BRICS RS-07-3 Aceto et al.: Ready To Preorder: Get Your BCCSP Axiomatization for Free!

BRICS

Basic Research in Computer Science

Ready To Preorder: Get Your BCCSP Axiomatization for Free!

Luca Aceto
Willem Jan Fokkink
Anna Ingólfsdóttir

BRICS Report Series **RS-07-3**

ISSN 0909-0878 **February 2007**

See back inner page for a list of recent BRICS Report Series publications.
Copies may be obtained by contacting:

BRICS
Department of Computer Science
University of Aarhus
IT-parken, Aabogade 34
DK–8200 Aarhus N
Denmark

Telephone: +45 8942 9300
Telefax: +45 8942 5601
Internet: BRICS@brics.dk

BRICS publications are in general accessible through the World Wide Web and anonymous FTP through these URLs:

http://www.brics.dk
ftp://ftp.brics.dk
This document in subdirectory RS/07/3/

Ready to Preorder: Get Your BCCSP Axiomatization for Free! ⋆

Luca Aceto[1], Wan Fokkink[2,3], and Anna Ingolfsdottir[1]

[1] Reykjavík University, School of Science and Engineering
Ofanleiti 2, 103 Reykjavík, Iceland
[2] Vrije Universiteit, Section Theoretical Computer Science
Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
[3] CWI, Embedded Systems Group
Kruislaan 413, 1098 SJ Amsterdam, The Netherlands
luca@ru.is,wanf@cs.vu.nl,annai@ru.is

Abstract. This paper contributes to the study of the equational theory of the semantics in van Glabbeek's linear time - branching time spectrum over the language BCCSP, a basic process algebra for the description of finite synchronization trees. It offers an algorithm for producing a complete (respectively, ground-complete) equational axiomatization of a behavioral congruence lying between ready simulation equivalence and partial traces equivalence from a complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence—that is, of the precongruence whose kernel is the equivalence. The algorithm preserves finiteness of the axiomatization when the set of actions is finite. It follows that each equivalence in the spectrum whose discriminating power lies in between that of ready simulation and partial traces equivalence is finitely axiomatizable over the language BCCSP if so is its defining preorder.

1 Introduction

The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. In his by now classic paper [12], van Glabbeek presented the linear time - branching time spectrum of behavioural preorders and equivalences for finitely branching, concrete, sequential processes. The semantics in this spectrum are based on simulation notions and on decorated traces. Figure 1 in Appendix A depicts the linear time - branching time spectrum.

Van Glabbeek [12] studied the semantics in his spectrum in the setting of the process algebra BCCSP, which contains only the basic process algebraic operators from CCS [17] and CSP [16], but is sufficiently powerful to express all finite synchronization trees. In the aforementioned reference, van Glabbeek

⋆ The first and third author were partly supported by the project "The Equational Logic of Parallel Processes" (nr. 060013021) of The Icelandic Research Fund.

gave, amongst a wealth of other results, (in)equational axiomatizations for the preorders and equivalences in the spectrum, such that two closed BCCSP terms can be equated by the axioms if, and only if, they are related by the preorder or equivalence in question. Groote [13] obtained ω-completeness results for most of the axiomatizations, in case the alphabet of actions is infinite. (An axiomatization E is *ω-complete* when an equation can be derived from E if, and only if, all of its closed instantiations can be derived from E.) The series of papers [2, 5, 7–9] offers positive and negative results on the existence of finite (in)equational axiomatizations for several behavioural equivalences and preorders in the spectrum over the language BCCSP, both in the setting of finite and infinite sets of actions.

The work we present in this paper stems from the observation that all of the extant axiomatization results presented in the aforementioned studies are based on separate, and often rather similar, developments for preorders and equivalences. For the semantics in the spectrum lying between 2-nested simulation semantics and partial traces semantics, the equivalences are the *kernels* of the preorders—meaning that two processes are considered equivalent if, and only if, each is a refinement of the other with respect to the preorder—, which are therefore more basic than the equivalences. Since the equivalences are defined in terms of the preorders in a canonical fashion, it would be very satisfying, in order to achieve a higher degree of generality and to highlight the commonalities in the technical developments pertaining to axiomatization results for the semantics in the spectrum, to develop a general strategy for obtaining complete axiomatizations of the equivalences in the spectrum from complete axiomatizations of the preorders. This is the aim of this paper.

Our contribution We offer an algorithm for producing an ω-complete (respectively, ground-complete) equational axiomatization of a behavioral congruence lying between ready simulation equivalence and partial traces equivalence from an ω-complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence—that is, of the precongruence whose kernel is the equivalence. The algorithm we give in this paper preserves finiteness of the axiomatization when the set of actions is finite. It follows that each equivalence in the spectrum whose discriminating power lies in between that of ready simulation and partial traces equivalence is finitely axiomatizable over the language BCCSP if so is its defining preorder.

Our algorithm may be seen as isolating and axiomatizing the ingredients that all of the extant proofs of completeness results for the class of behavioural equivalences we study have in common. It also eliminates the need to reprove, essentially from scratch, completeness results for a large fragment of behavioural equivalences in the spectrum once a completeness result has been obtained for their underlying preorders. The axiomatizations that are automatically generated by our algorithm are very similar, when not identical, to those presented in the literature. (See, for instance, the three specific examples of applications of our algorithm that are provided in Section 6.)

Our algorithm takes as input a sound and ω-complete (respectively, ground-complete) inequational axiomatization E for BCCSP modulo a preorder in the linear time - branching time spectrum that includes the ready simulation preorder. Without loss of generality, we assume that the four classic equations from [15] that completely axiomatize bisimulation equivalence [17] are contained in E, and that so do the defining inequational axioms for ready simulation for each action a:

$$ax \preccurlyeq ax + ay \ .$$

The axiomatization $\mathcal{A}(E)$ generated by our algorithm from E contains the axioms for bisimulation equivalence together with the following equations, for each inequational axiom $t \preccurlyeq u$ in E:

- $t + u \approx u$; and
- $b(t + x) + b(u + x) \approx b(u + x)$ (for each action b, and some variable x that does not occur in $t + u$).

The main technical result in the paper is a theorem to the effect that the axiomatization $\mathcal{A}(E)$ is ω-complete (respectively, ground-complete) for the equivalence if E is ω-complete (respectively, ground-complete) for the preorder (Theorem 1). The proof of this statement is non-trivial, and relies on a careful analysis of the so-called *cover equations* [9] for the semantics in the linear time - branching time spectrum we consider in this study. Cover equations give us an explicit description of the equational theory for a particular semantics in terms of equations having a rather simple, and canonical, form.

Roadmap of the paper The paper is organized as follows. Section 2 reviews the syntax and the operational semantics for the language BCCSP, introduces the linear time time - branching time spectrum, and discusses the very basic notions of (in)equational logic used in this study. (The full details of the definitions of the semantics in the spectrum may be found in Appendix A.) We present our algorithm in Section 3, where we also state the main theorem in the paper (Theorem 1) to the effect that the algorithm is guaranteed to produce an ω-complete (respectively, ground-complete) equational axiomatization of a behavioral congruence lying between ready simulation equivalence and partial traces equivalence from an ω-complete (respectively, ground-complete) inequational axiomatization of its underlying precongruence. The bulk of the rest of the paper (Sections 4–5 and Appendix B) is devoted to a proof of our main result. Section 6 presents applications of our algorithm in the setting of simulation, failures and partial traces semantics. We end the paper with some concluding remarks, and a detailed comparison with related work (Section 7).

2 Preliminaries

Syntax of BCCSP BCCSP(A) is a basic process algebra for expressing finite process behaviour. Its syntax consists of closed (process) terms p, q that are constructed from a constant $\mathbf{0}$, a binary operator $\_+\_$ called *alternative composition*, and unary *prefix* operators $a\_$, where a ranges over some nonempty set A of *actions* (with typical elements a, b, c, d). (We write $|A|$ for the cardinality of the set A.) Open terms p, q, r, s, t, u can moreover contain occurrences of variables from a countably infinite set V (with typical elements w, x, y, z).

A (closed) substitution maps variables in V to (closed) terms. For every term t and (closed) substitution σ, the (closed) term $\sigma(t)$ is obtained by replacing every occurrence of a variable x in t by $\sigma(x)$. We often write t^σ in lieu of $\sigma(t)$.

A *context* $C[]$ is a BCCSP(A) term with exactly one occurrence of a hole $[]$ in it. For every context $C[]$ and term p, we write $C[p]$ for the term that results by placing p in the hole in $C[]$.

Transition rules Intuitively, closed BCCSP(A) terms represent finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p+q$ is the nondeterministic choice between the behaviours of p and q, and ap executes action a to transform into p. This intuition is captured, in the style of Plotkin, by the transition rules below, which give rise to A-labelled transitions between closed terms.

$$\frac{}{ax \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'}$$

The operational semantics is extended to open terms by assuming that variables do not exhibit any behaviour. A sequence of actions $a_1 \cdots a_n$, with $n \geq 0$, is a *trace* of a term t_0 if there is a sequence of transitions $t_0 \xrightarrow{a_1} t_1 \xrightarrow{a_2} \cdots t_{n-1} \xrightarrow{a_n} t_n$. The *depth* of a term t, denoted by $depth(t)$, is the length of a longest trace of t.

Linear time - branching time spectrum Van Glabbeek [12] presented the linear time - branching time spectrum of behavioural preorders and equivalences. The semantics in this spectrum are based on simulation notions and on decorated traces. The definitions of the equivalences and preorders in the spectrum are collected in Appendix A. In what follows, we use $\precsim$ to denote a preorder in this spectrum, and $\simeq$ to denote the corresponding equivalence (i.e., $\precsim \cap \precsim^{-1}$). The equivalence induced by a preorder is also known as its *kernel*. When we want to refer to a specific preorder in the spectrum, we shall subscribe the symbol $\precsim$ with the initials of the intended semantics. For instance, we shall use $\precsim_{\mathrm{RS}}$ to denote the ready simulation preorder, $\precsim_{\mathrm{S}}$ for the simulation preorder, $\precsim_{\mathrm{F}}$ for the failures preorder, $\precsim_{\mathrm{CT}}$ for the completed traces preorder, and $\precsim_{\mathrm{PT}}$ for the partial traces preorder. A similar notational convention applies to the kernels of the preorders.

Each preorder in the linear time - branching time spectrum is a *precongruence* over the algebra of closed BCCSP(A) terms. That is, $p_1 \precsim q_1$ and $p_2 \precsim q_2$ imply $ap_1 \precsim aq_1$, for each $a \in A$, and $p_1 + p_2 \precsim q_1 + q_2$. Likewise, the equivalences in the spectrum constitute a *congruence* over closed BCCSP(A) terms.

Given a preorder $\precsim$ over closed terms, for open terms t and u, we define $t \precsim u$ if $\rho(t) \precsim \rho(u)$ for each closed substitution ρ; the corresponding equivalence $\simeq$ is lifted to open terms likewise.

Equations and inequations An *(in)equational axiomatization* (often abbreviated to *axiomatization*) E is a collection of either inequations $t \preccurlyeq u$ or equations $t \approx u$, where t and u are BCCSP(A) terms. We write $E \vdash t \preccurlyeq u$ or $E \vdash t \approx u$ if this (in)equation can be derived from the (in)equations in E using the standard rules of (in)equational logic, where the rule for symmetry can be applied for equational derivations but not for inequational ones. An axiomatization E is *sound* modulo $\precsim$ (or $\simeq$) if, for all open terms t, u, from $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$) it follows that $t \precsim u$ (or $t \simeq u$). An axiomatization E is *ground-complete* modulo $\precsim$ (or $\simeq$) if $p \precsim q$ (or $p \simeq q$) implies $E \vdash p \preccurlyeq q$ (or $E \vdash p \approx q$), for all closed terms p and q. We say that E is *ω-complete* if for all open terms t, u with $E \vdash \rho(t) \preccurlyeq \rho(u)$ (or $E \vdash \rho(t) \approx \rho(u)$) for all closed substitutions ρ, we have $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$).

The core axioms A1–4 for BCCSP(A) given below are ω-complete [18], and sound and ground-complete [15, 17] modulo bisimulation equivalence, which is the finest semantics in the linear time - branching time spectrum. (See Definition 1 in Appendix A for the definition of bisimulation equivalence.)

$$\begin{array}{ll}
\text{A1} & x + y \approx y + x \\
\text{A2} & (x + y) + z \approx x + (y + z) \\
\text{A3} & x + x \approx x \\
\text{A4} & x + \mathbf{0} \approx x
\end{array}$$

In the remainder of this paper, process terms are considered modulo A1–4. A term x or at is a *summand* of each term $x + u$ or $at + u$, respectively. We use *summation* $\sum_{i=1}^{n} t_i$ (with $n \geq 0$) to denote $t_1 + \cdots + t_n$, where the empty sum denotes $\mathbf{0}$. As binding convention, alternative composition and summation bind weaker than prefixing. Modulo the equations A1–4 each BCCSP(A) term t can be written in the form $\sum_{i=1}^{n} t_i$, where each t_i is either a variable or is of the form at' for some action a and term t'.

In his paper [12], van Glabbeek offered, amongst a host of other results, (in)equational axiomatizations for the preorders and equivalences in the spectrum. The proofs of the completeness results in that reference mostly employ the method of graph transformations. Groote [13] obtained ω-completeness results for most of the axiomatizations, in case the alphabet of actions is infinite.

In the remainder of this paper, in case of an infinite alphabet, occurrences of action names in axioms should be interpreted as action variables.

