}) \text{ iff } \forall j \in \{i+1, \dots, n-1\}. t_j \ I \ t_i,$$

where I is the independence relation in X . If $t_i \in BEn(\bar{t})$ we define $\delta(i, \bar{t})$ to be the result of deleting the event corresponding to t_i , i.e.

$$\delta(i, \bar{t}) = t_0 \cdots t_{i-1} s_{i+1} \cdots s_{n-1},$$

where $s_{i+1} \prec t_{i+1}, \dots, s_{n-1} \prec t_{n-1}$ as in the following figure in which the squares are the unique completions defined in Section 2.



□

The backtracking game on transition systems with independence is a simple extension of the previously defined (forward) game. By introducing rules like

$$(\bar{t}, \bar{t}') \triangleright (i, \delta(i, \bar{t}), \bar{t}') \text{ if } t_i \in BEn(\bar{t})$$

we allow Opponent to backtrack on transitions which are backwards enabled. The index i is a request to Player to play backwards on the i 'th transition of the longer of the sequences. So the only way Player can respond to such moves is to use the rule

$$(i, \delta(i, \bar{t}), \bar{t}') \triangleright (\delta(i, \bar{t}), \delta(i, \bar{t}')) \text{ if } t'_i \in BEn(\bar{t}'),$$

provided, of course, that t'_i is backwards enabled in \bar{t}' . Formally, we define the backtracking game $\Gamma(X, X')$ on transition systems with independence X and X' to be the structure $(C, c_0, \triangleright, \lambda, W)$:

- $C = \omega \times Seqs(X) \times Seqs(X') \cup Seqs(X) \times Seqs(X')$. Conventionally, writing configurations $(i, \delta(i, \bar{t}), \bar{t}')$ and $(i, \bar{t}, \delta(i, \bar{t}'))$ implicitly means that $|\bar{t}| = |\bar{t}'|$.
- $c_0 = (\varepsilon, \varepsilon)$.
- $\lambda : Pos(\Gamma(X, X')) \rightarrow \{O, P\}$ is defined by taking $\lambda(\bar{t}, \bar{t}') = O$ and $\lambda(\bar{t}\bar{t}, \bar{t}') = \lambda(\bar{t}, \bar{t}'\bar{t}') = \lambda(i, \delta(i, \bar{t}), \bar{t}') = \lambda(i, \bar{t}, \delta(i, \bar{t}')) = P$.
- $W = \emptyset$.
- $\triangleright \subseteq C^2$ is defined by the following rules:

$$\begin{array}{lll}
(\bar{t}, \bar{t}') \triangleright (\bar{t}\bar{t}, \bar{t}') & \text{if } \bar{t}\bar{t} \in Seqs(X) \\
(\bar{t}, \bar{t}') \triangleright (\bar{t}, \bar{t}'\bar{t}') & \text{if } \bar{t}'\bar{t}' \in Seqs(X') \\
(\bar{t}\bar{t}, \bar{t}') \triangleright (\bar{t}\bar{t}, \bar{t}'\bar{t}') & \text{if } \bar{t}'\bar{t}' \in Seqs(X') \ \& \ \ell(\bar{t}') = \ell(\bar{t}) \\
(\bar{t}, \bar{t}'\bar{t}') \triangleright (\bar{t}\bar{t}, \bar{t}'\bar{t}') & \text{if } \bar{t}\bar{t} \in Seqs(X) \ \& \ \ell(\bar{t}) = \ell(\bar{t}') \\
\\
(\bar{t}, \bar{t}') \triangleright (i, \delta(i, \bar{t}), \bar{t}') & \text{if } t_i \in BEn(\bar{t}) \\
(\bar{t}, \bar{t}') \triangleright (i, \bar{t}, \delta(i, \bar{t}')) & \text{if } t'_i \in BEn(\bar{t}') \\
(i, \delta(i, \bar{t}), \bar{t}') \triangleright (\delta(i, \bar{t}), \delta(i, \bar{t}')) & \text{if } t'_i \in BEn(\bar{t}') \\
(i, \bar{t}, \delta(i, \bar{t}')) \triangleright (\delta(i, \bar{t}), \delta(i, \bar{t}')) & \text{if } t_i \in BEn(\bar{t})
\end{array}$$

Definition 3.5 Two transition systems with independence X and X' are Γ -equivalent, written $X \sim_{\Gamma} X'$, iff Player has a winning strategy in $\Gamma(X, X')$. \square

With a simple example we will now illustrate how backtracking distinguishes parallelism from non-deterministic interleaving.

Example 3.6 Consider the transition systems with independence representing the CCS-processes $a \parallel b$ and $a.b + b.a$:



These systems are interleaving bisimilar but *not* Γ -equivalent, as we are able to define a winning counter-strategy τ as follows (here, p ranges over

all appropriate positions):

$$\begin{aligned}\tau(\varepsilon, \varepsilon) &= (i \xrightarrow{a} s_1, \varepsilon) \\ \tau(p \cdot (i \xrightarrow{a} s_1, i' \xrightarrow{a} s'_1)) &= (i \xrightarrow{a} s_1 \xrightarrow{b} u, i' \xrightarrow{a} s'_1) \\ \tau(p \cdot (i \xrightarrow{a} s_1 \xrightarrow{b} u, i' \xrightarrow{a} s'_1 \xrightarrow{b} u')) &= (0, i \xrightarrow{b} s_2, i' \xrightarrow{a} s'_1 \xrightarrow{b} u')\end{aligned}$$

The point is, of course, that Player is unable to backtrack on index 0 in the sequence $i' \xrightarrow{a} s'_1 \xrightarrow{b} u'$, as these transitions are dependent. \square

Distinguishing the transition systems with independence of Example 3.6 is, in fact, a minimum demand on any reasonable generalization of bisimulation to independence models. And following the reasoning of Example 3.6, the reader should not be surprised that backtracking may be used by Opponent to detect the partial order structures of configurations. However, it is more surprising that Γ -equivalence coincides exactly with the abstractly derived \mathbf{Pom}_L -bisimilarity of Section 2. As a more interesting example, the reader may check that Opponent has a winning counterstrategy in the game associated with the systems of Example 2.5.

Theorem 3.7 *Γ -equivalence coincides with \mathbf{Pom}_L -bisimilarity.*

Proof See Appendix B. \square

We also have a relational characterization of \mathbf{Pom}_L -bisimilarity.

Definition 3.8 A relation $\mathcal{T} \in \text{Seqs}(X) \times \text{Seqs}(X')$ is a δ -bisimulation between X and X' iff it satisfies the following axioms:

$$\mathbf{A}_{\text{init}}: \varepsilon \mathcal{T} \varepsilon$$

$$\mathbf{A}_{\text{bisim}}:$$

1. $\bar{t} \mathcal{T} \bar{t}' \ \& \ \bar{t}t \in \text{Seqs}(X) \Rightarrow \exists t'. (\ell'(t') = \ell(t) \ \& \ \bar{t}t \mathcal{T} \bar{t}'t')$
2. $\bar{t} \mathcal{T} \bar{t}' \ \& \ \bar{t}'t' \in \text{Seqs}(X') \Rightarrow \exists t. (\ell(t) = \ell'(t') \ \& \ \bar{t}t \mathcal{T} \bar{t}'t')$

$$\mathbf{A}_{\delta}: \bar{t} \mathcal{T} \bar{t}' \Rightarrow \left\{ \begin{array}{l} (t_i \in \text{BEn}(\bar{t}) \Leftrightarrow t'_i \in \text{BEn}(\bar{t}')) \ \& \\ (t_i \in \text{BEn}(\bar{t}) \Rightarrow \delta(i, \bar{t}) \mathcal{T} \delta(i, \bar{t}')) \end{array} \right.$$

Two transition systems with independence X and X' are δ -bisimilar, written $X \sim_{\delta} X'$, iff there exists a δ -bisimulation between them. \square

The axiom $\mathbf{A}_{\text{bisim}}$ is the usual “bisimulation axiom”, here formulated on sequences rather than states. So this formulation is also a simple extension of a well-known concept.