3 Producing an Axiomatization

Consider a preorder $\precsim$ in the linear time - branching time spectrum that includes the ready simulation preorder. Let E be a sound and ground-complete inequational axiomatization for BCCSP(A) modulo $\precsim$. We give an algorithm to produce an axiomatization $\mathcal{A}(E)$ that is sound and ground-complete for BCCSP(A) modulo $\simeq$, namely the kernel of the preorder $\precsim$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

Without loss of generality, we assume that the axioms A1–4 are present in E, together with the defining inequational axioms for ready simulation equivalence for each $a \in A$:

$$ax \preccurlyeq ax + ay \ .$$

The axiomatization $\mathcal{A}(E)$ is constructed as follows. The axioms A1–4 are by default included in $\mathcal{A}(E)$. Furthermore, for each inequational axiom $t \preccurlyeq u$ in E, we add to $\mathcal{A}(E)$:

A. $t + u \approx u$; and
B. $b(t+x) + b(u+x) \approx b(u+x)$ (for all $b \in A$, and some x that does not occur in $t + u$).

Note that $\mathcal{A}(E)$ is finite whenever A and E are finite. Moreover, using an action variable in step B in lieu of a concrete action $b \in A$, the axiomatization $\mathcal{A}(E)$ contains only finitely many axiom schemas when E does, even in the presence of an infinite collection of actions.

Remark 1. Since $ax \preccurlyeq ax + ay$ is assumed to be present in E for each $a \in A$, by step B of the algorithm, the defining axioms for ready simulation from [5], namely

$$b(ax + z) + b(ax + ay + z) \approx b(ax + ay + z) \ ,$$

are present in $\mathcal{A}(E)$, for all $a, b \in A$.

We are now ready to present the main result of the paper to the effect that the algorithm defined above delivers axiomatizations for the kernels of the preorders that are sound, and ground- or ω-complete.

Theorem 1. *Let $\precsim$ be a preorder in the linear time - branching time spectrum with $\precsim_{\mathrm{RS}} \subseteq \precsim$. Let E be a sound and ground-complete inequational axiomatization for* BCCSP(A) *modulo $\precsim$. Then the equational axiomatization $\mathcal{A}(E)$ is sound and ground-complete for* BCCSP(A) *modulo $\simeq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.*

Since the algorithm presented above preserves finiteness of the axiomatization when the set of actions A is finite, it follows that each equivalence in the spectrum whose discriminating power lies in between that of ready simulation and partial traces equivalence is finitely axiomatizable over the language BCCSP(A) if so is its defining preorder.

The remainder of the paper will be essentially devoted to a proof of the above theorem. Our proof of Theorem 1 relies on the isolation of a collection of equations, the so-called *cover equations*, that have a simple form and completely characterize the equational theory of BCCSP(A) modulo any of the behavioural equivalences whose discriminating power lies in between that of ready simulation and partial traces equivalence. Restricting ourselves to cover equations will help us overcome the technical complications in the proof-theoretic argument we shall use in Section 5 to complete the proof of Theorem 1.

In light of the key role cover equations play in the proof of Theorem 1, we now proceed to introduce them and to analyze the properties that make them a crucial ingredient in our proof of that result.

4 Cover Equations

For bisimulation semantics, and thus for all process semantics in the linear time - branching time spectrum, axiom A3 is sound. So if an equation $t \approx u$ is sound, then $u + t \approx t$ and $t + u \approx u$ are sound too; and from the last two equations one can derive $t \approx u$. Furthermore, for all process semantics in the linear time - branching time spectrum, if $t_1+t_2+u \approx u$ is sound, then $t_1+u \approx u$ and $t_2+u \approx u$ are sound; and from the last two equations one can derive $t_1+t_2+u \approx u$. Hence, from the point of view of provability, it suffices only to consider sound equations of the form $at + u \approx u$ and $x + u \approx u$. We call these the *cover equations*. We present three lemmas that limit the form that cover equations can have for the semantics in the spectrum we study in this paper. (In the statements of the lemmas below, t and u range over the collection of open BCCSP(A) terms.)

Lemma 1. *If $t + x \precsim u$, and either $\precsim \subseteq \precsim_{\mathrm{CT}}$, or $\precsim \subseteq \precsim_{\mathrm{PT}}$ and $|A| > 1$, then x is a summand of u.*

Proof. We distinguish the two cases.
CASE 1: $\precsim \subseteq \precsim_{\mathrm{CT}}$.
Let $\sigma(x) = a^{depth(u)+1}\mathbf{0}$ for some $a \in A$, and $\sigma(y) = \mathbf{0}$ for $y \neq x$. Then $a^{depth(u)+1}$ is a completed trace of $(t + x)^\sigma$, so it must be a completed trace of u^σ. This implies that x is a summand of u.
CASE 2: $\precsim \subseteq \precsim_{\mathrm{PT}}$ and $|A| > 1$.
Let $\sigma(x) = a^{depth(u)}b\mathbf{0}$ for some distinct $a, b \in A$, and $\sigma(y) = \mathbf{0}$ for $y \neq x$. Then $a^{depth(u)}b$ is a partial trace of $(t + x)^\sigma$, so it must be a partial trace of u^σ. This implies that x is a summand of u. □

Remark 2. If $|A| = 1$, then the partial traces preorder and the simulation preorder coincide—see, e.g., [3]. For this special case, Lemma 1 fails. Namely, let $A = \{a\}$. Then $x \precsim ax$ is sound for the partial traces (and simulation) preorder.

Lemma 2. *Let $\simeq$ be an equivalence in the linear time - branching time spectrum. If $at + u + bv \simeq u + bv$ with $a \neq b$, then $at + u \simeq u$.*

This lemma is trivial to check for each of the equivalences in the linear time - branching time spectrum. The key idea is that since $a \neq b$, the non-empty (decorated) traces of at and bu are disjoint, and bu cannot (ready/completed) simulate at.

The following lemma states a kind of cancellation result for the preorders in the spectrum.

Lemma 3. *Let $\precsim$ be a preorder in the linear time - branching time spectrum. If $t + x \precsim u + x$, and x is not a summand of $t + u$, then $t \precsim u$.*

Lemma 3 needs to be proved separately for each preorder in the linear time - branching time spectrum. Despite the naturalness of its statement, which appears obvious, these proofs are not trivial, and quite technical. Fokkink and Nain [9] proved such a lemma for failures semantics, with the aim to obtain an

ω-completeness result for this semantics, and their proof is rather delicate. Since the details of the proof of Lemma 3 are not necessary to understand the main result of the paper, we have collected the proof of that lemma in Appendix B.

Remark 3. The condition in Lemma 3 that x is not a summand of $t + u$ is essential. For instance, $x + x \precsim_{\mathrm{PT}} \mathbf{0} + x$, but $x \not\precsim_{\mathrm{PT}} \mathbf{0}$. And $\mathbf{0} + x \precsim_{\mathrm{CT}} x + x$, but $\mathbf{0} \not\precsim_{\mathrm{CT}} x$.

From the three lemmas above, one can conclude that in order to prove ω-completeness (or ground-completeness) of an equational axiomatization, it suffices to derive all sound equations (or all sound closed equations) of the form

$$at + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \qquad (n \geq 1)$$

and, only for the case of partial traces semantics with $|A| = 1$, all sound equations of the form

$$x + u \approx u \ .$$

In our proof of Theorem 1, we shall therefore focus on showing that the equational axiomatization $\mathcal{A}(E)$ generated by our algorithm is powerful enough to prove all of the sound equations of the above two forms.

5 Proof of Theorem 1

Proof. Let $\precsim$ be a preorder in the linear time - branching time spectrum, with $\precsim_{\mathrm{RS}} \subseteq \precsim$. Let E be a sound and ground-complete inequational axiomatization for BCCSP(A) modulo $\precsim$.

It is not hard, albeit tedious, to see that the equational axiomatization $\mathcal{A}(E)$ is sound for BCCSP(A) modulo $\simeq$. We prove that ω-completeness of E implies ω-completeness of $\mathcal{A}(E)$. The proof that $\mathcal{A}(E)$ is ground-complete is identical, but assumes that all terms that occur in the proof below are closed. (It is well known that if an axiomatization proves a closed (in)equation, then there is a closed proof for that (in)equation.)

We note that, for each of the preorders in the linear time - branching time spectrum, $ar + as + t \precsim u$ if, and only if, both $ar + t \precsim u$ and $as + t \precsim u$. This, together with the presence of the axiom A3, implies that the inequational axiomatization E that we start with can be pre-processed so that there are no multiple a-summands on the left-hand sides of the inequational axioms in E.

Moreover, in view of Lemmas 1 and 3, if $\precsim \subseteq \precsim_{\mathrm{CT}}$ or $|A| > 1$, then variable summands on the left-hand sides of inequational axioms can be omitted. Concluding, in this case we can assume that the inequational axiomatization E that we start with only contains inequational axioms of the form $ap \preccurlyeq \sum_{i=1}^{n} aq_i$ (with $n \geq 1$) or $\mathbf{0} \preccurlyeq q$.

For the case of partial traces semantics with $|A| = 1$, Lemma 1 does not apply. Note, however, that $r + s \precsim_{\mathrm{PT}} u$ if, and only if, both $r \precsim_{\mathrm{PT}} u$ and $s \precsim_{\mathrm{PT}} u$.

Hence, for this special case it suffices to allow also for inequational axioms of the form $x \preccurlyeq q$.

We start with showing that all cover equations of the form $at + u \approx u$ can be derived from $\mathcal{A}(E)$. (Cover equations of the form $x + u \approx u$ will be considered later.) In view of Lemmas 2 and 3, it suffices to only consider those equations where u is of the form $\sum_{i=1}^{n} au_i$ with $n \geq 1$. Let

$$at + \sum_{i=1}^{n} au_i \simeq \sum_{i=1}^{n} au_i \ \ .$$

We show that the corresponding cover equation can be derived from $\mathcal{A}(E)$. It is not hard to see that, for the semantics in the linear time - branching time spectrum, the above equivalence implies

$$at \precsim \sum_{i=1}^{n} au_i \ \ .$$

So by ω-completeness of E,

$$E \vdash at \preccurlyeq \sum_{i=1}^{n} au_i \ \ .$$

We prove, using induction on the length of such a derivation, not counting applications of axioms A1–4, that

$$\mathcal{A}(E) \vdash at + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \ \ .$$

Base case: $t = u_i$ for some i. Trivial using A1–3.

Inductive case: We distinguish two cases, which deal with instantiations of inequational axioms in context.

Case 1: The first step of the derivation is

$$E \vdash aC[p^\sigma] \preccurlyeq aC[q^\sigma] \ \ .$$

That is, $t = C[p^\sigma]$ for some context $C[]$, substitution σ, and inequational axiom $p \preccurlyeq q$. Then clearly $aC[p^\sigma]$ is of the form $D[b(p^\sigma + r)]$ and $aC[q^\sigma]$ is of the form $D[b(q^\sigma + r)]$ for some context $D[]$, action b, and term r.

Since $E \vdash aC[q^\sigma] \preccurlyeq \sum_{i=1}^{n} au_i$ by a shorter derivation, by induction,

$$\mathcal{A}(E) \vdash aC[q^\sigma] + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i \ \ .$$

Furthermore,

$$\mathcal{A}(E) \vdash aC[p^\sigma] + aC[q^\sigma] \approx aC[q^\sigma] \ \ .$$

This equation can indeed be derived from the axiom $b(p+x)+b(q+x) \approx b(q+x)$, which is present in $\mathcal{A}(E)$ for each $b \in A$ according to step B in the algorithm, together with the defining axiom for ready simulation, $b(cx+z)+b(cx+cy+z) \approx b(cx+cy+z)$, which by assumption is present in $\mathcal{A}(E)$ for all $b, c \in A$ (see Remark 1). The derivation of the above equation is by induction on the depth of the occurrence of the context symbol $[]$ within $C[]$.

- Let $[]$ occur at depth zero in $C[]$, i.e., $C[] = [] + r$ for some term r. Let the substitution ρ coincide with σ on variables in p and q, and let $\rho(x) = r$. (Recall that an assumption in step B of the algorithm was that x does not occur in $p+q$.) The derivation simply consists of applying the substitution ρ to the axiom $a(p+x)+a(q+x) \approx a(q+x)$.
- Let $C[] = dC'[] + s$. By induction on the depth of the occurrence of $[]$, $\mathcal{A}(E) \vdash dC'[p^\sigma] + dC'[q^\sigma] \approx dC'[q^\sigma]$. So

$$\begin{array}{rcl}
\mathcal{A}(E) & \vdash & aC[p^\sigma] + aC[q^\sigma] \;=\; a(dC'[p^\sigma]+s) + a(dC'[q^\sigma]+s) \\
& \approx & a(dC'[p^\sigma]+s) + a(dC'[p^\sigma]+dC'[q^\sigma]+s) \\
& \approx & a(dC'[p^\sigma]+dC'[q^\sigma]+s) \\
& \approx & a(dC'[q^\sigma]+s) \;=\; aC[q^\sigma] \;\; .
\end{array}$$

Hence,

$$\begin{array}{rcl}
\mathcal{A}(E) & \vdash & aC[p^\sigma] + \sum_{i=1}^{n} au_i \;\approx\; aC[p^\sigma] + aC[q^\sigma] + \sum_{i=1}^{n} au_i \\
& \approx & aC[q^\sigma] + \sum_{i=1}^{n} au_i \;\approx\; \sum_{i=1}^{n} au_i \;\; ,
\end{array}$$

which was to be shown.

Case 2: The first step of the derivation is

$$E \vdash ap^\sigma \preccurlyeq \sum_{j=1}^{m} aq_j^\sigma \qquad (m \geq 1) \;\; .$$

That is, $t = p^\sigma$ for some substitution σ and inequational axiom $ap \preccurlyeq \sum_{j=1}^{m} aq_j$.