Theorem 3.9 *δ -equivalence coincides with \mathbf{Pom}_L -bisimilarity.*

Proof See Appendix B. □

4 A Path Logic

Just as interleaving bisimulations can be characterized as a relation over paths, we can interpret the Hennessy-Milner logic over paths rather than states. Following [HS85], we may add a past tense modality \textcircled{a} , where a is a label, and obtain a logic which still characterizes bisimulation for ordinary transition systems. However, interpreted over transition systems with independence, we obtain a logic which is easily shown to be sound for \mathbf{Pom}_L -bisimilarity. Furthermore, the logic is *complete* if we restrict to systems which do not exhibit *auto-concurrency*, i.e. systems in which no two consecutive and equally labelled transitions are independent.

Let \mathbf{Assn} be the following language of assertions:

$$A ::= \neg A \mid \bigwedge_{j \in J} A_j \mid \langle a \rangle A \mid \textcircled{a}A.$$

By convention, *true* is the conjunction over the empty set.

Definition 4.1 Let X be a transition system with independence and suppose $\bar{t}, \bar{r} \in \text{Seqs}(X)$. Define

$$\begin{aligned} \bar{r} \xrightarrow{a} \bar{t} & \quad \text{iff} \quad \bar{r}(s \xrightarrow{a} s_1) = \bar{t} \\ \bar{r} \rightsquigarrow^a \bar{t} & \quad \text{iff} \quad \exists i. (t_i \in \text{BEn}(\bar{t}) \ \& \ \ell(t_i) = a \ \& \ \bar{r} = \delta(i, \bar{t})) \end{aligned}$$

□

In ordinary transition systems, $\textcircled{a}A$ is interpreted as “it was the case at the last moment – just before a – that A ”. It seems natural for transition systems with independence to interpret $\textcircled{a}A$ as “ a could have been the last action, and at the moment before a it was the case that A ”.

Formally, let X be a transition system with independence and define the *satisfaction* relation $\models_X \subseteq \text{Seqs}(X) \times \mathbf{Assn}$ by structural induction on assertions:

$$\begin{aligned}
\bar{t} \models_X \neg A & \text{ iff } \bar{t} \not\models_X A \\
\bar{t} \models_X \bigwedge_{j \in J} A_j & \text{ iff } \forall j \in J. \bar{t} \models_X A_j \\
\bar{t} \models_X \langle a \rangle A & \text{ iff } \exists \bar{r}. (\bar{t} \xrightarrow{a} \bar{r} \ \& \ \bar{r} \models_X A) \\
\bar{t} \models_X @A & \text{ iff } \exists \bar{r}. (\bar{r} \xrightarrow{a} \bar{t} \ \& \ \bar{r} \models_X A)
\end{aligned}$$

An assertion is *satisfied by* X , written $\models_X A$, iff $\varepsilon \models_X A$.

Definition 4.2 Two transition systems with independence X and X' are **Assn**-equivalent iff they satisfy the same assertions, i.e. iff

$$\forall A \in \mathbf{Assn}. (\models_X A \Leftrightarrow \models_{X'} A).$$

□

For ordinary transition systems (without independence) this logic is characteristic for bisimulation.

Theorem 4.3 *Two transition systems are bisimilar iff they are **Assn**-equivalent.*

Proof See [HS85].

□

Example 4.4 To see the logic in action on transition systems with independence, let us return to Example 2.5. An assertion distinguishing the two systems is $\langle a \rangle (\langle c \rangle \text{true} \wedge \langle b \rangle @ \langle d \rangle)$. This assertion is satisfied by the right-hand system, but not by the left-hand system. □

As usual, the soundness proof is straightforward.

Proposition 4.5 *If two transition systems with independence are **Pom_L**-bisimilar, then they are also **Assn**-equivalent.*

Proof See Appendix C.

□

Restriction to systems without auto-concurrency is essential for completeness.

Example 4.6 Consider two systems X and X'



which are identical except that the square in X is an independence square, whereas the square in X' is not. These systems satisfy the same assertions, but are certainly *not* \mathbf{Pom}_L -bisimilar. \square

Proposition 4.7 *If two non-auto-concurrent transition systems with independence are \mathbf{Assn} -equivalent, then they are also \mathbf{Pom}_L -bisimilar.*

Proof See Appendix C. \square

As mentioned in the introduction, the restriction to systems without auto-concurrency has to do with the logic being based on labels. Replacing the backwards modalities $@$, where a is label, with modalities \textcircled{i} , where i is an *index*, and defining

$$\bar{t} \models_X \textcircled{i}A \text{ iff } t_i \in BEn(\bar{t}) \ \& \ \delta(i, \bar{t}) \models_X A,$$

we obtain a logic which is complete for \mathbf{Pom}_L -bisimilarity without restrictions.

5 Conclusion

We have given concrete characterizations of \mathbf{Pom}_L -bisimilarity on transition systems with independence. Our characterizations are easy to understand and appear as conservative extensions of the corresponding characterizations of interleaving bisimulation.

The present work leaves open the decidability of \mathbf{Pom}_L -bisimilarity for finite state systems. One approach would be to look for set W of winning positions, generalizing the notion of duplicates in our argument for decidability of bisimulation for ordinary transition systems. However, it is not quite clear what the appropriate generalization should be in the setting of transition systems with independence.

References

- [AJ92] S. Abramsky and R. Jagadeesan. Games and Full Completeness for Multiplicative Linear Logic. DoC 92/24, Imperial College of Science, Technology and Medicine, 1992.
- [Bed88] M. A. Bednarczyk. *Categories of asynchronous systems*. PhD thesis, University of Sussex, 1988. Technical report no. 1/88.
- [Bed91] M. A. Bednarczyk. Heredity History Preserving Bisimulations. Unpublished, Draft of 1991.
- [Ben84] J. Van Benthem. Correspondence theory. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, volume 2. Reidel, 1984.
- [GG89] R. van Glaabek and U. Goltz. Equivalence Notions for Concurrent Systems and Refinement of Actions. In *MFCS '89*. Springer-Verlag *LNCS* 379, 1989.
- [GKP92] U. Goltz, R. Kuiper, and W. Penczek. Propositional Temporal Logics and Equivalences. In *Concur '92*. Springer-Verlag *LNCS* 630, 1992.
- [HM85] M. C. Hennessy and A. J. R. G. Milner. Algebraic Laws for Non-determinism and Concurrency. *Journal of ACM*, 32(1), 1985.
- [HS85] M. Hennessy and C. P. Stirling. The power of the future perfect in program logics. In A. R. Meyer, editor, *Information and Control*, volume 67. Academic Press, Inc. , 1985.
- [JNW93] A. Joyal, M. Nielsen, and G. Winskel. Bisimulation and open maps. In *LICS '93*, 1993. To appear in *Information and Computation*.
- [Mil89] A. J. R. G. Milner. *Communication and Concurrency*. Prentice Hall, 1989.
- [Old91] E.-R. Olderog. *Nets, Terms and Formulas*. Cambridge University Press, 1991.

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- [Par81] D. M. R. Park. Concurrency and Automata on Infinite Sequences. In *Theoretical Computer Science, 5th GL-conference*. Springer-Verlag LNCS 104, 1981.
- [Pra86] V. R. Pratt. Modelling concurrency with partial orders. *International Journal of Parallel Programming*, 15(1), 1986.
- [RT88] A. Rabinovich and B. Traktenbrot. Behaviour structures and nets. *Fundamenta Informatica*, 11(4), 1988.
- [Shi85] M. W. Shields. Concurrent machines. *Computer Journal*, 88, 1985.
- [Sti93] C. Stirling. Modal and Temporal Logics for Processes, 1993. Notes for Summer School in Logic Methods in Concurrency, Department of Computer Science, Aarhus University.
- [WN94] G. Winskel and M. Nielsen. Models for Concurrency. In S. Abramsky and D. Gabbay, editors, *Handbook of Logic in Computer Science*, volume 3. Oxford University Press, 1994.

A Every Game Has a Winner

Proposition A.1 (Proposition 3.1 restated) *For any game, there is a winning strategy iff there is no winning counter-strategy.*

Proof Let Γ be a game. Obviously, if there is a winning strategy in Γ , there can be no winning counter-strategy.

For the converse direction, assume there is no winning counter-strategy. Say τ a *winning counter-strategy from (wcsf)* $p \in Pos(\Gamma)$ iff

$$\lambda(p) = O \ \& \ \forall \sigma. (p \in Plays(\sigma, \tau) \Rightarrow \langle \sigma | \tau \rangle \text{ is lost}).$$

Suppose now that $\lambda(p) = O$. It can then be shown that if Opponent has no winning counter-strategy from p , then

$$\forall p \cdot c \in Pos(\Gamma). \left(\begin{array}{l} p \cdot c \in W, \text{ or} \\ \exists p \cdot c \cdot c' \in Pos(\Gamma). (O \text{ has no wcsf } p \cdot c \cdot c'). \end{array} \right)$$

Using this property, we define a strategy σ as follows. Whenever Opponent has no winning counter-strategy from $p \in Pos(\Gamma)$, and $p \cdot c \in Pos(\Gamma) \setminus W$, we choose a configuration c' such that $p \cdot c \cdot c' \in Pos(\Gamma)$ and Opponent has no winning counter-strategy from $p \cdot c \cdot c'$. Define $\sigma(p \cdot c) = c'$.

By definition, σ is a strategy. In order to argue that σ is winning, let τ be an arbitrary counter-strategy. Inductively in the length of p , it is easy to show the following property:

$$\forall p \in Plays(\sigma, \tau). \left(\begin{array}{l} \lambda(p) = O \ \& \ \text{Opponent has no wcsf } p, \text{ or} \\ \lambda(p) = P \ \& \ (\sigma \text{ is defined on } p \text{ or } p \in W) \end{array} \right).$$

(Notice that our assumption that Opponent has no winning counter-strategy amounts to the fact that Opponent has no winning counter-strategy from the initial configuration c_0 .) But then, since σ is defined whenever it is Player's turn to move, the play $\langle \sigma | \tau \rangle$ must be won. This finishes the proof of the proposition. \square

B Proofs From Section 3

As an intermediate step we introduce yet another equivalence, denoted **iso**- δ -bisimilarity. The main theorems 3.7 and 3.9 will be established by completing the following picture in which \hookrightarrow denotes set inclusion:

$$\begin{array}{ccc}
 \sim_{\delta} & \xrightarrow{B.10} & \sim_{\mathbf{Pom}_L} \\
 \left. \begin{array}{c} \uparrow \\ B.1 \\ \downarrow \end{array} \right\} & & \left. \begin{array}{c} \downarrow \\ B.13 \\ \uparrow \end{array} \right\} \\
 \sim_{\Gamma} & \xrightarrow{B.15} & \sim_{\mathbf{iso}\text{-}\delta}
 \end{array}$$

The inclusions are labeled with references to corresponding propositions.

B.1 From δ -bisimilarity to Γ -equivalence

Proposition B.1 $\sim_{\delta} \subseteq \sim_{\Gamma}$.

Proof Let $\mathcal{T} \subseteq \text{Seqs}(X) \times \text{Seqs}(X')$ be a δ -bisimulation between X and X' , and define a partial function $\sigma : \text{Pos}(\Gamma(X, X')) \rightarrow C$ in the following way:

- Whenever $\bar{t} \mathcal{T} \bar{t}'$ and $\bar{t}\bar{t} \in \text{Seqs}(X)$, choose t' such that $\bar{t}'t' \in \text{Seqs}(X')$ and $\bar{t}\bar{t} \mathcal{T} \bar{t}'t'$. Then, for all $p \cdot (\bar{t}\bar{t}, \bar{t}')$ $\in \text{Pos}(\Gamma(X, X'))$, define

$$\sigma(p \cdot (\bar{t}\bar{t}, \bar{t}')) = (\bar{t}\bar{t}, \bar{t}'t').$$

In a similar way, σ is defined to respond to moves on the right-hand side.

- Whenever $\bar{t} \mathcal{T} \bar{t}'$ and $t_i \in \text{BEn}(\bar{t})$, we know, since \mathcal{T} satisfies \mathbf{A}_{δ} , that $t'_i \in \text{BEn}(\bar{t}')$ and $\delta(i, \bar{t}) \mathcal{T} \delta(i, \bar{t}')$. For all $p \cdot (i, \delta(i, \bar{t}), \bar{t}')$ $\in \text{Pos}(\Gamma(X, X'))$ we define

$$\sigma(p \cdot (i, \delta(i, \bar{t}), \bar{t}')) = (\delta(i, \bar{t}), \delta(i, \bar{t}')).$$

Here, too, there is a symmetric definition.

By construction, σ is a strategy. We now argue that σ is winning: For all counter-strategies τ it follows by induction in the length of p that

$$p \cdot (\bar{t}, \bar{t}') \in \text{Plays}(\sigma, \tau) \ \& \ \lambda(p \cdot (\bar{t}, \bar{t}')) = O \Rightarrow \bar{t} \mathcal{T} \bar{t}'.$$

Hence, from construction it follows that $\langle \sigma | \tau \rangle$ is won. □

B.2 Permutations as Pomsets

An **iso**- δ -bisimulation is a δ -bisimulation which furthermore respect isomorphisms between observations, i.e. between pomsets. Actually, we present **iso**- δ -bisimulations without mentioning pomsets at all. Instead, we use *permutations* of transition sequences. As we shall soon see, these permit us to reason inductively.

If two consecutive transitions t_i and t_{i+1} of a transition sequence \bar{t} are independent, they can – due to Axiom 2 of Definition 2.2 – be swapped, resulting in the (unique) sequence

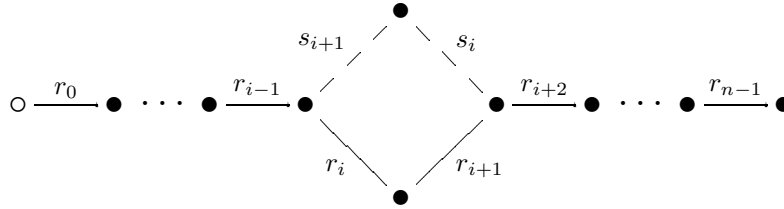
$$t_0 \cdots t_{i-1} s_{i+1} s_i t_{i+2} \cdots t_{n-1},$$

where $s_{i+1} s_i$ is the unique completion of $t_i t_{i+1}$. A permutation is a series of such “swappings”.

Formally, we define a permutation (in a given transition system with independence X) to be a partial function $\bullet : \omega^* \times \text{Seqs}(X) \rightarrow \text{Seqs}(X)$, where ω^* is the set of natural numbers (here, including 0). We shall use infix notation for \bullet , and for $\pi \in \omega^*$ and $\bar{t} \in \text{Seqs}(X)$ we define $\pi \bullet \bar{t}$ recursively in the length $|\pi|$ of π :

$\varepsilon \bullet \bar{t}$ is always defined and equals \bar{t} .

$i\pi \bullet \bar{t}$ is defined iff $\pi \bullet \bar{t}$ is defined, $i \in \{0, \dots, |\bar{t}| - 2\}$, and $r_i \perp r_{i+1}$, where $\pi \bullet \bar{t} = \bar{r}$. If defined, $i\pi \bullet \bar{t}$ equals $r_0 \cdots r_{i-1} s_{i+1} s_i r_{i+2} \cdots r_{n-1}$, where $s_{i+1} s_i$ is the unique completion of $r_i r_{i+1}$:



It is straightforward to show that \bullet possesses the “group action property” $\pi\pi' \bullet \bar{t} = \pi \bullet (\pi' \bullet \bar{t})$

Definition B.2 A sequence $\pi \in \omega^*$ is a *permutation* of a sequence $\bar{t} \in \text{Seqs}(X)$ provided $\pi \bullet \bar{t}$ is defined. Write $\Pi_X(\bar{t})$ for the set of permutations of \bar{t} .

Given a permutation π of a sequence \bar{t} , we want $\langle \pi, \bar{t} \rangle(i) = j$ to express that the i 'th transition of \bar{t} by π is swapped to position j . This intuition is captured in the following definition.