By the soundness of E, clearly $aq_j^\sigma \precsim \sum_{i=1}^{n} au_i$ for $j = 1, \ldots, m$. So by ω-completeness, $E \vdash aq_j^\sigma \preccurlyeq \sum_{i=1}^{n} au_i$ for $j = 1, \ldots, m$. By one of our assumptions, the inequational axioms in E do not contain multiple occurrences of a-summands on their left-hand sides. This implies that each of these derivations is not longer than the derivation of $E \vdash \sum_{j=1}^{m} aq_j^\sigma \preccurlyeq \sum_{i=1}^{n} au_i$. So by induction,

$$\mathcal{A}(E) \vdash aq_j^\sigma + \sum_{i=1}^{n} au_i \approx \sum_{i=1}^{n} au_i$$

for $j = 1, \ldots, m$. Furthermore, according to step A of the algorithm, the axiom $p + \sum_{j=1}^{m} aq_j \approx \sum_{j=1}^{m} aq_j$ is present in $\mathcal{A}(E)$. Hence,

$$\begin{aligned}\mathcal{A}(E) \quad \vdash \quad & ap^\sigma + \sum_{i=1}^{n} au_i \;\approx\; ap^\sigma + \sum_{j=1}^{m} aq_j^\sigma + \sum_{i=1}^{n} au_i \\ \approx \quad & \sum_{j=1}^{m} aq_j^\sigma + \sum_{i=1}^{n} au_i \;\approx\; \sum_{i=1}^{n} au_i \ .\end{aligned}$$

This completes the proof for the case of cover equations of the form $at + \sum_{i=1}^{n} au_i \simeq \sum_{i=1}^{n} au_i$.

It remains to prove that cover equations of the form $x+u \approx u$ can be derived from $\mathcal{A}(E)$. If $\precsim \subseteq \precsim_{\mathrm{CT}}$ or $|A| > 1$, then in view of Lemma 1, such cover equations can be derived using A3. So we are left to consider the special case that $\precsim = \precsim_{\mathrm{PT}}$ and $|A| = 1$. Let

$$x + u \simeq_{\mathrm{PT}} u \ .$$

Clearly, this implies

$$x \precsim_{\mathrm{PT}} u \ .$$

So, by ω-completeness of E,

$$E \vdash x \preccurlyeq u \ .$$

We prove, using induction on the length of such a derivation, not counting applications of A1–4, that

$$\mathcal{A}(E) \vdash x + u \approx u \ .$$

Base case: x is a summand of u. Trivial.

Inductive case: The first step of the derivation is

$$E \vdash y^\sigma \preccurlyeq q^\sigma \ .$$

That is, $\sigma(y) = x$ for some substitution σ and inequational axiom $y \preccurlyeq q$ in E.

By the soundness of E, clearly $r \precsim_{\mathrm{PT}} u$ for each summand r of q^σ. So by ω-completeness, $E \vdash r \preccurlyeq u$. By assumption, the inequational axioms in E are all of the form $as \preccurlyeq \sum_{i=1}^{n} as_i$ (with $n \geq 1$) or $\mathbf{0} \preccurlyeq s$ or $z \preccurlyeq s$, for some variable z. This implies that each of these derivations is not longer than the derivation of $E \vdash q^\sigma \preccurlyeq u$. So by induction and A3,

$$\mathcal{A}(E) \vdash q^\sigma + u \approx u \ .$$

Furthermore, according to step A of the algorithm, the axiom $y+q \approx q$ is present in $\mathcal{A}(E)$. Hence,

$$\mathcal{A}(E) \ \vdash \ y^\sigma + u \ \approx \ y^\sigma + q^\sigma + u \ \approx \ q^\sigma + u \ \approx \ u \ .$$

The proof of the theorem is now complete. □

6 Examples

We show how our algorithm produces equational axiomatizations for three equivalences in the linear time - branching time spectrum—namely simulation, failures and partial traces equivalence—from the inequational axiomatizations for the corresponding preorders. For the simulation and partial traces preorders, we leave out the pre-supposed inequational axiom $ax \preccurlyeq ax + ay$, since it can be derived from the defining inequational axioms for these preorders.

6.1 Simulation

Let $|A| > 1$. Then A1–4 plus one inequational axiom

$$\mathbf{0} \preccurlyeq x$$

is a sound and ground-complete axiomatization for BCCSP(A) modulo the simulation preorder [12].

Step A of the algorithm produces the already present axiom A4:

$$\mathbf{0} + x \approx x \ \ .$$

Step B of the algorithm produces the defining axioms for simulation equivalence for each $b \in B$:

$$b(\mathbf{0} + y) + b(x + y) \approx b(x + y) \ \ .$$

6.2 Failures

Let $|A| \geq 1$. The axiomatization consisting of A1–4 plus one inequational axiom

$$a(x + y) \preccurlyeq ax + a(y + z)$$

for each $a \in A$ is sound and ground-complete for BCCSP(A) modulo the failures preorder [12].

Step A of the algorithm produces, for all $a \in A$:

$$a(x + y) + ax + a(y + z) \approx ax + a(y + z) \ \ .$$

This axiom is one of the two defining axioms for failures equivalence. (The second defining axiom for failures equivalence is the ready simulation axiom, which is assumed to be present from the start.)

Step B of the algorithm produces, for all $a, b \in A$:

$$b(a(x + y) + w) + b(ax + a(y + z) + w) \approx b(ax + a(y + z) + w) \ \ .$$

This axiom is redundant; it can be derived from the other axioms as follows. (The subterm to which an axiom is applied is underlined.)

$$\begin{array}{ll} & b(\underline{ax + a(y+z)} + w) \\ \approx & \underline{b(a(x+y) + ax + a(y+z) + w)} \\ \approx & \underline{b(a(x+y) + a(y+z) + w)} + b(a(x+y) + ax + a(y+z) + w) \\ \approx & b(a(x+y) + w) + \underline{b(a(x+y) + a(y+z) + w) + b(a(x+y) + ax + a(y+z) + w)} \\ \approx & b(a(x+y) + w) + b(\underline{a(x+y) + ax + a(y+z)} + w) \\ \approx & b(a(x+y) + w) + b(ax + a(y+z) + w) \end{array}$$

6.3 Partial traces

Let $|A| > 1$. The axiomatization consisting of A1–4 plus

$$\begin{array}{c} \mathbf{0} \preccurlyeq x \quad \text{and} \\ ax + ay \approx a(x+y) \quad \text{(one axiom for each } a \in A) \end{array}$$

is sound and ground-complete for BCCSP(A) modulo the partial traces preorder [12].

Each axiom of the latter form, for $a \in A$, is split into two inequational axioms:

$$\begin{array}{r} ax \preccurlyeq a(x+y) \\ a(x+y) \preccurlyeq ax + ay \ \ . \end{array}$$

Step A of the algorithm produces, for each $a \in A$:

$$\begin{array}{r} \mathbf{0} + x \approx x \\ ax + a(x+y) \approx a(x+y) \\ a(x+y) + ax + ay \approx ax + ay \ \ . \end{array}$$

The first of these axioms coincides with A4, which is assumed to be present from the start.

Step B of the algorithm produces, for all $a, b \in A$:

$$\begin{array}{r} b(\mathbf{0} + y) + b(x+y) \approx b(x+y) \\ b(ax + z) + b(a(x+y) + z) \approx b(a(x+y) + z) \\ b(a(x+y) + z) + b(ax + ay + z) \approx b(ax + ay + z) \ \ . \end{array}$$

Note that the defining axiom for partial traces equivalence, namely $ax + ay \approx a(x+y)$ for $a \in A$, can be derived from the second and third axiom that were produced in step A as follows:

$$\underline{ax + ay} \ \approx \ \underline{a(x+y) + ax} + ay \ \approx \ \underline{a(x+y) + ay} \ \approx \ a(x+y) \ \ .$$

In turn, from this defining axiom and A3–4, one can derive the second and third axiom that were produced in step A, and the three axioms that were produced in step B, for all $a, b \in A$.

7 Conclusions and Comparison with Related Work

In this paper, we have offered an algorithm for generating a ground-complete (respectively, ω-complete) axiomatization for behavioural equivalences in the linear time - branching time spectrum starting from a ground-complete (respectively, ω-complete) axiomatization for their underlying preorders—that is, of the preorders that have the equivalences as their kernels. Our algorithm applies to all of the process semantics in the spectrum whose discriminating power lies in between that of ready simulation semantics and of partial traces semantics. Moreover, in the presence of a finite set of actions, our procedure preserves finiteness of axiomatizations, and thus can be used to obtain automatically finite basis results for behavioural equivalences in the spectrum from similar results for their underlying preorders. In fact, our results apply to *any* behavioural precongruence whose discriminating power lies in between that of the ready simulation preorder and of the partial traces preorder, provided that Lemmas 1–3 hold for the precongruence in question.

Our algorithm may thus be considered as isolating and axiomatizing the ingredients that all of the extant proofs of completeness results for the class of behavioural equivalences we study have in common. (See, for example, the references [4, 5, 7–9, 12, 13] for a sample of such results.) It also eliminates the need to reprove, essentially from scratch, completeness results for a large fragment of behavioural equivalences in the spectrum once a completeness result has been obtained for their underlying preorders. As witnessed by the examples we provided in Section 6, the axiomatizations that are automatically generated by our algorithm are very similar, when not identical, to those presented in the literature. In this respect, this study may be seen as a companion to [1]. That paper offered an algorithm that generates a finite, ground-complete axiomatization for bisimulation equivalence from an operational specification of a language in GSOS format [6]. That procedure relies on the axiomatization of bisimulation equivalence over the language BCCSP. Here we have focused on the algorithmic generation of complete axiomatizations for other equivalences in the spectrum over the language BCCSP.

The spirit of our study is also very similar to the one in the unpublished paper [11]. In that reference, independently of our work and building on their previous paper [10], de Frutos Escrig and Rodriguez show, amongst other things, how to generate an inequational axiomatization for preorders in the spectrum from equational axiomatizations for the corresponding equivalence. They generate this inequational axiomatization by simply adding the defining inequational axioms for the ready simulation preorder to the axiomatization for the equivalence—see Theorem 6 in [11]. That result applies to behavioural equivalences in the linear time - branching time spectrum that

1. include ready simulation equivalence, and
2. whose underlying preorders only equate processes having the same set of initial actions.

The latter condition is not met by completed simulation, simulation, completed traces and partial traces semantics. Furthermore, the aforementioned result from [11] only applies to ground-complete axiomatizations.

There are some interesting general connections between the technical developments in this paper and those in [11]. For instance, Lemma 1 in [11] gives a soundness proof for the equations generated by step A in our algorithm for the preorders in the spectrum that satisfy condition 2 above. However, the equations generated by step A are sound also for completed simulation, simulation, completed traces and partial traces semantics. So Lemma 1 in [11] is not as general as it could be.

It would also be interesting to investigate the possible relation between the cover equations approach, used in this paper to reduce the class of equations to be considered in the proof of completeness, and the condition of action factorization mentioned in the statement of Theorem 1 of [11]. (Action factorization means that if $p \precsim q$, then, for each action a, the sum of the a-summands of p is also dominated by the sum of the a-summands of q with respect to $\precsim$.)

In summary, our work differs from that in [11] in the following fundamental ways.

- We show how to produce automatically an equational axiomatization for an equivalence from an inequational axiomatization of its underlying preorder. Since the equivalences in the linear time - branching time spectrum that include ready simulation equivalence are the kernels of their underlying preorders, to our mind, the preorders are a more basic notion to build on in this setting.
- Unlike Theorem 1 of [11], our main result applies to all of the semantics in the spectrum whose discriminating power lies in between that of ready simulation semantics and partial traces semantics.
- Last, but not least, unlike Theorem 1 of [11], our results apply to ω-complete as well as to ground-complete axiomatizations.

It would be interesting to extend our algorithm so that it applies also to nested simulation semantics [14] and to possible futures semantics [19]. However, as shown in [2], unlike the semantics we have considered in this study, nested simulation and possible futures semantics afford no finite ground-complete axiomatization over BCCSP even in the presence of a single action. This indicates that such a generalization of our results will not be easy to achieve without recourse to conditional equations. We leave such generalizations of our results and proof techniques as a topic for future investigations.

References

1. L. Aceto, B. Bloom and F.W. Vaandrager. Turning SOS rules into equations. *Information and Computation*, 111(1):1–52, 1994.
2. L. Aceto, W. Fokkink, R.J. van Glabbeek, and A. Ingolfsdottir. Nested semantics over finite trees are equationally hard. *Information and Computation*, 191(2):203–232, 2004.

3. L. Aceto, W. Fokkink, and A. Ingolfsdottir. A menagerie of non-finitely based process semantics over BPA*—from ready simulation to completed traces. *Mathematical Structures in Computer Science*, 8(3):193–230, 1998.
4. L. Aceto, W. Fokkink, A. Ingolfsdottir, and B. Luttik. Finite equational bases in process algebra: Results and open questions. In *Processes, Terms and Cycles: Steps on the Road to Infinity*, LNCS 3838, pp. 338–367. Springer, 2005.
5. S. Blom, W. Fokkink, and S. Nain. On the axiomatizability of ready traces, ready simulation and failure traces. In *Proc. ICALP'03*, LNCS 2719, pp. 109–118. Springer, 2003.
6. B. Bloom, S. Istrail, and A. Meyer. Bisimulation can't be traced. *Journal of the ACM*, 42:232–268, 1995.
7. T. Chen and W. Fokkink. On finite alphabets and infinite bases III: Simulation. *Proc. CONCUR'06*, LNCS 4137, pp. 421–434. Springer, 2006.
8. T. Chen, W. Fokkink and S. Nain. On finite alphabets and infinite bases II: Completed and ready simulation. In *Proc. FOSSACS'06*, LNCS 3921, pp. 1–15. Springer, 2006.
9. W. Fokkink and S. Nain. A finite basis for failure semantics. In *Proc. ICALP'05*, LNCS 3580, pp. 755–765. Springer, 2005.
10. D. de Frutos Escrig and C.G. Rodriguez. Bisimulations up-to for the linear time branching time spectrum. *Proc. CONCUR 2005*, LNCS 3653, pp. 278–292. Springer, 2005.
11. D. de Frutos Escrig and C.G. Rodriguez. Axiomatization of canonical preorders via simulations up-to. Unpublished manuscript, 2006. (Private communication.)
12. R.J. van Glabbeek. The linear time – branching time spectrum I. The semantics of concrete, sequential processes. In J.A. Bergstra, A. Ponse, and S.A. Smolka, eds, *Handbook of Process Algebra*, pp. 3–99. Elsevier, 2001.
13. J.F. Groote. A new strategy for proving ω-completeness with applications in process algebra. In *Proc. CONCUR'90*, LNCS 458, pp. 314–331. Springer, 1990.
14. J.F. Groote and F.W. Vaandrager, Structured operational semantics and bisimulation as a congruence, *Information and Computation*, 100 (1992), pp. 202–260.
15. M.C.B. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.
16. C.A.R. Hoare. *Communicating Sequential Processes.* Prentice-Hall, 1985.
17. R. Milner. *Communication and Concurrency.* Prentice-Hall, 1989.
18. F. Moller. *Axioms for Concurrency.* PhD thesis, Department of Computer Science, University of Edinburgh, July 1989. Report CST-59-89. Also published as ECS-LFCS-89-84.
19. W. Rounds and S. Brookes. Possible futures, acceptances, refusals and communicating processes. In *Proc. FOCS'81*, pp. 140–149. IEEE, 1981.