Definition B.3 Suppose π is a permutation of \bar{t} . Then, we define $\langle \pi, \bar{t} \rangle : \{0, \dots, |\bar{t}| - 1\} \rightarrow \{0, \dots, |\bar{t}| - 1\}$ inductively in $|\pi|$.

$\langle \varepsilon, \bar{t} \rangle$ is the identity function.

$$\langle k\pi, \bar{t} \rangle(i) = \begin{cases} k + 1 & \text{if } k = \langle \pi, \bar{t} \rangle(i) \\ k & \text{if } k + 1 = \langle \pi, \bar{t} \rangle(i) \\ \langle \pi, \bar{t} \rangle(i) & \text{otherwise} \end{cases}$$

It is easy to see that $\langle \pi, \bar{t} \rangle$ is a bijection. The inverse function is $\langle \pi^{-1}, \pi \bullet \bar{t} \rangle$, where π^{-1} is π reversed.

The following lemmata express two very natural properties of permutations.

Lemma B.4 Let $\bar{t} \in \text{Seqs}(X)$ and assume $\langle \pi, \bar{t} \rangle(i) = j$ for some permutation π and indices i and j . Then $t_i \in \text{BEn}(\bar{t})$ iff $(\pi \bullet \bar{t})_j \in \text{BEn}(\pi \bullet \bar{t})$.

Proof By induction on $|\pi|$ it can be shown that

$$\langle \pi, \bar{t} \rangle(i) = j \ \& \ t_i \in \text{BEn}(\bar{t}) \ \text{implies} \ (\pi \bullet \bar{t})_j \in \text{BEn}(\pi \bullet \bar{t}). \quad (1)$$

For the converse direction, we simply apply (1) to $\langle \pi^{-1}, \pi \bullet \bar{t} \rangle(j) = i$ & $(\pi \bullet \bar{t})_j \in \text{BEn}(\pi \bullet \bar{t})$. \square

When two sequences \bar{t} and \bar{t}' have the same length, are equally labelled, and have the same permutations, they obviously represent isomorphic pomsets. This leads to the following definition.

Definition B.5 For $\bar{t} \in \text{Seqs}(X)$ and $\bar{t}' \in \text{Seqs}(X')$ we define

$$\bar{t} \text{ iso } \bar{t}' \ \text{iff} \ |\bar{t}| = |\bar{t}'| \ \& \ \forall i. (\ell(t_i) = \ell'(t'_i)) \ \& \ \Pi_X(\bar{t}) = \Pi_{X'}(\bar{t}').$$

Definition B.6 Let $\bar{t}, \bar{r} \in \text{Seqs}(X)$. Define $\bar{t} \simeq \bar{r}$ iff there exists a permutation π such that $\pi \bullet \bar{t} = \bar{r}$. Furthermore, if $\bar{t}', \bar{r}' \in \text{Seqs}(X')$ and $\pi \bullet \bar{t}' = \bar{r}'$, we write $(\bar{t}, \bar{t}') \simeq (\bar{r}, \bar{r}')$.

Lemma B.7 Suppose $\pi \bullet \bar{t} = \bar{r}$ and $\pi \bullet \bar{t}' = \bar{r}'$, and assume $\langle \pi, \bar{t} \rangle(i) = j$ and, similarly, $\langle \pi, \bar{t}' \rangle(i) = j$. Then

$$(\delta(i, \bar{t}), \delta(i, \bar{t}')) \simeq (\delta(j, \bar{r}), \delta(j, \bar{r}')),$$

provided all deletions are well-defined.

Proof Induction in $|\pi|$. □

Corollary B.8 *With the assumptions of the previous lemma,*

$$\delta(i, \bar{t}) \text{ iso } \delta(i, \bar{t}') \text{ implies } \delta(j, \bar{r}) \text{ iso } \delta(j, \bar{r}').$$

A \mathbf{Pom}_L -open morphism $f : X \rightarrow X'$ extends to a function $\text{Seqs}(X) \rightarrow \text{Seqs}(X')$ in the obvious way. By abuse of notation, this function will also be denoted by f . We can now state the fact that \mathbf{Pom}_L -open maps respect permutations.

Lemma B.9 *If $f : X \rightarrow X'$ is \mathbf{Pom}_L -open, $\bar{t} \in \text{Seqs}(X)$, and $\pi \in \omega^*$, then*

$$\pi \bullet \bar{t} \text{ is defined iff } \pi \bullet f(\bar{t}) \text{ is defined,}$$

and when both are defined,

$$f(\pi \bullet \bar{t}) = \pi \bullet f(\bar{t}).$$

Proof The two statements are proved simultaneously by induction on $|\pi|$ using the group action property of \bullet . The kernel of the argument is that open maps respect completion of independence squares. □

B.3 From \mathbf{Pom}_L -bisimilarity to δ -bisimilarity

Before turning to the proof that $\sim_{\mathbf{Pom}_L} \subseteq \sim_\delta$, we notice that backwards enablings can be expressed in terms of permutations. Using Axiom 3 of Definition 2.2 it is easy to see that $t_i \in \text{BEn}(\bar{t})$ iff $[n-2, \dots, i+1, i]$ is a permutation of \bar{t} . Furthermore, $\delta(i, \bar{t})$ is the one-step prefix of $[n-2, \dots, i+1, i] \bullet \bar{t}$.

Proposition B.10 $\sim_{\mathbf{Pom}_L} \subseteq \sim_\delta$.

Proof Suppose X and X' are \mathbf{Pom}_L -bisimilar. Then there is a transition system with independence Y and a span of \mathbf{Pom}_L -open maps:

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow f' \\ X & & X' \end{array}$$

Using the extensions of f and f' to sequences, we now define a relation $\mathcal{T} \subseteq \text{Seqs}(X) \times \text{Seqs}(X')$ by

$$\mathcal{T} = \{(f(\bar{r}), f'(\bar{r})) \mid \bar{r} \in \text{Seqs}(Y)\}.$$

The claim is now that \mathcal{T} is a δ -bisimulation. There are three axioms to check.

A_{init}: Trivial.

A_{bisim}: Suppose $\bar{t} \mathcal{T} \bar{t}'$ and $\bar{t}\bar{t} \in \text{Seqs}(X)$. Then there exists $\bar{r} \in \text{Seqs}(Y)$ such that $\bar{t} = f(\bar{r})$ and $\bar{t}' = f'(\bar{r})$. Since f is zig-zag and $\text{tgt}(\bar{t}) = \text{tgt}(f(\bar{r})) = f(\text{tgt}(\bar{r}))$ there is a transition r such that $\bar{r}r \in \text{Seqs}(Y)$ and $f(r) = \bar{t}$. Define $t' = f'(r)$. Then $f(\bar{r}r) = \bar{t}\bar{t}$ and $f'(\bar{r}r) = \bar{t}'t'$ so we conclude that $\bar{t}\bar{t} \mathcal{T} \bar{t}'t'$. Obviously, $\ell'(t') = \ell(\bar{t}\bar{t})$. The proof of the second part of the axiom is similar and therefore omitted.

A _{δ} : Let $\bar{t} \mathcal{T} \bar{t}'$ and assume $t_i \in \text{BEn}(\bar{t})$. Choose \bar{r} such that $\bar{t} = f(\bar{r})$ and $\bar{t}' = f'(\bar{r})$. Using the above characterization of $t_i \in \text{BEn}(\bar{t})$ we see that $\pi = [n-2, \dots, i+1, i]$ is a permutation of \bar{t} . Lemma B.9 now states that

$$\begin{aligned} f(\pi \bullet \bar{r}) &= \pi \bullet f(\bar{r}) = \pi \bullet \bar{t} \quad \text{and} \\ f'(\pi \bullet \bar{r}) &= \pi \bullet f'(\bar{r}) = \pi \bullet \bar{t}'. \end{aligned}$$

Especially, $\pi \bullet \bar{t}'$ is defined, ensuring $t'_i \in \text{BEn}(\bar{t}')$. Moreover, we see that $\pi \bullet \bar{t} \mathcal{T} \pi \bullet \bar{t}'$ and since \mathcal{T} is obviously closed under simultaneous prefix, we conclude that $\delta(i, \bar{t}) \mathcal{T} \delta(i, \bar{t}')$, as required. By symmetry, this completes the proof of **A _{δ}** . □

B.4 From iso- δ -bisimilarity to Pom_L -bisimilarity

As earlier mentioned, an **iso- δ -bisimulation** is just a δ -bisimulation which also respects isomorphism:

Definition B.11 A relation $\mathcal{T} \in \text{Seqs}(X) \times \text{Seqs}(X')$ is an **iso- δ -simulation** between X and X' iff it is a δ -bisimulation which satisfies the following axiom:

$$A_{\text{iso}} \quad \bar{t} \mathcal{T} \bar{t}' \Rightarrow \bar{t} \text{ iso } \bar{t}'$$

We write $X \sim_{\mathbf{iso}\text{-}\delta} X'$ iff there is an **iso**- δ -bisimulation between X and X' , and say that X and X' are **iso**- δ -bisimilar.

In the proof that an **iso**- δ -bisimulation induces a span of **Pom** _{L} -open maps, we need the **iso**- δ -bisimulation to be \simeq -closed. But this is no problem, as the \simeq -closure of *any* **iso**- δ -bisimulation is itself an **iso**- δ -bisimulation.

Lemma B.12 *Let \mathcal{T} be an **iso**- δ -bisimulation. Then the \simeq -closure*

$$\{(\bar{r}, \bar{r}') \mid \exists(\bar{t}, \bar{t}') \in \mathcal{T}. (\bar{t}, \bar{t}') \simeq (\bar{r}, \bar{r}')\}$$

*is an **iso**- δ -bisimulation, too.*

Proof Only the axiom **A** _{δ} can cause troubles. But fortunately, we are rescued by Lemma B.4 and Lemma B.7. \square

Proposition B.13 $\sim_{\mathbf{iso}\text{-}\delta} \subseteq \sim_{\mathbf{Pom}_L}$.

Proof Let $\mathcal{T} \subseteq \text{Seqs}(X) \times \text{Seqs}(X')$ be an **iso**- δ -bisimulation. According to Lemma B.12 we can without loss of generality assume that \mathcal{T} is \simeq -closed. As a candidate for a common domain of a span of **Pom** _{L} -open maps, we define the following structure which will be proved to be a transition system with independence. Define

$$Y = (S_Y, i_Y, L, \text{Tran}_Y, I_Y),$$

where

- $S_Y = \{[(\bar{t}, \bar{t}')]_{\simeq} \mid \bar{t} \mathcal{T} \bar{t}'\}$, i.e. \simeq -equivalence classes²,
- $i_Y = [(\varepsilon, \varepsilon)]$, and
- Tran_Y is defined by

$$y_0 \xrightarrow{a} y_1 \text{ iff } \exists \bar{t}, \bar{t}', s_1, s'_1. (y_0 = [(\bar{t}, \bar{t}')] \ \& \ y_1 = [(\bar{t}(s_0 \xrightarrow{a} s_1), \bar{t}'(s'_0 \xrightarrow{a} s'_1))]).$$

Before defining I_Y we introduce “projection” morphisms $f : Y \rightarrow X$ and $f' : Y \rightarrow X'$ as follows:

$$f([(\bar{t}, \bar{t}')]) = \text{tgt}(\bar{t}) \quad \text{and} \quad f'([(\bar{t}, \bar{t}')]) = \text{tgt}(\bar{t}').$$

²For convenience we shall from now on drop subscripts \simeq .

Thus, every transition $[(\bar{t}, \bar{t}')] \xrightarrow{a} [(\bar{t}(s_0 \xrightarrow{a} s_1), \bar{t}'(s'_0 \xrightarrow{a} s'_1))] = y_0 \xrightarrow{a} y_1$ of Y induces transitions $s_0 \xrightarrow{a} s_1 = f(y_0) \xrightarrow{a} f(y_1)$ and $s'_0 \xrightarrow{a} s'_1 = f'(y_0) \xrightarrow{a} f'(y_1)$ of X and X' , respectively. Since also $f([\varepsilon, \varepsilon])$ and $f'([\varepsilon, \varepsilon])$ denote the initial states of X and X' , respectively, f and f' are clearly well-defined morphisms. Using the extensions of f and f' to transitions we are now able to express when two transitions in Y are independent:

- $r_0 I_Y r_1$ iff $f(r_0) I_X f(r_1) \ \& \ f'(r_0) I_{X'} f'(r_1)$.

When it is clear from the context which transition system we refer to, we will drop the subscripts. It should be noticed that exactly \mathcal{T} being \simeq -closed ensures the well-definedness of the components of Y .

We now show that Y is a transition system with independence and that f and f' are **Pom_L**-open morphisms.

Y is a transition system with independence: Before proving the characterizing axioms we notice that the relations \prec_Y and \sim_Y between transitions in Y can be expressed by the corresponding relations in X and X' : For any transitions r_0 and r_1 in Y , we have

$$r_0 \prec_Y r_1 \text{ iff } f(r_0) \prec_X f(r_1) \ \& \ f'(r_0) \prec_{X'} f'(r_1), \quad (2)$$

and similarly for \sim_Y :

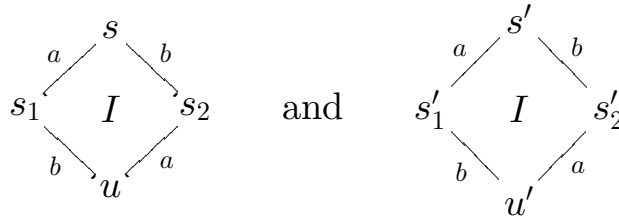
$$r_0 \sim_Y r_1 \text{ iff } f(r_0) \sim_X f(r_1) \ \& \ f'(r_0) \sim_{X'} f'(r_1). \quad (3)$$

We are now ready to prove the axioms.

1. is a consequence of (3) in conjunction with Axiom (1) holding for X and X' .
2. Assume two independent consecutive transitions in Y ,

$$[(\bar{t}, \bar{t}')] \xrightarrow{a} [(\bar{t}(s \xrightarrow{a} s_1), \bar{t}'(s' \xrightarrow{a} s'_1))] \xrightarrow{b} [(\bar{t}(s \xrightarrow{a} s_1)(s_1 \xrightarrow{b} u), \bar{t}'(s' \xrightarrow{a} s'_1)(s'_1 \xrightarrow{b} u'))].$$

Since independence in Y is defined component-wise, there are independence squares



in X and X' , respectively. By definition of S_Y , we see that

$$\bar{t}(s \xrightarrow{a} s_1)(s_1 \xrightarrow{b} u) \mathcal{T} \bar{t}'(s' \xrightarrow{a} s'_1)(s'_1 \xrightarrow{b} u'),$$

and since \mathcal{T} is δ -closed, it must be the case that

$$\bar{t}(s \xrightarrow{b} s_2) \mathcal{T} \bar{t}'(s' \xrightarrow{b} s'_2)$$

where s_2 and s'_2 are determined uniquely above. Hence the equivalence class $[(\bar{t}(s \xrightarrow{b} s_2), \bar{t}'(s' \xrightarrow{b} s'_2))]$ is a state of Y and, obviously, there are transitions

$$\begin{aligned} [(\bar{t}, \bar{t}')] &\xrightarrow{b} [(\bar{t}(s \xrightarrow{b} s_2), \bar{t}'(s' \xrightarrow{b} s'_2))] \\ &\xrightarrow{a} [(\bar{t}(s \xrightarrow{b} s_2)(s_2 \xrightarrow{a} u), \bar{t}'(s' \xrightarrow{b} s'_2)(s'_2 \xrightarrow{a} u'))]. \end{aligned}$$

Now, since $s \xrightarrow{a} s_1 \ I \ s_1 \xrightarrow{b} u$ and $s' \xrightarrow{a} s'_1 \ I \ s'_1 \xrightarrow{b} u'$, it follows that

$$\begin{aligned} (\bar{t}(s \xrightarrow{a} s_1)(s_1 \xrightarrow{b} u), \bar{t}'(s' \xrightarrow{a} s'_1)(s'_1 \xrightarrow{b} u')) \\ \simeq (\bar{t}(s \xrightarrow{b} s_2)(s_2 \xrightarrow{a} u), \bar{t}'(s' \xrightarrow{b} s'_2)(s'_2 \xrightarrow{a} u')). \end{aligned}$$

This completes the square in Y . The required independencies follow immediately, as I_Y is defined component-wise.