A The Linear Time - Branching Time Spectrum

Van Glabbeek presented in [12] the linear time - branching time spectrum of behavioural semantics for finitely branching, concrete processes. In this section, for the sake of completeness, we define the semantics in this spectrum.

A *labelled transition system* contains a set of *states*, with typical element s, and a set of transitions $s \xrightarrow{a} s'$, where a ranges over some set of labels A. The set $\mathcal{I}(s)$ of *initial actions* of s consists of those labels a for which there exists a transition $s \xrightarrow{a} s'$.

First we define four variations on the notion of simulation.

Definition 1 (Simulations). *Assume a labelled transition system.*

- *A binary relation* $\mathcal{R}$ *on states is a* simulation *if* $s_0 \,\mathcal{R}\, s_1$ *and* $s_0 \xrightarrow{a} s_0'$ *imply* $s_1 \xrightarrow{a} s_1'$ *for some* s_1' *with* $s_0' \,\mathcal{R}\, s_1'$.
- *A simulation* $\mathcal{R}$ *is a* completed simulation *if* $s_0 \,\mathcal{R}\, s_1$ *and* $\mathcal{I}(s_0) = \emptyset$ *imply* $\mathcal{I}(s_1) = \emptyset$.
- *A simulation* $\mathcal{R}$ *is a* ready simulation *if* $s_0 \,\mathcal{R}\, s_1$ *and* $a \notin \mathcal{I}(s_0)$ *imply* $a \notin \mathcal{I}(s_1)$.
- *A* bisimulation *is a symmetric simulation.*

Next we define six types of decorated versions of execution traces.

Definition 2 (Decorated Traces). *Assume a labelled transition system.*

- *A sequence* $a_1 \cdots a_n$, *with* $n \geq 0$, *is a* (partial) trace *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots s_{n-1} \xrightarrow{a_n} s_n$. *It is a* completed trace *of* s_0 *if moreover* $\mathcal{I}(s_n) = \emptyset$.
- *A pair* $(a_1 \cdots a_n, X)$, *with* $n \geq 0$ *and* $X \subseteq A$, *is a* ready pair *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots s_{n-1} \xrightarrow{a_n} s_n$ *with* $\mathcal{I}(s_n) = X$. *It is a* failure pair *of* s_0 *if* $\mathcal{I}(s_n) \cap X = \emptyset$.
- *A sequence* $X_0 a_1 X_1 \ldots a_n X_n$, *with* $n \geq 0$ *and* $X_i \subseteq A$, *is a* ready trace *of a state* s_0 *if there is a sequence of transitions* $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots s_{n-1} \xrightarrow{a_n} s_n$ *with* $\mathcal{I}(s_i) = X_i$ *for* $i = 0, \ldots, n$. *It is a* failure trace *of* s_0 *if* $\mathcal{I}(s_i) \cap X_i = \emptyset$ *for* $i = 0, \ldots, n$.

In what follows, we shall often write $s_0 \xrightarrow{a_1 \ldots a_n} s_n$ if there is a sequence of transitions $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots s_{n-1} \xrightarrow{a_n} s_n$, and $s_0 \xrightarrow{a_1 \ldots a_n}$ if there is some s_n such that $s_0 \xrightarrow{a_1 \ldots a_n} s_n$.

Finally, we define the notion of a possible world of a process term.

Definition 3 (Possible Worlds). *Assume a labelled transition system. A state* s *is* deterministic *if for each* $a \in \mathcal{I}(s)$ *there is exactly one state* s' *such that* $s \xrightarrow{a} s'$, *and moreover* s' *is deterministic.*

A state s *is a* possible world *of a state* s_0 *if* s *is deterministic and* $s \,\mathcal{R}\, s_0$ *for some ready simulation* $\mathcal{R}$.

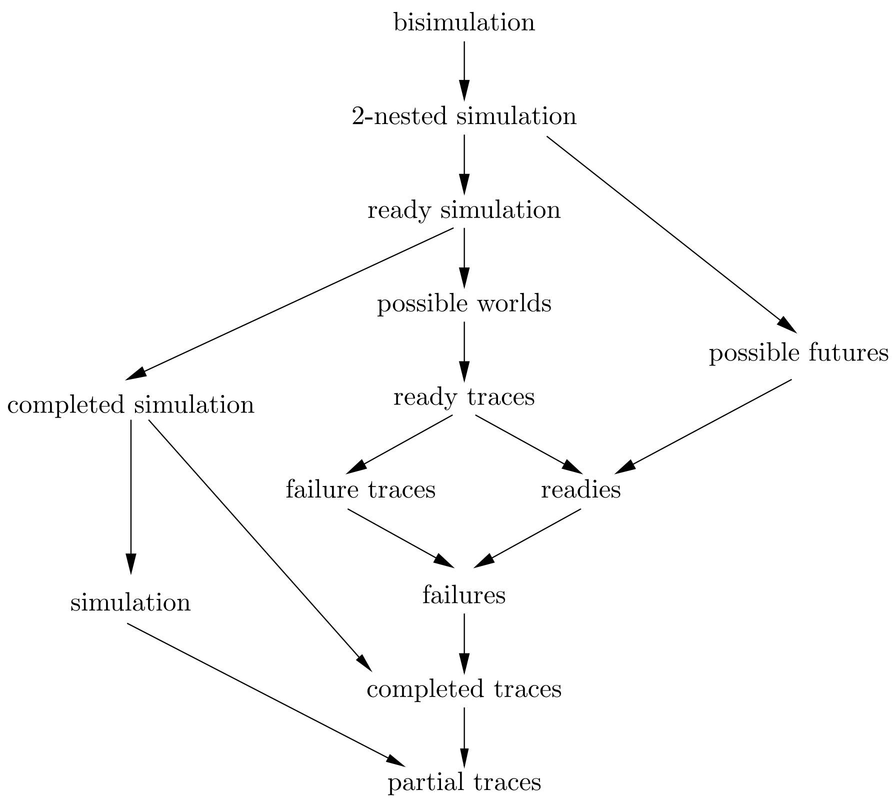

Fig. 1. The linear time - branching time spectrum

Two states s and s' are related by the *simulation*, *ready simulation*, or *completed simulation preorder* if there exists a simulation, ready simulation, or completed simulation $\mathcal{R}$, respectively, with $s \, \mathcal{R} \, s'$. They are *bisimilar* if there is a bisimulation that relates them. They are related by the *possible worlds*, *ready traces*, *failure traces*, *readies*, *failures*, *completed traces*, or *partial traces preorder* if the set of possible worlds, ready traces, failure traces, ready pairs, failure pairs, completed traces, or traces of the former is included in that of the latter, respectively.

Figure 1 depicts the linear time - branching time spectrum, where a directed edge from one semantics to another means that the source of the edge is finer than the target. We use $\precsim$ to denote a preorder in this spectrum, and $\simeq$ to denote the corresponding equivalence. When we want to refer to a specific preorder in the spectrum, we shall subscribe the symbol $\precsim$ with the initials of the intended semantics in the spectrum.

We note that for each of the preorders in the spectrum, if $p \precsim q$, then $depth(p) \leq depth(q)$.

B Proof of Lemma 3

In this section, we collect the proof of Lemma 3 in the main body of the paper for each of the behavioural preorders in the linear time - branching time spectrum ranging between the ready simulation and partial traces preorders. Throughout this section, we use $\sigma_{\mathbf{0}}$ to stand for the substitution mapping each variable to $\mathbf{0}$. For each closed substitution σ, variable x, and closed term p, we use the notation $\sigma[x \mapsto p]$ to stand for the substitution mapping x to p, and acting like σ on all of the other variables.

B.1 Proof of Lemma 3 for $\precsim_{\mathrm{CT}}$

We begin our proof of Lemma 3 for $\precsim_{\mathrm{CT}}$ by stating a couple of useful lemmas.

Lemma 4. *Let $t = \sum_{i\in I} x_i$ and $u = \sum_{k\in K} b_k.u_k + \sum_{j\in J} y_j$. Then $t \precsim_{\mathrm{CT}} u$ iff $K = \emptyset$ and $\{x_i \mid i \in I\} = \{y_j \mid j \in J\}$.*

Proof. The "if" implication is trivial, since then t and u are bisimilar. We therefore focus on establishing the implication from left to right. First note that K must be empty because otherwise $\sigma_{\mathbf{0}}(u)$ would not have the empty string ε as one of its completed traces, contradicting $t \precsim_{\mathrm{CT}} u$. We now prove that $\{x_i \mid i \in I\} = \{y_j \mid j \in J\}$.

To this end, we begin by observing that each x_i must occur as a summand of u by Lemma 1. We are therefore left to prove that each y_j is also a summand of t. To see that this does hold, pick an action $a \in A$, and consider the closed substitution $\sigma = \sigma_{\mathbf{0}}[y_j \mapsto a\mathbf{0}]$. The only completed trace of $\sigma(u)$ is a. It follows that y_j must be a summand of t. Indeed, if y_j is not a summand of t, then $\sigma(t) = \sigma_{\mathbf{0}}(t) = \mathbf{0}$ has only the empty string ε as completed trace, contradicting $t \precsim_{\mathrm{CT}} u$. □

Lemma 5. *Let σ be a closed substitution. Then $\sigma(t) \stackrel{a_1\ldots a_n}{\rightarrow} \mathbf{0}$, for some sequence of actions $a_1 \ldots a_n$ and $n \geq 0$, iff there are a $j \leq n$ and a term t' such that $t \stackrel{a_1\ldots a_j}{\rightarrow} t'$ and*

1. *either $j = n$ and $\sigma(t') = \mathbf{0}$*
2. *or $j < n$ and $\sigma(x) \stackrel{a_{j+1}\ldots a_n}{\rightarrow} \mathbf{0}$, for some summand x of t'.*

Proof. Both statements can be shown by induction on the structure of t. The details are tedious, but not hard, and are therefore omitted. □

We are now ready to prove that Lemma 3 holds for $\precsim_{\mathrm{CT}}$.

Proof of Lemma 3 for $\precsim_{\mathrm{CT}}$ Assume that $t + x \precsim_{\mathrm{CT}} u + x$, and x is not a summand of $t+u$. Let σ be a closed substitution. We prove that each completed trace of $\sigma(t)$ is also a completed trace of $\sigma(u)$. This is immediate from the proviso of the lemma if $\sigma(x) = \mathbf{0}$. Assume therefore that $\sigma(x) \neq \mathbf{0}$.

Let $a_1 \ldots a_n$ be a completed trace of $\sigma(t)$—that is, $\sigma(t) \stackrel{a_1\ldots a_n}{\rightarrow} \mathbf{0}$. If $n = 0$, then $\sigma(t) = \mathbf{0}$. This means that $t = \sum_{i\in I} x_i$ for some set of variables $\{x_i \mid i \in I\}$

such that $\sigma(x_i) = \mathbf{0}$ for each $i \in I$. Note that, by the proviso of the lemma, $x \neq x_i$ for each $i \in I$. Since $t + x \precsim_{\mathrm{CT}} u + x$, Lemma 4 yields that $u = t$, and therefore $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$.

Assume now that $n \geq 1$. Since $\sigma(t) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, Lemma 5 yields that there are a $j \leq n$ and a term t' such that $t \stackrel{a_1 \ldots a_j}{\longrightarrow} t'$ and

1. either $j = n$ and $\sigma(t') = \mathbf{0}$
2. or $j < n$ and $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow} \mathbf{0}$, for some summand y of t'.

In the former case, $t' = \sum_{m \in M} z_m$ for some collection $\{z_m \mid m \in M\}$ of variables such that $\sigma(z_m) = \mathbf{0}$ for each $m \in M$. By assumption, $\sigma(x) \neq \mathbf{0}$, so $z_m \neq x$ for each $m \in M$. We wish to argue that $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$. Let $\ell > n$. By Lemma 5,

$$\sigma[x \mapsto a^\ell \mathbf{0}](t) \stackrel{a_1 \ldots a_n}{\longrightarrow} \sigma[x \mapsto a^\ell \mathbf{0}](t') = \sigma(t') = \mathbf{0}\ .$$

Since $t + x \precsim_{\mathrm{CT}} u + x$ and $n \geq 1$, $\sigma[x \mapsto a^\ell \mathbf{0}](u + x)$ also affords $a_1 \ldots a_n$ as one of its completed traces. As $\ell > n$, it follows that $\sigma[x \mapsto a^\ell \mathbf{0}](u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$. Using Lemma 5 and the assumption that $\ell > n$, we may conclude that $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, which was to be shown.