3. Both axioms are easy to establish using (2) and the definition of I_Y .

This finishes the proof that Y is a transition system with independence. It remains to show that f and f' are \mathbf{Pom}_L -open morphisms. The arguments are identical, so we present only the proof that f is \mathbf{Pom}_L -open.

f is zig-zag: Let $f([\bar{t}, \bar{t}']) \xrightarrow{a} s$ be a transition in X where, by definition, $f([\bar{t}, \bar{t}']) = \text{tgt}(\bar{t})$. Since $\bar{t} \mathcal{T} \bar{t}'$, there exists a transition $\text{tgt}(\bar{t}') \xrightarrow{a} s'$ for some s such that

$$\bar{t}(\text{tgt}(\bar{t}) \xrightarrow{a} s) \mathcal{T} \bar{t}'(\text{tgt}(\bar{t}') \xrightarrow{a} s').$$

Thus $[(\bar{t}(\text{tgt}(\bar{t}) \xrightarrow{a} s), \bar{t}'(\text{tgt}(\bar{t}') \xrightarrow{a} s'))]$ is a state of Y which by f is mapped to s . Moreover, there is an a -transition from $[(\bar{t}, \bar{t}')]$ into it. This proves that f is zig-zag.

f reflects consecutive independence: Suppose

$$\begin{aligned} [(\bar{t}, \bar{t}')] &\xrightarrow{a} [(\bar{t}(s \xrightarrow{a} s_1), \bar{t}'(s' \xrightarrow{a} s'_1))] \\ &\xrightarrow{b} [(\bar{t}(s \xrightarrow{a} s_1)(s_1 \xrightarrow{b} u), \bar{t}'(s' \xrightarrow{a} s'_1)(s'_1 \xrightarrow{b} u))] \end{aligned}$$

and assume $s \xrightarrow{a} s_1 \ I_X \ s_1 \xrightarrow{b} u$. Since

$$\bar{t}(s \xrightarrow{a} s_1)(s_1 \xrightarrow{b} u) \ \mathcal{T} \ \bar{t}'(s' \xrightarrow{a} s'_1)(s'_1 \xrightarrow{b} u)$$

and \mathcal{T} respects **iso**, we also have $s' \xrightarrow{a} s'_1 \ I_{X'} \ s'_1 \xrightarrow{b} u'$. Then, by definition of I_Y , the above transitions in Y must be independent, too.

We have now constructed a transition system with independence and equipped it with **Pom**_L-open morphisms to X and X' . This finishes the proof of the proposition. \square

B.5 From Γ -equivalence to iso- δ -bisimilarity

The key observation underlying this result is that any winning strategy in the backtracking game $\Gamma(X, X')$ maintains isomorphisms between observations of the two systems.

Lemma B.14 *A winning strategy σ in $\Gamma(X, X')$ respects **iso** in the sense that if $p \cdot (\bar{t}, \bar{t}')$ is a play respecting σ , then \bar{t} **iso** \bar{t}' .*

Proof Assume towards contradiction that there are plays $p \cdot (\bar{t}\bar{t}, \bar{t}'t')$ respecting σ such that $\neg(\bar{t}\bar{t} \ \mathbf{iso} \ \bar{t}'t')$. Choose $p, \bar{t}\bar{t}, \bar{t}'t'$ with this property and such that $\bar{t}\bar{t}$ is of minimal length with that property. Since every play respects length and labels, there are sequences $i\pi \in \omega^*$ such that

$$i\pi \bullet \bar{t}\bar{t} \text{ defined} \ \& \ i\pi \bullet \bar{t}'t' \text{ undefined} \tag{4}$$

(or vice versa). Choose $i\pi$ of minimal length with this property. Then, furthermore,

$$\pi \bullet \bar{t}\bar{t} \text{ defined} \ \& \ \pi \bullet \bar{t}'t' \text{ defined.} \tag{5}$$

Suppose $|\bar{t}\bar{t}| = n$ and write

$$\begin{aligned} \pi \bullet \bar{t}\bar{t} &= r_0 r_1 \cdots r_i r_{i+1} \cdots r_{n-1} \quad \text{and} \\ \pi \bullet \bar{t}'t' &= r'_0 r'_1 \cdots r'_i r'_{i+1} \cdots r'_{n-1}, \end{aligned} \tag{6}$$

where, by (4) and (5), $r_i \ I \ r_{i+1}$ and $\neg(r'_i \ I \ r'_{i+1})$.

We will now argue that in (6), $i = n - 2$, meaning that r_{i+1} and r'_{i+1} are the last transitions. Assume towards contradiction that $i < n - 2$ and let k be such that $\langle \pi, \bar{t}t \rangle(k) = n - 1$ and hence $\langle \pi, \bar{t}'t' \rangle(k) = n - 1$. Then, by Lemma B.4, $(\bar{t}t)_k$ is backwards enabled in $\bar{t}t$ making it possible for Opponent to do the backwards move

$$(\bar{t}t, \bar{t}'t') \triangleright (k, \delta(k, \bar{t}t), \bar{t}'t').$$

Since σ is winning, Player is able match this move, and we end up in $(\delta(k, \bar{t}t), \delta(k, \bar{t}'t'))$. Now, since $\bar{t}t$ was chosen minimal, we see that $\delta(k, \bar{t}t) \mathbf{iso} \delta(k, \bar{t}'t')$. Since $r_i I r_{i+1}$ and $\neg(r'_i I r'_{i+1})$, the only values of s for which $\delta(s, \bar{r}) \mathbf{iso} \delta(s, \bar{r}')$ can be true, are i and $i + 1$. By Corollary B.8, $\delta(n - 1, \bar{r}) \mathbf{iso} \delta(n - 1, \bar{r}')$, so we conclude that $n - 1 \in \{i, i + 1\}$. But $n - 1 = i$ contradicts $i + 1$ being a legal index of \bar{r} . Hence, $n - 1 = i + 1$ or, equivalently, $i = n - 2$.

Finally we can now rewrite (6) using the information that $i = n - 2$:

$$\begin{aligned} \pi \bullet \bar{t}t &= r_0 r_1 \cdots r_i r_{i+1} \quad \text{and} \\ \pi \bullet \bar{t}'t' &= r'_0 r'_1 \cdots r'_i r'_{i+1}. \end{aligned}$$

Let now l be such that $\langle \pi, \bar{t}t \rangle(l) = i$ and hence $\langle \pi, \bar{t}'t' \rangle(l) = i$. Since $r_i I r_{i+1}$, it follows from Lemma B.4 that $(\bar{t}t)_l$ is backwards enabled in $\bar{t}t$. Hence, Opponent can do the backwards move

$$(\bar{t}t, \bar{t}'t') \triangleright (l, \delta(l, \bar{t}t), \bar{t}'t').$$

But, since $\neg(r'_i I r'_{i+1})$, it follows from Lemma B.4 that $(\bar{t}'t')_l$ is *not* backwards enabled in $\bar{t}'t'$, so Player is stuck. This contradicts σ being winning. \square

The proof of the proposition below is now straightforward.

Proposition B.15 $\sim_\Gamma \subseteq \sim_{\mathbf{iso}-\delta}$.

Proof Let σ be a winning strategy in $\Gamma(X, X')$ and assume, in accordance with Lemma B.14, that σ respects **iso**. Define a relation $\mathcal{T} \subseteq \text{Seqs}(X) \times \text{Seqs}(X')$ by

$\bar{t} \mathcal{T} \bar{t}'$ iff $p \cdot (\bar{t}, \bar{t}') \in \text{Plays}(\sigma, \tau)$ for some play p and counter-strategy τ .

We show that \mathcal{T} is an **iso**- δ -bisimulation:

A_{iso}: $\bar{t} \mathcal{T} \bar{t}' \Rightarrow \bar{t} \mathbf{iso} \bar{t}'$ follows directly from the fact that σ respects **iso**.