In the latter case, it suffices to show that $u \stackrel{a_1 \ldots a_j}{\longrightarrow} u'$ for some u' that has y as a summand. Let $N > \mathit{depth}(u)$. By Lemma 5, $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](t)$ affords the completed trace $a_1 \ldots a_j a^N$. Since $j + N \geq 1$, $a_1 \ldots a_j a^N$ is also a completed trace of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](t + x)$, and therefore of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](u + x)$. Note that if $j = 0$, then $y \neq x$ because x is not a summand of t by the proviso of the lemma. Hence it follows that $a_1 \ldots a_j a^N$ is also a completed trace of $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](u)$. Let $b_1 \ldots b_{N+j} = a_1 \ldots a_j a^N$. Since $N > \mathit{depth}(u)$, by Lemma 5, $u \stackrel{b_1 \ldots b_k}{\longrightarrow} u'$ and $\sigma_{\mathbf{0}}[y \mapsto a^N \mathbf{0}](z) \stackrel{b_{k+1} \ldots b_{N+j}}{\longrightarrow} \mathbf{0}$ for some term u', variable z and $k < N$, where u' has z as a summand. Since $N + j > k$, it follows that $z = y$, $k = j$ and $b_{k+1} \ldots b_{N+j} = a^N$. Concluding, $u \stackrel{a_1 \ldots a_j}{\longrightarrow} u'$ where u' has y as a summand. Since $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow} \mathbf{0}$ and $j < n$, by Lemma 5, $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow} \mathbf{0}$, which was to be shown.

This concludes the proof. □

B.2 Proof of Lemma 3 for the Simulation Preorders

In this section, we collect the proof of Lemma 3 for the ready simulation, completed simulation and simulation preorders.

Proof of Lemma 3 for $\precsim_{\mathbf{RS}}$ Assume that $t + x \precsim_{\mathrm{RS}} u + x$, and x is not a summand of $t + u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{RS}} \sigma(u)$.

In order to prove that $\sigma(t) \precsim_{\mathrm{RS}} \sigma(u)$, we need to show the following two claims:

1. if $\sigma(t) \xrightarrow{a} p$, then $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{RS}} q$, and
2. $\mathcal{I}(\sigma(u)) \subseteq \mathcal{I}(\sigma(t))$.

We prove these two claims separately.

Proof of Claim 1. Suppose that $\sigma(t) \xrightarrow{a} p$. Either this transition is due to a variable summand y of t such that $\sigma(y) \xrightarrow{a} p$, or there is a summand at' of t such that $p = \sigma(t')$. In the former case, $y \neq x$ by the proviso of the lemma. Therefore, by Lemma 1, y is also a summand of u. It follows that $\sigma(u) \xrightarrow{a} p$, and we are done.

Suppose now that there is a summand at' of t such that $p = \sigma(t')$. If $\sigma(y) \xrightarrow{a} q$ for some variable summand y of u and closed term q such that $p \precsim_{\mathrm{RS}} q$, we are done. Assume therefore that, for each r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{RS}} r \ . \tag{1}$$

We claim that $p \precsim_{\mathrm{RS}} \sigma(u')$ for some summand au' of u. We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

– Case x does not occur in t'. Let $N \geq \mathit{depth}(\sigma(t))$. Define the closed term s as follows:

$$s = \sum_{b \in \mathcal{I}(\sigma(x))} ba^N\mathbf{0} \ .$$

Since at' is a summand of t and x does not occur in t',

$$\sigma[x \mapsto s](t + x) \xrightarrow{a} \sigma[x \mapsto s](t') = \sigma(t') = p \ .$$

Therefore

$$\sigma[x \mapsto s](u + x) \xrightarrow{a} q \ ,$$

for some q such that $p \precsim_{\mathrm{RS}} q$. Note that $p \not\precsim_{\mathrm{RS}} a^N\mathbf{0}$, because $\mathit{depth}(p) < N$ and $a^N\mathbf{0}$ affords only one completed trace whose length is N. Hence, by assumption (1), u must have a summand of the form au' such that

$$p \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u') \ .$$

We now prove, by induction on the depth of p, that

$$p \precsim_{\mathrm{RS}} \sigma(u') \ .$$

First of all, note that $\mathcal{I}(\sigma(x)) = \mathcal{I}(s)$ implies $\mathcal{I}(\sigma(u')) = \mathcal{I}(\sigma[x \mapsto s](u')) = \mathcal{I}(p)$.

Suppose that $p \xrightarrow{b} p'$. We prove that $\sigma(u') \xrightarrow{b} q'$ for some q' such that $p' \precsim_{\mathrm{RS}} q'$. Since $p \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u')$, there is a q'' such that $\sigma[x \mapsto s](u') \xrightarrow{b} q''$ and $p' \precsim_{\mathrm{RS}} q''$. Note that this transition cannot be due to a summand x of u', because $\mathit{depth}(p') \leq N - 2$. If the transition $\sigma[x \mapsto s](u') \xrightarrow{b} q''$ is due to a variable summand $y \neq x$ of u', then $\sigma(u') \xrightarrow{b} q''$ also holds, and we are done. Otherwise, there is a summand bu'' of u' such that $q'' = \sigma[x \mapsto s](u'')$. As $p' \precsim_{\mathrm{RS}} \sigma[x \mapsto s](u'')$, the induction hypothesis yields that $p' \precsim_{\mathrm{RS}} \sigma(u'')$. Since $\sigma(u') \xrightarrow{b} \sigma(u'')$, we are done.

Therefore $p \precsim_{\mathrm{RS}} \sigma(u')$, as claimed above. Since $\sigma(u) \xrightarrow{a} \sigma(u')$, we are done.

– CASE x OCCURS IN t'. In this case,

$$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \ .$$

Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. Since $\sigma(t + x) \precsim_{\mathrm{RS}} \sigma(u + x)$ and $\sigma(t + x) \xrightarrow{a} p$, it follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{RS}} \sigma(u')$ for some summand au' of u.

PROOF OF CLAIM 2. Assume that $a \in \mathcal{I}(\sigma(u))$. Since x is not a summand of u by the proviso of the lemma, $a \in \mathcal{I}(\sigma[x \mapsto \mathbf{0}](u + x))$. As $t + x \precsim_{\mathrm{RS}} u + x$, it follows that $a \in \mathcal{I}(\sigma[x \mapsto \mathbf{0}](t + x))$. Using the assumption that x is not a summand of t, we may conclude that $a \in \mathcal{I}(\sigma(t))$, which was to be shown.

This concludes the proof. $\square$

Proof of Lemma 3 for $\precsim_{\mathrm{CS}}$ Assume that $t + x \precsim_{\mathrm{CS}} u + x$, and x is not a summand of $t + u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{CS}} \sigma(u)$.

In order to prove that $\sigma(t) \precsim_{\mathrm{CS}} \sigma(u)$, we need to show the following two claims:

1. if $\sigma(t) \xrightarrow{a} p$, then $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{CS}} q$, and
2. if $\sigma(t) = \mathbf{0}$, then $\sigma(u) = \mathbf{0}$.

We prove these two claims separately.

PROOF OF CLAIM 1. Suppose that $\sigma(t) \xrightarrow{a} p$. We show that $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{CS}} q$. This is immediate if there is a variable summand y of u such that $\sigma(y) \xrightarrow{a} q$ and $p \precsim_{\mathrm{CS}} q$.

Assume therefore that, for each r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{CS}} r \ . \tag{2}$$

By Lemma 1, it follows that $\sigma(t) \xrightarrow{a} p$ because t has a summand at' such that $\sigma(t') = p$.

We claim that $p \precsim_{\mathrm{CS}} \sigma(u')$ for some summand au' of u. We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

– CASE x DOES NOT OCCUR IN t'. Let $N > depth(\sigma(t))$. Since at' is a summand of t and x does not occur in t,

$$\sigma[x \mapsto a^N\mathbf{0}](t + x) \xrightarrow{a} \sigma[x \mapsto a^N\mathbf{0}](t') = \sigma(t') = p \ .$$

Therefore

$$\sigma[x \mapsto a^N\mathbf{0}](u + x) \xrightarrow{a} q \ ,$$

for some q such that $p \precsim_{\mathrm{CS}} q$. Note that $p \not\precsim_{\mathrm{CS}} a^{N-1}\mathbf{0}$, because $depth(p) < N - 1$ and $a^{N-1}\mathbf{0}$ affords only a completed trace of length $N - 1$. Hence, by assumption (2), u must have a summand of the form au' such that

$$p \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u') \ .$$

We now prove, by induction on the depth of p, that

$$p \precsim_{\mathrm{CS}} \sigma(u') \ .$$

First of all, if $p = \mathbf{0}$, then $\sigma[x \mapsto a^N\mathbf{0}](u') = \mathbf{0}$, which yields that x does not occur in u'. Therefore, $\sigma[x \mapsto a^N\mathbf{0}](u') = \sigma(u')$, and we are done.
Suppose that $p \xrightarrow{b} p'$. We prove that $\sigma(u') \xrightarrow{b} q'$ for some q' such that $p' \precsim_{\mathrm{CS}} q'$. Since $p \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u')$, there is a q'' such that $\sigma[x \mapsto a^N\mathbf{0}](u') \xrightarrow{b} q''$ and $p' \precsim_{\mathrm{CS}} q''$. Note that this transition cannot be due to a summand x of u' because $depth(p') < N - 2$. If the transition $\sigma[x \mapsto a^N\mathbf{0}](u') \xrightarrow{b} q''$ is due to a variable summand $y \neq x$ of u', then $\sigma(u') \xrightarrow{b} q''$ also holds, and we are done. Otherwise, there is a summand bu'' of u' such that

$$q'' = \sigma[x \mapsto a^N\mathbf{0}](u'') \ .$$

As $p' \precsim_{\mathrm{CS}} \sigma[x \mapsto a^N\mathbf{0}](u'')$, the induction hypothesis yields that $p' \precsim_{\mathrm{CS}} \sigma(u'')$. Since $\sigma(u') \xrightarrow{b} \sigma(u'')$, we are done.
Therefore $p \precsim_{\mathrm{CS}} \sigma(u')$, as claimed above. Since $\sigma(u) \xrightarrow{a} \sigma(u')$, we are done.

– Case x occurs in t'. In this case,

$$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \ .$$

Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. As $\sigma(t + x) \precsim_{\mathrm{CS}} \sigma(u + x)$ by the proviso of the lemma, and $\sigma(t + x) \xrightarrow{a} p$, it follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{CS}} \sigma(u')$ for some summand au' of u.

Proof of Claim 2. Assume that $\sigma(t) = \mathbf{0}$. This means that t is a sum of variables, and $t = u$ by Lemma 4. (Recall that $\precsim_{\mathrm{CS}} \subseteq \precsim_{\mathrm{CT}}$; see Figure 1 on page 18.) Hence $\sigma(u) = \mathbf{0}$.

This concludes the proof. □

Proof of Lemma 3 for $\precsim_{\mathrm{S}}$ Assume that $t+x \precsim_{\mathrm{S}} u+x$, and x is not a summand of $t + u$. Let σ be a closed substitution. We prove that $\sigma(t) \precsim_{\mathrm{S}} \sigma(u)$.

Suppose that $\sigma(t) \xrightarrow{a} p$. We show that $\sigma(u) \xrightarrow{a} q$ for some q such that $p \precsim_{\mathrm{S}} q$.

Assume, first of all, that the transition $\sigma(t) \xrightarrow{a} p$ is due to a variable summand y of t, that is $\sigma(y) \xrightarrow{a} p$. In this case, $y \neq x$ by the proviso of the lemma. If $|A| > 1$, then y is also a variable summand of u (Lemma 1), and we are done because $\sigma(u) \xrightarrow{a} p$. If $A = \{a\}$, then Lemma 1.3 in [7] yields that y occurs in u. Therefore, $u \xrightarrow{a^n} u'$ for some $n \geq 0$ and term u' having y as a summand. If $n = 0$, then $\sigma(u) \xrightarrow{a} p$ as above, and we are done. If instead $u \xrightarrow{a} u_1 \xrightarrow{a^{n-1}} u'$ for some u_1, then $\sigma(u) \xrightarrow{a} \sigma(u_1)$. Moreover, we claim that $p \precsim_{\mathrm{S}} \sigma(u_1)$. Indeed, since $u_1 \xrightarrow{a^{n-1}} u'$, the variable y occurs in u_1. This yields that

$$depth(p) < depth(\sigma(y)) \leq depth(\sigma(u_1)) \ .$$

The claim now follows because, in the presence of a single action a, the partial traces preorder and the simulation preorder coincide—see, e.g., [3]—, and therefore

$$depth(p) < depth(\sigma(u_1)) \text{ implies } p \precsim_{\mathrm{S}} \sigma(u_1) \ .$$

We are now left to examine the case in which $\sigma(t) \xrightarrow{a} p$ because t has a summand at' such that $p = \sigma(t')$. If $\sigma(y) \xrightarrow{a} q$ for some variable summand y of u and closed term q such that $p \precsim_{\mathrm{S}} q$, we are done. Assume therefore that, for each r and variable summand y of u,

$$\sigma(y) \xrightarrow{a} r \text{ implies } p \not\precsim_{\mathrm{S}} r \ . \tag{3}$$

We claim that $p \precsim_{\mathrm{S}} \sigma(u')$ for some summand au' of u.

We proceed with the proof of this claim by distinguishing two cases, depending on whether x occurs in t' or not.