A_{bisim}: Suppose $\bar{t} \mathcal{T} \bar{t}'$ and $\bar{t}t \in Seqs(X)$. Let p and τ be such that $p \cdot (\bar{t}, \bar{t}') \in Plays(\sigma, \tau)$ and define

$$\tau' = [(\bar{t}t, \bar{t}')/p \cdot (\bar{t}, \bar{t}')],$$

which is the function equal to τ everywhere except at $p \cdot (\bar{t}, \bar{t}')$ at which it yields $(\bar{t}t, \bar{t}')$. Now³,

$$p' = p \cdot (\bar{t}, \bar{t}') \cdot (\bar{t}t, \bar{t}') \in Plays(\sigma, \tau'),$$

forcing σ to be defined on p' . Inspecting the rules of $\Gamma(X, X')$ we see that $\sigma(p')$ equals $(\bar{t}t, \bar{t}'t')$ for some t' . But then $\bar{t}t \mathcal{T} \bar{t}'t'$, as required.

A _{δ} : Let $\bar{t} \mathcal{T} \bar{t}'$ and assume $t_i \in BEn(\bar{t})$. Choose p and τ such that $p \cdot (\bar{t}, \bar{t}') \in Plays(\sigma, \tau)$ and define

$$\tau'' = [(i, \delta(i, \bar{t}), \bar{t}')/p \cdot (\bar{t}, \bar{t}')].$$

Observe that

$$p'' \stackrel{\text{def}}{=} p \cdot (\bar{t}, \bar{t}') \cdot (i, \delta(i, \bar{t}), \bar{t}') \in Plays(\sigma, \tau'').$$

Being winning, σ is able to respond to this move with $\sigma(p'') = (\delta(i, \bar{t}), \delta(i, \bar{t}'))$. Hence, $t'_i \in BEn(\bar{t}')$, by the definition of $\Gamma(X, X')$. Furthermore, $\delta(i, \bar{t}) \mathcal{T} \delta(i, \bar{t}')$, since $p'' \cdot (\delta(i, \bar{t}), \delta(i, \bar{t}')) \in Plays(\sigma, \tau)$. By a symmetric argument, this finishes the proof. \square

³Here we use that strategies are defined on positions rather than configurations.

C Proofs From Section 4

Proposition C.1 (Proposition 4.5 restated) *If two transition systems with independence are \mathbf{Pom}_L -bisimilar, then they are also \mathbf{Assn} -equivalent.*

Proof Let \mathcal{T} be a δ -bisimulation between X and X' . By structural induction on assertions we show that for all $A \in \mathbf{Assn}$, $\bar{t} \in \mathit{Seqs}(X)$, and $\bar{t}' \in \mathit{Seqs}(X')$,

$$\bar{t} \mathcal{T} \bar{t}' \Rightarrow (\bar{t} \models_X A \Leftrightarrow \bar{t}' \models_{X'} A). \quad (7)$$

$\neg A$:

$$\begin{aligned} \bar{t} \mathcal{T} \bar{t}' &\Rightarrow (\bar{t} \models_X A \Leftrightarrow \bar{t}' \models_{X'} A) && \text{(inductively)} \\ &\Leftrightarrow (\bar{t} \not\models_X A \Leftrightarrow \bar{t}' \not\models_{X'} A) \\ &\Leftrightarrow (\bar{t} \models_X \neg A \Leftrightarrow \bar{t}' \models_{X'} \neg A) && \text{(by definition)} \end{aligned}$$

$\bigwedge_{j \in J} A_j$:

$$\begin{aligned} \bar{t} \mathcal{T} \bar{t}' &\Rightarrow \forall j \in J. (\bar{t} \models_X A_j \Leftrightarrow \bar{t}' \models_{X'} A_j) && \text{(inductively)} \\ &\Rightarrow (\forall j \in J. (\bar{t} \models_X A_j) \Leftrightarrow \forall j \in J. (\bar{t}' \models_{X'} A_j)) \\ &\Leftrightarrow (\bar{t} \models_X \bigwedge_{j \in J} A_j \Leftrightarrow \bar{t}' \models_{X'} \bigwedge_{j \in J} A_j) && \text{(by definition)} \end{aligned}$$

$\langle a \rangle A$: Suppose $\bar{t} \mathcal{T} \bar{t}'$ and $\bar{t} \models_X \langle a \rangle A$. Then there exists \bar{r} such that $\bar{t} \xrightarrow{a} \bar{r}$ and $\bar{r} \models_X A$. Since $\bar{t} \mathcal{T} \bar{t}'$ there exists \bar{r}' such that $\bar{t}' \xrightarrow{a} \bar{r}'$ and $\bar{r} \mathcal{T} \bar{r}'$. By the induction hypothesis, $\bar{r}' \models_{X'} A$, so we conclude that $\bar{t}' \models_{X'} \langle a \rangle A$.

$@A$: Suppose $\bar{t} \mathcal{T} \bar{t}'$ and $\bar{t} \models_X @A$. Then there exists $t_i \in BEn(\bar{t})$ such that $\ell(t_i) = a$ and $\delta(i, \bar{t}) \models_X A$. Since $\bar{t} \mathcal{T} \bar{t}'$ we also have $t'_i \in BEn(\bar{t}')$ and $\delta(i, \bar{t}) \mathcal{T} \delta(i, \bar{t}')$, so inductively, $\delta(i, \bar{t}') \models_{X'} A$. Since \mathcal{T} respects labels, $\bar{t}' \models_{X'} @A$, as required.

This finishes the proof of (7). The proposition follows by taking $\bar{t} = \bar{t}' = \varepsilon$.
□

Lemma C.2 *Suppose X has no auto-concurrency and let $\bar{t} \in \mathit{Seqs}(X)$. Then, if $t_i, t_j \in BEn(\bar{t})$ and $\ell(t_i) = \ell(t_j)$, the transitions are equal, i.e. $i = j$.*

Proof Assume towards contradiction that $i \neq j$. Without loss of generality we may assume that $i < j$. Since $t_i \in BEn(\bar{t})$, by definition, $t_i I t_k$ for all $k > i$. This clearly contradicts X having no auto-concurrency. □

For non-auto-concurrent systems, there is a simpler game, Γ' , characterizing **Pom**_L-bisimilarity. Backtracking on indices can be substituted by backtracking on *labels*, yielding rules like

$$(\bar{t}, \bar{t}') \triangleright (a, \bar{r}, \bar{t}') \text{ if } \bar{r} \xrightarrow{a} \bar{t},$$

where the a is a directive to Player to play backwards with an a -move in the longer of the sequences. More concretely, Player must reply with an application of the rule

$$(a, \bar{r}, \bar{t}') \triangleright (\bar{r}, \bar{r}') \text{ if } \bar{r}' \xrightarrow{a} \bar{t}'.$$

Say that two transition systems with independence X and X' are Γ' -equivalent iff Player has a winning strategy in $\Gamma'(X, X')$.

Proposition C.3 *Two non-auto-concurrent transition systems with independence are Γ' -equivalent iff they are Γ -equivalent.*

Proof Let X and X' be transition systems with independence. First we notice that there is a bijection $\mathcal{C} : Pos(\Gamma(X, X')) \rightarrow Pos(\Gamma'(X, X'))$ defined inductively in length of the position:

$$\begin{aligned} \mathcal{C}(\varepsilon, \varepsilon) &= (\varepsilon, \varepsilon) \\ \mathcal{C}(p \cdot (\bar{t}, \bar{t}')) &= \mathcal{C}(p) \cdot (\bar{t}, \bar{t}') \\ \mathcal{C}(p \cdot (\bar{t}\bar{t}, \bar{t}')) &= \mathcal{C}(p) \cdot (\bar{t}\bar{t}, \bar{t}') \\ \mathcal{C}(p \cdot (\bar{t}, \bar{t}'\bar{t}')) &= \mathcal{C}(p) \cdot (\bar{t}, \bar{t}'\bar{t}') \\ \mathcal{C}(p \cdot (\bar{t}, \bar{t}') \cdot (i, \delta(i, \bar{t}), \bar{t}')) &= \mathcal{C}(p \cdot (\bar{t}, \bar{t}')) \cdot (\ell(t_i), \delta(i, \bar{t}), \bar{t}') \\ \mathcal{C}(p \cdot (\bar{t}, \bar{t}') \cdot (i, \bar{t}, \delta(i, \bar{t}')) &= \mathcal{C}(p \cdot (\bar{t}, \bar{t}')) \cdot (\ell(t'_i), \bar{t}, \delta(i, \bar{t}')) \end{aligned}$$

Using that the plays of $\Gamma(X, X')$ respect labels (i.e. that whenever $(\bar{t}, \bar{t}') \in Pos(\Gamma(X, X'))$, $\forall i. (\ell(t_i) = \ell(t'_i))$), it is easy to show that \mathcal{C} is well-defined. The inverse function \mathcal{C}^{-1} of \mathcal{C} is given inductively by very similar clauses. As an example, let us see how $\mathcal{C}^{-1}(p \cdot (\bar{t}, \bar{t}') \cdot (a, \bar{r}, \bar{t}'))$ (where $\bar{r} \xrightarrow{a} \bar{t}$) is defined. By definition 4.1, there is an index i such that $t_i \in BEn(\bar{t})$, $\ell(t_i) = a$, and $\bar{r} = \delta(i, \bar{t})$. Moreover, this index is unique. We can thus define

$$\mathcal{C}^{-1}(p \cdot (\bar{t}, \bar{t}') \cdot (a, \bar{r}, \bar{t}')) = \mathcal{C}^{-1}(p \cdot (\bar{t}, \bar{t}')) \cdot (i, \delta(i, \bar{t}), \bar{t}').$$

The well-definedness of \mathcal{C}^{-1} follows from Lemma C.2 using that plays of $\Gamma(X, X')$ respect labels. It is straightforward to show that \mathcal{C} and \mathcal{C}^{-1} are each others inverse.

For one direction, assume that σ is a winning strategy in $\Gamma(X, X')$. We then define a winning strategy σ' in $\Gamma(X, X')$ as follows: For each $p \in Pos(\Gamma'(X, X'))$, let

$$\sigma'(p) = \sigma(\mathcal{C}^{-1}(p)).$$

Using the bijection, it can be shown that σ' is a winning strategy.

The other direction is managed in a similar way. \square

Proposition C.4 (Proposition 4.7 restated) *If two non-auto-concurrent transition systems with independence are **Assn**-equivalent, then they are also **Pom_L**-bisimilar.*

Proof Assume towards contradiction that X and X' are **Assn**-equivalent but not **Pom_L**-bisimilar. Then there can be no winning strategy in the modified game $\Gamma'(X, X')$. Hence, by Proposition 3.1, there is a winning counter-strategy τ in $\Gamma'(X, X')$. Using τ , we construct an assertion $\mathcal{A}(p)$ for any play p such that

- p respects τ , and
- $\lambda(p) = O$.

Define the following partial order on such plays:

$$p' \prec p \text{ iff } \exists c, c'. (p' = p \cdot c \cdot c').$$

As τ is winning, there can be no infinite plays respecting τ . Hence, \prec is well-founded. We now define $\mathcal{A}(p)$ by well-founded recursion on p . Let p be a play of the above kind and suppose $\mathcal{A}(p')$ is defined for all $p' \prec p$. Since p respects τ , and $\lambda(p) = O$, $\tau(p)$ must be defined – otherwise τ would not be winning. Assuming the last configuration of p is (\bar{t}, \bar{t}') , consider now the following cases of $\tau(p)$:

$\tau(p) = (\bar{r}, \bar{t}')$ **where** $\bar{t} \xrightarrow{a} \bar{r}$:

$$\mathcal{A}(p) = \langle a \rangle (\wedge \{ \mathcal{A}(p') \mid \exists \bar{r}'. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p) \})$$

$\tau(p) = (\bar{t}, \bar{r}')$ **where** $\bar{t}' \xrightarrow{a} \bar{r}'$:

$$\mathcal{A}(p) = \neg \langle a \rangle (\wedge \{ \neg \mathcal{A}(p') \mid \exists \bar{r}. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p) \})$$

$\tau(p) = (a, \bar{r}, \bar{t}')$ **where** $\bar{r} \xrightarrow{a} \bar{t}$:

$$\mathcal{A}(p) = @(\wedge\{\mathcal{A}(p') \mid \exists \bar{r}'. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p)\})$$

$\tau(p) = (a, \bar{t}, \bar{r}')$ **where** $\bar{r}' \xrightarrow{a} \bar{t}'$:

$$\mathcal{A}(p) = \neg @(\wedge\{\neg \mathcal{A}(p') \mid \exists \bar{r}. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p)\})$$

The claim is now that if p is a play of the above kind, and the last configuration of p is (\bar{t}, \bar{t}') , then

$$\bar{t} \models_X \mathcal{A}(p) \quad \& \quad \bar{t}' \models_{X'} \neg \mathcal{A}(p). \quad (8)$$

Instantiating p to $(\varepsilon, \varepsilon)$ yields $\models_X \mathcal{A}(\varepsilon, \varepsilon)$ and $\models_{X'} \neg \mathcal{A}(\varepsilon, \varepsilon)$ which contradicts the assumption $X \sim_{\text{Assn}} X'$.

The proof of the claim is by well-founded induction on p . We consider the following four cases of $\tau(p)$:

$\tau(p) = (\bar{r}, \bar{t}')$ **where** $\bar{t} \xrightarrow{a} \bar{r}$: To establish $\bar{t} \models_X \mathcal{A}(p)$ we show

$$\bar{r} \models_X \wedge\{\mathcal{A}(p') \mid \exists \bar{r}'. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p)\}.$$

Assume $p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p$ for some \bar{r}' . We are then required to show $\bar{r} \models_X \mathcal{A}(p')$. But this follows inductively, since p' respects τ and the last configuration of p' is (\bar{r}, \bar{r}') .

To establish $\bar{t}' \models_{X'} \neg \mathcal{A}(p)$ we assume towards contradiction that $\bar{t}' \models_{X'} \mathcal{A}(p)$. Then there exists a transition sequence \bar{r}' such that $\bar{t}' \xrightarrow{a} \bar{r}'$ and

$$\bar{r}' \models_{X'} \wedge\{\mathcal{A}(p') \mid \exists \bar{r}'. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p)\}.$$

Now there is a move $\tau(p) \triangleright (\bar{r}, \bar{r}')$, so $p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p$. Hence, $\bar{r}' \models_{X'} \mathcal{A}(p')$, but this contradicts the induction hypothesis.

$\tau(p) = (\bar{t}, \bar{r}')$ **where** $\bar{t}' \xrightarrow{a} \bar{r}'$: Similar to the previous case.

$\tau(p) = (a, \bar{r}, \bar{t}')$ **where** $\bar{r} \xrightarrow{a} \bar{t}$: To establish $\bar{t} \models_X \mathcal{A}(p)$ we simply show

$$\bar{r} \models_X \wedge\{\mathcal{A}(p') \mid \exists \bar{r}'. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p)\}.$$

Suppose $p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p$. We are then required to show $\bar{r} \models_X \mathcal{A}(p')$, but this follows directly from the induction hypothesis.

We still need to show $\bar{t}' \models_{X'} \neg \mathcal{A}(p)$. Assume towards contradiction that $\bar{t}' \models_{X'} \mathcal{A}(p)$. Then there exists \bar{r}' such that $\bar{r}' \overset{a}{\rightsquigarrow} \bar{t}'$ and

$$\bar{r}' \models_{X'} \bigwedge \{ \mathcal{A}(p') \mid \exists \bar{r}. (p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p) \}.$$

Now there is a move $\tau(p) \triangleright (\bar{r}, \bar{r}')$, so, in fact, $p' = p \cdot \tau(p) \cdot (\bar{r}, \bar{r}') \prec p$. Hence, $\bar{r}' \models_{X'} \mathcal{A}(p')$, which contradicts the induction hypothesis.

$\tau(p) = (a, \bar{t}, \bar{r}')$ **where** $\bar{r}' \overset{a}{\rightsquigarrow} \bar{t}'$: Symmetric. □

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