- Case x does not occur in t'. In this case,

  $$\sigma[x \mapsto \mathbf{0}](t + x) \xrightarrow{a} \sigma[x \mapsto \mathbf{0}](t') = \sigma(t') = p \ .$$

  Since $t + x \precsim_{\mathrm{S}} u + x$, there is a q such that $\sigma[x \mapsto \mathbf{0}](u) \xrightarrow{a} q$ and $p \precsim_{\mathrm{S}} q$. By (3), it follows that $q = \sigma[x \mapsto \mathbf{0}](u')$ for some summand au' of u. Since $\mathbf{0} \precsim_{\mathrm{S}} \sigma(x)$ and $\precsim_{\mathrm{S}}$ is a precongruence, we may conclude that

  $$p \precsim_{\mathrm{S}} q = \sigma[x \mapsto \mathbf{0}](u') \precsim_{\mathrm{S}} \sigma(u') \ .$$

  It follows that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{S}} \sigma(u')$, and we are done.
- Case x occurs in t'. In this case,

  $$depth(p) = depth(\sigma(t')) \geq depth(\sigma(x)) \ .$$

  Since each a-derivative of $\sigma(x)$ has depth smaller than that of p, it cannot simulate p. As $\sigma(t + x) \precsim_{\mathrm{S}} \sigma(u + x)$ by the proviso of the lemma, and $\sigma(t + x) \xrightarrow{a} p$, it follows by (3) that $\sigma(u) \xrightarrow{a} \sigma(u')$ and $p \precsim_{\mathrm{S}} \sigma(u')$ for some summand au' of u.

This concludes the proof. □

B.3 Proof of Lemma 3 for $\precsim_{\mathbf{PT}}$

Assume that $t + x \precsim_{\mathrm{PT}} u + x$, and x is not a summand of $t + u$. We shall show that $t \precsim_{\mathrm{PT}} u$. This follows from the result for the simulation preorder if $|A| = 1$. Indeed, in that case, the partial traces and the simulation preorders coincide—see, e.g., [3].

Assume therefore that $|A| > 1$, so that there are two distinct actions a, b in A. Let σ be a closed substitution. We prove that each trace $a_1 \ldots a_n$ of $\sigma(t)$ is also a trace of $\sigma(u)$.

Since $\sigma(t) \xrightarrow{a_1 \ldots a_n}$, we have that

1. either $t \stackrel{a_1 \ldots a_n}{\longrightarrow}$
2. or there are a $j < n$, a variable y and a term t' such that $t \stackrel{a_1 \ldots a_j}{\longrightarrow} t'$, y is a summand of t', and $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow}$.

In the former case, clearly $\sigma_{\mathbf{0}}(t+x) \stackrel{a_1 \ldots a_n}{\longrightarrow}$. Thus also $\sigma_{\mathbf{0}}(u+x) \stackrel{a_1 \ldots a_n}{\longrightarrow}$. It follows that $u \stackrel{a_1 \ldots a_n}{\longrightarrow}$, and thus $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow}$, which was to be shown.

Consider the latter case. If $j = 0$, then $y \neq x$ is a summand of t. By Lemma 1, it is also a summand of u, and therefore $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow}$, which was to be shown. Assume therefore that $j \geq 1$. Let $N \geq \mathit{depth}(u)$. Clearly $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](t + x) \stackrel{a_1 \ldots a_j a^N b}{\longrightarrow}$, so also $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](u + x) \stackrel{a_1 \ldots a_j a^N b}{\longrightarrow}$. Since $j \geq 1$, this implies $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](u) \stackrel{a_1 \ldots a_j a^N b}{\longrightarrow}$. Let $b_1 \ldots b_{N+j+1} = a_1 \ldots a_j a^N b$. Since $N \geq \mathit{depth}(u)$, clearly $u \stackrel{b_1 \ldots b_k}{\longrightarrow} u'$ and $\sigma_{\mathbf{0}}[y \mapsto a^N b\mathbf{0}](z) \stackrel{b_{k+1} \ldots b_{N+j+1}}{\longrightarrow}$ for some term u', variable z and $k \leq N$, where u' has z as a summand. Since $N + j + 1 > k$, it follows that $z = y$, $k = j$ and $b_{k+1} \ldots b_{N+j+1} = a^N b$. Concluding, $u \stackrel{a_1 \ldots a_j}{\longrightarrow} u'$ where u' has y as a summand. Since $\sigma(y) \stackrel{a_{j+1} \ldots a_n}{\longrightarrow}$ and $j < n$, we infer that $\sigma(u) \stackrel{a_1 \ldots a_n}{\longrightarrow}$, which was to be shown.

This conclues the proof. $\square$

B.4 Proof of Lemma 3 for $\precsim_{\mathbf{RT}}$

We begin by stating a useful lemma relating the ready traces of a term $\sigma(t)$, where σ is a closed substitution, to the action transitions and ready traces of the term t and of the terms $\sigma(x)$ for each variable x occurring in t.

Lemma 6. *Let σ be a closed substitution, and t a term.*

1. *Assume that $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(t)$. Then*
   (a) *either there are terms $t_1, \ldots, t_k$ such that*
   $$t = t_0 \stackrel{b_1}{\to} t_1 \cdots t_{k-1} \stackrel{b_k}{\to} t_k$$
   and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$,
   (b) *or $t = t_0 \stackrel{b_1}{\to} t_1 \cdots t_{i-1} \stackrel{b_i}{\to} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
      i. *$\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and*
      ii. *$t_i = y + t'$ for some variable y and term t' such that*
      $$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \ldots b_k X_k$$
      is a ready trace of $\sigma(y)$.
2. *Assume that $t = t_0 \stackrel{b_1}{\to} t_1 \cdots t_{k-1} \stackrel{b_k}{\to} t_k$ and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$. Then $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(t)$.*
3. *Assume that $t = t_0 \stackrel{b_1}{\to} t_1 \cdots t_{i-1} \stackrel{b_i}{\to} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*

(a) $\mathcal{I}(\sigma(t_j)) = X_j$, *for each* $0 \leq j \leq i$, *and*
(b) $t_i = y + t'$ *for some variable* y *and term* t' *such that*

$$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$$

is a ready trace of $\sigma(y)$.
Then $X_0 b_1 X_1 \dots b_k X_k$ *is a ready trace of* $\sigma(t)$.

We are now ready to prove Lemma 3 for $\precsim_{\mathrm{RT}}$. Following the structure of the proof of this statement for the failures preorder offered in [9], we establish the contrapositive statement. To this end, assume that $t \not\precsim_{\mathrm{RT}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{RT}} u + x$.

Since $t \not\precsim_{\mathrm{RT}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{RT}} \sigma(u)$. This means that there is a ready trace $X_0 b_1 X_1 \dots b_k X_k$ of $\sigma(t)$ that is not a ready trace of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t + x) \not\precsim_{\mathrm{RT}} \rho(u + x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{RT}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{RT}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{RT}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u)) = X_0$. In particular this implies that $k > 0$.

Our order of business now will be to construct a closed substitution ρ with the following properties:

1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
2. $\rho(x)$ and $\sigma(x)$ have the same ready traces of length smaller than k, and
3. $\rho(x)$ does not have any ready pairs of the form $(c_1 \dots c_k, X_k)$.

Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t + x) \not\precsim_{\mathrm{RT}} \rho(u + x) \ .$$

Observe, first of all, that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$. To see this, recall that, as $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\sigma(t)$, by Lemma 6(1) we have that

1. either there are terms $t_1, \dots, t_k$ such that

   $$t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$$

   and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$,
2. or $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \dots, t_i$ such that
   (a) $\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and
   (b) $t_i = y + t'$ for some variable y and term t' such that

   $$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$$

   is a ready trace of $\sigma(y)$.

(Note that, in light of Lemma 1 and our assumptions that $X_0 b_1 X_1 \dots b_k X_k$ is not a ready trace of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $X_0 b_1 X_1 \dots b_k X_k$ would be a ready trace of $\sigma(u)$.) We proceed to prove that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$ by considering the two possibilities above separately.

- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$ and $\mathcal{I}(\sigma(t_i)) = X_i$, for each $0 \leq i \leq k$. By property 1 of ρ, $\mathcal{I}(\rho(t_i)) = \mathcal{I}(\sigma(t_i)) = X_i$ for each $0 \leq i \leq k$. So $X_0 b_1 X_1 \dots b_k X_k$ is also a ready trace of $\rho(t)$. By property 1 of ρ, $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$, as claimed.
- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \dots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 < i < k$, and terms $t_1, \dots, t_i$ such that
  1. $\mathcal{I}(\sigma(t_j)) = X_j$, for each $0 \leq j \leq i$, and
  2. $t_i = y + t'$ for some variable y and term t' such that
  $$\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$$
  is a ready trace of $\sigma(y)$.

  If $y \neq x$, then $\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$ is a ready trace of $\rho(y)$, by property 1 of ρ. By Lemma 6(3) and property 1 of ρ, $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t)$. Since $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, we may conclude that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$, as claimed.
  If $y = x$, then $\mathcal{I}(\sigma(y))\, b_{i+1} X_{i+1} \dots b_k X_k$ is a ready trace of $\rho(x)$, by property 2 of ρ because $i > 0$. By Lemma 6(3) and property 1 of ρ, $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t)$. Since $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, we may again conclude that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(t + x)$, as claimed.

We now prove that $(X_0 \cup \mathcal{I}(\sigma(x)))\, b_1 X_1 \dots b_k X_k$ is *not* a ready trace of $\rho(u + x)$. Since $k > 0$, this follows if we can argue that $\mathcal{I}(\sigma(x))\, b_1 X_1 \dots b_k X_k$ is not a ready trace of $\rho(x)$ and $X_0 b_1 X_1 \dots b_k X_k$ is not a ready trace of $\rho(u)$. To this end, note, first of all, that $\mathcal{I}(\sigma(x))\, b_1 X_1 \dots b_k X_k$ is not a ready trace of $\rho(x)$ by property 3 of ρ. Therefore, we are left to show that $X_0 b_1 X_1 \dots b_k X_k$ is not a ready trace of $\rho(u)$.

By Lemma 6(1), $X_0 b_1 X_1 \dots b_k X_k$ is a ready trace of $\rho(u)$ only if

1. either there are terms $u_1, \dots, u_k$ such that
$$u = u_0 \xrightarrow{b_1} u_1 \cdots u_{k-1} \xrightarrow{b_k} u_k$$
and $\mathcal{I}(\rho(u_i)) = X_i$, for each $0 \leq i \leq k$,
2. or $u = u_0 \xrightarrow{b_1} u_1 \cdots u_{i-1} \xrightarrow{b_i} u_i$ for some $0 \leq i < k$, and terms $u_1, \dots, u_i$ such that
   (a) $\mathcal{I}(\rho(u_j)) = X_j$, for each $0 \leq j \leq i$, and
   (b) $u_i = z + u'$ for some variable z and term u' such that
   $$\mathcal{I}(\rho(z))\, b_{i+1} X_{i+1} \dots b_k X_k$$
   is a ready trace of $\rho(z)$.

We now proceed to argue that both of these possibilities contradict our assumption that $X_0 b_1 X_1 \ldots b_k X_k$ is not a ready trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0 b_1 X_1 \ldots b_k X_k$ is a ready trace of $\sigma(u)$ using property 1 of ρ and Lemma 6(2). In the latter case, we could reach the same conclusion using properties 1 and 2 of ρ and Lemma 6(3).

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We begin by defining, for each closed term p, $n \geq 0$ and set of actions X, the closed term $\pi_n^X(p)$ as follows:

$$\pi_0^X(p) = \sum\{a\mathbf{0} \mid a \in \mathcal{I}(p) \cap X\} + \sum\{aa\mathbf{0} \mid a \in \mathcal{I}(p) - X\}$$
$$\pi_{n+1}^X(p) = \sum\{a\pi_n^X(p') \mid p \xrightarrow{a} p'\} \ .$$

Take $\rho = \sigma[x \mapsto \pi_{k-1}^{X_k}(\sigma(x))]$. By definition, $\mathcal{I}(\pi_n^X(p)) = \mathcal{I}(p)$, for each closed term p, $n \geq 0$ and $X \subseteq A$. Therefore ρ meets property 1.

We claim that $\rho(x)$ and $\sigma(x)$ have the same ready traces of length smaller than k. This follows immediately from the following two observations:

- $\mathcal{I}(\pi_n^{X_k}(p)) = \mathcal{I}(p)$, for each p and $n \geq 0$, and
- for all closed terms p, q, action c and $n > 0$,

$$p \xrightarrow{c} q \text{ iff } \pi_n^{X_k}(p) \xrightarrow{c} \pi_{n-1}^{X_k}(q) \ .$$

So ρ enjoys property 2.

Finally, to see that ρ meets property 3, assume that $\pi_{k-1}^{X_k}(\sigma(x)) \xrightarrow{c_1 \ldots c_k} q$ for some sequence $c_1 \ldots c_k$ of actions and closed term q. It is not hard to see that either $X_k \neq \emptyset$ and $q = \mathbf{0}$, or $q = a\mathbf{0}$ for some $a \notin X_k$. In both cases, $\mathcal{I}(q) \neq X_k$.

This concludes the proof. □

B.5 Proof of Lemma 3 for $\precsim_{\mathbf{FT}}$

We begin by stating a useful lemma relating the failure traces of a term $\sigma(t)$, where σ is a closed substitution, to the action transitions and failure traces of the term t and of the terms $\sigma(x)$ for each variable x occurring in t.

Lemma 7. *Let σ be a closed substitution, and t be a term.*

1. *Assume that $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$. Then*
   (a) *either there are terms $t_1, \ldots, t_k$ such that*

$$t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$$

   and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$,

   (b) *or $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that*
      i. *$\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and*

ii. $t_i = y + t'$ *for some variable* y *and term* t' *such that*

$$\emptyset\, b_{i+1}X_{i+1}\ldots b_kX_k$$

is a failure trace of $\sigma(y)$.

2. *Assume that* $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$ *and* $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, *for each* $0 \le i \le k$. *Then* $X_0b_1X_1\ldots b_kX_k$ *is a failure trace of* $\sigma(t)$.
3. *Assume that* $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{i-1} \xrightarrow{b_i} t_i$ *for some* $0 \le i < k$, *and terms* $t_1, \ldots, t_i$ *such that*
   (a) $\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, *for each* $0 \le j \le i$, *and*
   (b) $t_i = y + t'$ *for some variable* y *and term* t' *such that*

   $$\emptyset\, b_{i+1}X_{i+1}\ldots b_kX_k$$

   is a failure trace of $\sigma(y)$.

   Then $X_0b_1X_1\ldots b_kX_k$ *is a failure trace of* $\sigma(t)$.

We are now ready to prove Lemma 3 for $\precsim_{\mathrm{FT}}$. Following the structure of the proof of this statement for the readies preorder, we establish the contrapositive statement. To this end, assume that $t \not\precsim_{\mathrm{FT}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{FT}} u + x$.

Since $t \not\precsim_{\mathrm{FT}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{FT}} \sigma(u)$. This means that there is a failure trace $X_0b_1X_1\ldots b_kX_k$ of $\sigma(t)$ that is not a failure trace of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t + x) \not\precsim_{\mathrm{FT}} \rho(u + x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{FT}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{FT}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{FT}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$. In particular this implies that $k > 0$. We distinguish two cases, depending on whether $k = 1$ or $k > 1$.

- Case $k = 1$ Our order of business now will be to construct a closed substitution ρ with the following properties:
  1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x)) \cap X_1$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$, and
  2. $\rho(x)$ does not have any failure pairs of the form (c_1, X_1).

  Before giving the construction of ρ, we shall argue that from these two properties it follows that

  $$\rho(t + x) \not\precsim_{\mathrm{FT}} \rho(u + x) \ .$$

  Observe, first of all, that (b_1, X_1) is a failure pair of $\rho(t + x)$. To see this, recall that, as $X_0b_1X_1$ is a failure trace of $\sigma(t)$, by Lemma 7(1) we have two possibilities. Either there are is a term t' such that $t \xrightarrow{b_1} t'$ and $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$. Or $t = y + t'$ for some variable y and term t' such that (b_1, X_1) is a failure pair of $\sigma(y)$.

  In the second case, in light of Lemma 1, y would also be a variable summand of u, and $X_0b_1X_1$ would be a failure trace of $\sigma(u)$ because $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$. This contradicts one of our assumptions.

So we can assume that that $t \xrightarrow{b_1} t'$ with $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$. By property 1 of ρ, $\mathcal{I}(\sigma(t')) \cap X_1 = \emptyset$ implies that $\mathcal{I}(\rho(t')) \cap X_1 = \emptyset$. So (b_1, X_1) is a failure pair of $\rho(t)$. We conclude that (b_1, X_1) is a failure pair of $\rho(t+x)$, as claimed.
We now prove that (b_1, X_1) is *not* a failure pair of $\rho(u + x)$. This follows if we can argue that (b_1, X_1) is neither a failure pair of $\rho(x)$ nor a failure pair of $\rho(u)$. To this end, note, first of all, that (b_1, X_1) is not a failure pair of $\rho(x)$ by property 2 of ρ. Therefore, we are left to show that (b_1, X_1) is not a failure pair of $\rho(u)$.
By Lemma 7(1), (b_1, X_1) is a failure pair of $\rho(u)$ only if

- either there are is a term u' such that $u \xrightarrow{b_1} u'$ and $\mathcal{I}(\rho(u')) \cap X_1 = \emptyset$;
- or $u = z + u'$ for some variable z and term u' such that (b_1, X_1) is a failure pair of $\rho(z)$.

We now proceed to argue that both of these possibilities contradict our assumption that $X_0 b_1 X_1$ is not a failure trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0 b_1 X_1$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, property 1 of ρ and Lemma 7(2). In the latter case, by assumption $z \neq x$, so by property 1 of ρ, $\rho(z) = \sigma(z)$. So again we could conclude that $X_0 b_1 X_1$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$ and Lemma 7(3).
All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–2. We begin by defining, for each closed term p, the closed term $\mathrm{chop}^{X_1}(p)$ as follows:

$$\mathrm{chop}^{X_1}(p) = \sum\{aa\mathbf{0} \mid a \in \mathcal{I}(p) \cap X_1\} \ .$$

Take $\rho = \sigma[x \mapsto \mathrm{chop}^{X_1}(\sigma(x))]$. By definition, $\mathcal{I}(\mathrm{chop}^{X_1}(p)) = \mathcal{I}(p) \cap X_1$, for each closed term p. Therefore ρ meets property 1.
To see that ρ meets property 2, assume that $\mathrm{chop}^{X_1}(\sigma(x)) \xrightarrow{c_1} q$ for some action c_1 and closed term q. Then clearly $c_1 \in X_1$ and $q = c_1\mathbf{0}$. Therefore $\rho(x)$ does not have any failure pairs of the form (c_1, X_1).

– Case $k > 1$ Our order of business now will be to construct a closed substitution ρ with the following properties:
  1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
  2. $\rho(x)$ and $\sigma(x)$ have the same failure traces of the form $Y_0 c_1 Y_1 \ldots Y_{\ell-1} c_\ell X_k$ for $\ell < k$, and
  3. $\rho(x)$ does not have any failure pairs of the form $(c_1 \ldots c_k, X_k)$.

  Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t + x) \not\lesssim_{\mathrm{FT}} \rho(u + x) \ .$$

  Observe, first of all, that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t+x)$. To see this, recall that, as $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(t)$, by Lemma 7(1) we have that

1. either there are terms $t_1, \ldots, t_k$ such that

$$t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$$

and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$,
2. or $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 \leq i < k$, and terms $t_1, \ldots, t_i$ such that
   (a) $\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and
   (b) $t_i = y + t'$ for some variable y and term t' such that

$$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$

is a failure trace of $\sigma(y)$.

(Note that, in light of Lemma 1 and our assumptions that $X_0 b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $X_0 b_1 X_1 \ldots b_k X_k$ would be a failure trace of $\sigma(u)$ because $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$.) We proceed to prove that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$ by considering the two possibilities above separately.

- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{k-1} \xrightarrow{b_k} t_k$ and $\mathcal{I}(\sigma(t_i)) \cap X_i = \emptyset$, for each $0 \leq i \leq k$. By property 1 of ρ, $\mathcal{I}(\rho(t_i)) = \mathcal{I}(\sigma(t_i))$ for each $0 \leq i \leq k$. So $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.
- Suppose that $t = t_0 \xrightarrow{b_1} t_1 \cdots t_{i-1} \xrightarrow{b_i} t_i$ for some $0 < i < k$, and terms $t_1, \ldots, t_i$ such that
  1. $\mathcal{I}(\sigma(t_j)) \cap X_j = \emptyset$, for each $0 \leq j \leq i$, and
  2. $t_i = y + t'$ for some variable y and term t' such that

$$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$

  is a failure trace of $\sigma(y)$.

  If $y \neq x$, then $\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$ is a failure trace of $\rho(y)$, by property 1 of ρ. By Lemma 7(3) and property 1 of ρ, it follows that $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.

  If $y = x$, then $\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$ is a failure trace of $\rho(x)$, by property 2 of ρ, because $i > 0$. By Lemma 7(3) and property 1 of ρ, we have that $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t)$. We may again conclude that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(t + x)$, as claimed.

We now prove that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is *not* a failure trace of $\rho(u + x)$. This follows if we can argue that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is neither a failure trace of $\rho(x)$ nor a failure trace of $\rho(u)$. To this end, note, first of all, that $\emptyset\, b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\rho(x)$ by property 3 of ρ. Therefore, we are left to show that $\emptyset b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\rho(u)$.

By Lemma 7(1), $\emptyset b_1 X_1 \ldots b_k X_k$ is a failure trace of $\rho(u)$ only if

1. either there are terms $u_1, \ldots, u_k$ such that

$$u = u_0 \xrightarrow{b_1} u_1 \cdots u_{k-1} \xrightarrow{b_k} u_k$$

and $\mathcal{I}(\rho(u_i)) \cap X_i = \emptyset$, for each $1 \leq i \leq k$,
2. or $u = u_0 \xrightarrow{b_1} u_1 \cdots u_{i-1} \xrightarrow{b_i} u_i$ for some $0 \leq i < k$, and terms $u_1, \ldots, u_i$ such that
   (a) $\mathcal{I}(\rho(u_j)) \cap X_j = \emptyset$, for each $1 \leq j \leq i$, and
   (b) $u_i = z + u'$ for some variable z and term u' such that

$$\emptyset\, b_{i+1} X_{i+1} \ldots b_k X_k$$

is a failure trace of $\rho(z)$.

We now proceed to argue that both of these possibilities contradict our assumption that $X_0 b_1 X_1 \ldots b_k X_k$ is not a failure trace of $\sigma(u)$. Indeed, in the former case, we could conclude that $X_0 b_1 X_1 \ldots b_k X_k$ is a failure trace of $\sigma(u)$ using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, property 1 of ρ and Lemma 7(2). In the latter case, we could reach the same conclusion using our assumption that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$, properties 1 and 2 of ρ and Lemma 7(3).
All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We begin by defining, for each closed term p, and $n \geq 0$, the closed term $\mathrm{chop}_n^{X_k}(p)$ as follows:

$$\mathrm{chop}_0^{X_k}(p) = \sum\{aa\mathbf{0} \mid a \in \mathcal{I}(p) \cap X_k\}$$
$$\mathrm{chop}_{n+1}^{X_k}(p) = \sum\{a\,\mathrm{chop}_n^{X_k}(p') \mid p \xrightarrow{a} p'\} \ .$$

Take $\rho = \sigma[x \mapsto \mathrm{chop}_{k-1}^{X_k}(\sigma(x))]$. By definition, $\mathcal{I}(\mathrm{chop}_n^{X_k}(p)) = \mathcal{I}(p)$, for each closed term p, and $n > 0$. Since $k - 1 > 0$, ρ meets property 1.
We claim that $\rho(x)$ and $\sigma(x)$ have the same failure traces of length smaller than k. This follows immediately from the following three observations:

- for each closed term p and $n > 0$,

$$\mathcal{I}(\mathrm{chop}_n^{X_k}(p)) = \mathcal{I}(p) \ ,$$

- for all closed terms p, q, action c and $n > 0$,

$$p \xrightarrow{c} q \text{ iff } \mathrm{chop}_n^{X_k}(p) \xrightarrow{c} \mathrm{chop}_{n-1}^{X_k}(q) \ , \text{ and}$$

- for each closed term p,

$$\mathcal{I}(p) \cap X_k = \emptyset \text{ iff } \mathcal{I}(\mathrm{chop}_0^{X_k}(p)) \cap X_k = \emptyset \ .$$

So ρ enjoys property 2.
Finally, to see that ρ meets property 3, assume that $\mathrm{chop}_{k-1}^{X_k}(\sigma(x)) \xrightarrow{c_1 \ldots c_k} q$ for some sequence $c_1 \ldots c_k$ of actions and closed term q. It is not hard to see that then $c_k \in X_k$ and $q = c_k\mathbf{0}$. Therefore $\rho(x)$ does not have any failure pairs of the form $(c_1 \ldots c_k, X_k)$.

This concludes the proof. $\square$

B.6 Proof of Lemma 3 for $\precsim_{\mathrm{R}}$

We begin by stating a useful lemma relating the ready pairs of a closed term $\sigma(t)$, where σ is a closed substitution, to the action transitions and ready pairs of t and of the closed terms $\sigma(x)$ for each variable x occurring in t.

Lemma 8. *Let σ be a closed substitution, and let t be a term.*

1. *Assume that $(b_1 \dots b_k, X)$ is a ready pair of $\sigma(t)$. Then*
   (a) either $t \overset{b_1 \dots b_k}{\rightarrow} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t',
   (b) or $t \overset{b_1 \dots b_i}{\rightarrow} y + t'$ for some $i < k$, variable y and term t' such that $(b_{i+1} \dots b_k, X)$ is a ready pair of $\sigma(y)$.
2. *Assume that $t \overset{b_1 \dots b_k}{\rightarrow} t'$ for some t'. Then $(b_1 \dots b_k, \mathcal{I}(\sigma(t')))$ is a ready pair of $\sigma(t)$.*
3. *Assume that $t \overset{b_1 \dots b_i}{\rightarrow} y + t'$ for some $i < k$, variable y and term t' such that $(b_{i+1} \dots b_k, X)$ is a ready pair of $\sigma(y)$. Then $(b_1 \dots b_k, X)$ is a ready pair of $\sigma(t)$.*

We are now ready to prove Lemma 3 for $\precsim_{\mathrm{R}}$. Following the structure of the proof of this statement for the ready traces preorder, we establish the contrapositive statement. To this end, assume that $t \not\precsim_{\mathrm{R}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\precsim_{\mathrm{R}} u + x$.

Since $t \not\precsim_{\mathrm{R}} u$, there is a closed substitution σ such that $\sigma(t) \not\precsim_{\mathrm{R}} \sigma(u)$. This means that there is a ready pair $(b_1 \dots b_k, X)$ of $\sigma(t)$ that is not a ready pair of $\sigma(u)$. In the remainder of the proof, we use this information to construct a closed substitution ρ such that $\rho(t+x) \not\precsim_{\mathrm{R}} \rho(u+x)$, thus establishing our claim that $t + x \not\precsim_{\mathrm{R}} u + x$.

Suppose that $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$. As x is not a summand of $t+u$, then clearly $\sigma[x \mapsto \mathbf{0}](t + x) \not\precsim_{\mathrm{R}} \sigma[x \mapsto \mathbf{0}](u + x)$. Hence, $t + x \not\precsim_{\mathrm{R}} u + x$, which was to be shown.

So we may assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u)) = X$. In particular this implies that $k > 0$.

As in the proof for the ready traces preorder, we define the closed substitution ρ by $\rho = \sigma[x \mapsto \pi^X_{k-1}(\sigma(x))]$, where the closed term $\pi^X_{k-1}(\sigma(x))$ is defined as on page 28. We observed in the proof for the ready traces preorder that (stronger versions of) the following properties hold for ρ:

1. $\mathcal{I}(\rho(x)) = \mathcal{I}(\sigma(x))$, and $\rho(y) = \sigma(y)$ for each variable $y \neq x$,
2. $\rho(x)$ and $\sigma(x)$ have the same ready pairs of length smaller than k, and
3. $\rho(x)$ does not have any ready pairs of the form $(c_1 \dots c_k, X)$.

We shall argue that

$$\rho(t + x) \not\precsim_{\mathrm{R}} \rho(u + x) \ ,$$

showing that $t + x \not\precsim_{\mathrm{R}} u + x$, as claimed.

Observe, first of all, that $(b_1 \dots b_k, X)$ is a ready pair of $\rho(t+x)$. To see this, recall that, as $(b_1 \dots b_k, X)$ is a ready pair of $\sigma(t)$, by Lemma 8(1) we have that

– either $t \stackrel{b_1 \ldots b_k}{\rightarrow} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t',
– or $t \stackrel{b_1 \ldots b_i}{\rightarrow} y+t'$ for some $i < k$, variable y and term t' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\sigma(y)$.

(Note that, in light of Lemma 1 and our assumptions that $(b_1 \ldots b_k, X)$ is not a ready pair of $\sigma(u)$ and $k > 0$, in the latter case $i > 0$. Indeed, if $i = 0$, then y would also be a variable summand of u, and $(b_1 \ldots b_k, X)$ would be a ready pair of $\sigma(u)$.) We proceed to prove that $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$ by considering the two possibilities above separately.

– Suppose that $t \stackrel{b_1 \ldots b_k}{\rightarrow} t'$ and $\mathcal{I}(\sigma(t')) = X$, for some t'. Lemma 8(2) yields that $\rho(t) \stackrel{b_1 \ldots b_k}{\rightarrow} \rho(t')$. Moreover, by property 1 of ρ, $\mathcal{I}(\rho(t')) = \mathcal{I}(\sigma(t')) = X$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.
– Suppose that $t \stackrel{b_1 \ldots b_i}{\rightarrow} y + t'$ for some $0 < i < k$, variable y and term t' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\sigma(y)$. In this case,
$$\rho(t) \stackrel{b_1 \ldots b_i}{\rightarrow} \rho(y + t') \ .$$
If $y \neq x$, then $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\rho(y)$, by property 1 of ρ. Since $i < k$, by Lemma 8(3), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t)$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.
If $y = x$, then $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\rho(x)$, by property 2 of ρ because $i > 0$. Since $i < k$, by Lemma 8(3), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t)$. Since $k > 0$, $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(t + x)$, as claimed.

We now prove that $(b_1 \ldots b_k, X)$ is *not* a ready pair of $\rho(u + x)$. To this end, note, first of all, that $(b_1 \ldots b_k, X)$ is not a ready pair of $\rho(x)$ by property 3 of ρ. Since $k > 0$, it suffices to show that $(b_1 \ldots b_k, X)$ is not a ready pair of $\rho(u)$. By Lemma 8(1), $(b_1 \ldots b_k, X)$ is a ready pair of $\rho(u)$ only if

– either $u \stackrel{b_1 \ldots b_k}{\rightarrow} u'$ and $\mathcal{I}(\rho(u')) = X$, for some u',
– or $u \stackrel{b_1 \ldots b_i}{\rightarrow} y+u'$ for some $i < k$, variable y and term u' such that $(b_{i+1} \ldots b_k, X)$ is a ready pair of $\rho(y)$.

We now proceed to argue that both of these possibilities contradict our assumption that $(b_1 \ldots b_k, X)$ is not a ready pair of $\sigma(u)$. Indeed, in the former case, we could conclude that $(b_1 \ldots b_k, X)$ is a ready pair of $\sigma(u)$ using property 1 of ρ and Lemma 8(2). In the latter case, we could reach the same conclusion using properties 1 and 2 of ρ and Lemma 8(3).

This concludes the proof. □

B.7 Proof of Lemma 3 for $\precsim_{\mathbf{PW}}$

We begin by offering a reformulation of the definition of the possible worlds preorder that will be useful in the proof to follow.

Definition 4. *A closed term ap is a* prefixed possible world *of a closed term q if:*

1. *p is deterministic, and*
2. *$q \xrightarrow{a} q'$ for some closed term q' such that $p \precsim_{RS} q'$.*

For closed terms r and s, we define $r \sqsubseteq_{PW} s$ if:

1. *the prefixed possible worlds of r are also prefixed possible worlds of s, and*
2. *$\mathcal{I}(r) = \mathcal{I}(s)$.*

The relation $\sqsubseteq_{\mathrm{PW}}$ is lifted to open terms in the standard fashion; see page 4.

Lemma 9. *The preorders $\precsim_{\mathrm{PW}}$ and $\sqsubseteq_{\mathrm{PW}}$ coincide over* BCCSP(A).

Proof. It suffices to show the statement for closed terms. Assume that $r \precsim_{\mathrm{PW}} s$. We prove that $r \sqsubseteq_{\mathrm{PW}} s$ also holds. To this end, observe, first of all, that $\mathcal{I}(r) = \mathcal{I}(s)$, since $\precsim_{\mathrm{PW}}$ is included in the readies preorder. We are therefore left to show that the prefixed possible worlds of r are also prefixed possible worlds of s.

Suppose that ap is a prefixed possible world of r. It is not hard to see that $ap + p'$ is a possible world of r, for some p'. As $r \precsim_{\mathrm{PW}} s$, it follows that $ap + p'$ is also a possible world of s. We may therefore conclude that ap is a prefixed possible world of s, which was to be shown.

Assume now that $r \sqsubseteq_{\mathrm{PW}} s$. We prove that $r \precsim_{\mathrm{PW}} s$ also holds. Observe, first of all, that $\mathcal{I}(r) = \mathcal{I}(s)$ by our assumption that $r \sqsubseteq_{\mathrm{PW}} s$. Let p be a possible world of r. Then p is deterministic and $p \precsim_{\mathrm{RS}} r$. Since p is deterministic, for each $a \in \mathcal{I}(p)$ there is a unique closed term p_a such that $p \xrightarrow{a} p_a$. Moreover,

$$p = \sum_{a \in \mathcal{I}(p)} ap_a$$

and $\mathcal{I}(p) = \mathcal{I}(r) = \mathcal{I}(s)$. As $p \precsim_{\mathrm{RS}} r$, for each $a \in \mathcal{I}(p)$ there is a closed term r_a such that $p_a \precsim_{\mathrm{RS}} r_a$. Since p_a is itself deterministic, ap_a is a prefixed possible world of r, for each $a \in \mathcal{I}(p)$. As $r \sqsubseteq_{\mathrm{PW}} s$ by assumption, it follows that ap_a is also a prefixed possible world of s for each $a \in \mathcal{I}(p)$. We conclude that p is a possible world of s, which was to be shown. □

We are now ready to prove Lemma 3 for $\precsim_{\mathrm{PW}}$. In light of the above lemma, it suffices to prove this statement for $\sqsubseteq_{\mathrm{PW}}$. We establish the contrapositive statement. To this end, assume that $t \not\sqsubseteq_{\mathrm{PW}} u$, and x is not a summand of $t + u$. We shall show that $t + x \not\sqsubseteq_{\mathrm{PW}} u + x$.

Since $t \not\sqsubseteq_{\mathrm{PW}} u$, there is a closed substitution σ such that $\sigma(t) \not\sqsubseteq_{\mathrm{PW}} \sigma(u)$.

If $\mathcal{I}(\sigma(t)) \neq \mathcal{I}(\sigma(u))$, then, reasoning as in the proof for the readies preorder, it is easy to prove that $t+x \not\sqsubseteq_{\mathrm{PW}} u+x$. So we can assume that $\mathcal{I}(\sigma(t)) = \mathcal{I}(\sigma(u))$.

Because of this assumption, there is a prefixed possible world ap of $\sigma(t)$ that is not a prefixed possible world of $\sigma(u)$. Our order of business will now be to construct a closed substitution ρ with the following properties:

1. $\rho(y) = \sigma(y)$ for each variable $y \neq x$,

2. $\rho(x)$ and $\sigma(x)$ have the same prefixed possible worlds of depth at most *depth*(p), and
3. $\rho(x)$ does not have any completed traces of length *depth*$(p)+1$.

Before giving the construction of ρ, we shall argue that from these three properties it follows that

$$\rho(t+x) \not\sqsubseteq_{\mathrm{PW}} \rho(u+x) \ .$$

In view of properties 1 and 2, it is not hard to see that for any term r,

(i) $\sigma(r)$ and $\rho(r)$ have the same prefixed possible worlds of depth at most *depth*(p), and
(ii) if r does not have a summand x, then $\sigma(r)$ and $\rho(r)$ have the same prefixed possible worlds of depth at most *depth*$(p)+1$.

We prove these two claims by induction on *depth*(r). Suppose that

$$r = \sum_{i \in I} a_i r_i + \sum_{j \in J} y_j \ .$$

By induction, for $i \in I$, claim (i) yields that $\rho(r_i)$ and $\sigma(r_i)$ have the same prefixed possible worlds of depth at most *depth*(p). This implies (cf. Lemma 9) that $\rho(a_i r_i)$ and $\sigma(a_i r_i)$ have the same prefixed possible worlds of depth at most *depth*$(p)+1$. And for $j \in J$, if $y_j \neq x$, then by property 1, $\rho(y_j) = \sigma(y_j)$, so they have the same prefixed possible worlds. This completes the proof of claim (ii). Finally, if $y_j = x$, then by property 2, $\rho(x)$ and $\sigma(x)$ have the same prefixed possible worlds of depth at most *depth*(p). Hence we can conclude that claim (i) also holds.

By assumption, x is not a summand of t, and ap is a prefixed possible world of $\sigma(t)$. So by claim (ii), ap is a prefixed possible world of $\rho(t)$, and so also of $\rho(t+x)$.

By assumption, x is not a summand of u, and ap is not a prefixed possible world of $\sigma(u)$. So by claim (ii), ap is not a prefixed possible world of $\rho(u)$. Moreover, by property 3, ap is not a prefixed possible world of $\rho(x)$. Hence, ap is not a prefixed possible world of $\rho(u+x)$.

Since ap is a prefixed possible world of $\rho(t+x)$ and not of $\rho(u+x)$, we conclude that $t+x \not\sqsubseteq_{\mathrm{PW}} u+x$, which was to be proved.

All that we are left to do to complete the proof for this case is to construct a closed substitution ρ having properties 1–3. We define

$$\rho = \sigma[x \mapsto \pi^{\emptyset}_{\mathit{depth}(p)}(\sigma(x))] \ ,$$

where the closed term $\pi^{\emptyset}_{\mathit{depth}(p)}(\sigma(x))$ is defined as on page 28. Property 1 trivially holds. And property 2 follows immediately from the following two observations:

– $\mathcal{I}(\pi^{\emptyset}_n(q)) = \mathcal{I}(q)$, for each q and $n \geq 0$, and

– for all closed terms q, r, action c and $n > 0$,

$$q \xrightarrow{c} r \text{ iff } \pi^{\emptyset}_{n}(q) \xrightarrow{c} \pi^{\emptyset}_{n-1}(r) \ .$$

Finally, to see that ρ meets property 3, assume that $\pi^{\emptyset}_{depth(p)}(\sigma(x)) \stackrel{c_1 \ldots c_{depth(p)+1}}{\longrightarrow} r$ for some sequence $c_1 \ldots c_{depth(p)+1}$ of actions and closed term r. It is not hard to see that then $r = a\mathbf{0}$ for some $a \in A$.

This concludes the proof. $\square$

Recent BRICS Report Series Publications

RS-07-3 **Luca Aceto, Willem Jan Fokkink, and Anna Ingólfsdóttir.** ***Ready To Preorder: Get Your BCCSP Axiomatization for Free!*** **February 2007. 36 pp.**

RS-07-2 **Luca Aceto and Anna Ingólfsdóttir.** ***Characteristic Formulae: From Automata to Logic*****. January 2007. 18 pp.**

RS-07-1 **Daniel Andersson.** ***HIROIMONO is NP-complete*****. January 2007. 8 pp.**

RS-06-19 **Michael David Pedersen.** ***Logics for The Applied π Calculus*****. December 2006. viii+111 pp.**

RS-06-18 **Małgorzata Biernacka and Olivier Danvy.** ***A Syntactic Correspondence between Context-Sensitive Calculi and Abstract Machines*****. dec 2006. iii+39 pp. Extended version of an article to appear in TCS. Revised version of BRICS RS-05-22.**

RS-06-17 **Olivier Danvy and Kevin Millikin.** ***A Rational Deconstruction of Landin's J Operator*****. December 2006. ii+37 pp. Revised version of BRICS RS-06-4. A preliminary version appears in the proceedings of IFL 2005, LNCS 4015:55–73.**

RS-06-16 **Anders Møller.** ***Static Analysis for Event-Based XML Processing*****. October 2006. 16 pp.**

RS-06-15 **Dariusz Biernacki, Olivier Danvy, and Kevin Millikin.** ***A Dynamic Continuation-Passing Style for Dynamic Delimited Continuations*****. October 2006. ii+28 pp. Revised version of BRICS RS-05-16.**

RS-06-14 **Giorgio Delzanno, Javier Esparza, and Jiří Srba.** ***Monotonic Set-Extended Prefix Rewriting and Verification of Recursive Ping-Pong Protocols*****. July 2006. 31 pp. To appear in ATVA '06.**

RS-06-13 **Jiří Srba.** ***Visibly Pushdown Automata: From Language Equivalence to Simulation and Bisimulation*****. July 2006. 21 pp. To appear in CSL '06.**

RS-06-12 **Kristian Støvring.** ***Higher-Order Beta Matching with Solutions in Long Beta-Eta Normal Form*****. June 2006. 13 pp. To appear in *Nordic Journal of Computing*, 2006.**