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Kristian Støvring

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Extending the Extensional Lambda Calculus with Surjective Pairing is Conservative

Kristian Støvring

BRICS *
Department of Computer Science
University of Aarhus †

November 22, 2005

Abstract

We answer Klop and de Vrijer's question whether adding surjective-pairing axioms to the extensional lambda calculus yields a conservative extension. The answer is positive. As a byproduct we obtain the first "syntactic" proof that the extensional lambda calculus with surjective pairing is consistent.

1 Introduction

The theory $\lambda_{\beta\eta\mathrm{SP}}$ is obtained from the (untyped) extensional lambda calculus $\lambda_{\beta\eta}$ [2, p. 32], by adding three *surjective-pairing* axioms:

$$\begin{array}{lrcl}
(\pi_1) & \pi_1 \langle M, N\rangle & = & M \\
(\pi_2) & \pi_2 \langle M, N\rangle & = & N \\
(\textsc{sp}) & \langle \pi_1 M, \pi_2 M\rangle & = & M
\end{array}$$

A λ-term is called *pure* if it does not contain any of the new constructs π_i and $\langle \cdot, \cdot \rangle$. In this article we give a positive answer to the following question, asked by Klop and de Vrijer in 1989 [7, 15] and featured as Problem 5 on the original RTA list of open problems [4]:

*Basic Research in Computer Science (`www.brics.dk`), funded by the Danish National Research Foundation.
†IT-parken, Aabogade 34, DK-8200 Aarhus N, Denmark
E-mail: `kss@brics.dk`

Suppose that M and N are pure λ-terms. Does $M =_{\beta\eta\mathrm{SP}} N$ imply $M =_{\beta\eta} N$?

In other words, we show that the theory $\lambda_{\beta\eta\mathrm{SP}}$ is a *conservative extension* of the theory $\lambda_{\beta\eta}$. As a byproduct we obtain the (as far as the author knows) first proof of consistency of $\lambda_{\beta\eta\mathrm{SP}}$ which uses purely syntactic methods.

1.1 Background of the problem

The two perhaps most obvious attempts at showing conservativity of $\lambda_{\beta\eta\mathrm{SP}}$ fail because of two negative results: No surjective-pairing function (that is, no pairing function satisfying the three axioms on the preceding page) is definable in the lambda calculus [1], and the standard reduction relation for the lambda calculus with surjective pairing is not confluent [8]. Both results were shown for the *extensional* lambda calculus as well.

Klop [8] and Klop and de Vrijer [7] have considered a number of properties of the (non-extensional) lambda calculus with surjective pairing, $\lambda_{\beta\mathrm{SP}}$, which would have trivially followed from confluence of the standard reduction relation. In particular, de Vrijer has shown that $\lambda_{\beta\mathrm{SP}}$ is a conservative extension of the lambda calculus [15]. This result motivated the question answered here: whether surjective pairing also conservatively extends the *extensional* lambda calculus.

The proof of conservativity by de Vrijer is furthermore the first known "syntactic" consistency proof for $\lambda_{\beta\mathrm{SP}}$. A model-theoretic consistency proof for $\lambda_{\beta\eta\mathrm{SP}}$ (and hence for $\lambda_{\beta\mathrm{SP}}$) can be given using the inverse limit model construction [12].

The theory $\lambda_{\beta\eta\mathrm{SP}}$ has also been investigated from a categorical point of view. If $\mathcal{C}$ is a cartesian closed category with an object D such that

$$D \cong D \times D \cong D \to D,$$

then there are various ways of interpreting λ-terms as morphisms of $\mathcal{C}$ [2, 9]. Moreover, every extension of the theory $\lambda_{\beta\eta\mathrm{SP}}$ is the theory of a model arising in this way [9, 13].

1.2 Formalization

The author has formalized and verified the proof of the conservativity result using the Twelf system [11]. The formalized proof additionally serves as an implementation of a procedure transforming a formal derivation of $M =_{\beta\eta\mathrm{SP}} N$ into a formal derivation of $M =_{\beta\eta} N$ (for pure terms M and N). It is available from

`http://www.brics.dk/~kss/papers/SP/`

The formalized statement of the main result is presented in Appendix A.

2 Background and notation

The reader is assumed to be familiar with basic properties of the untyped lambda calculus, as presented for example in the first three chapters of Barendregt's book [2].

The syntax of λ-terms is extended with constructs for pairing and projection:

$$M \ ::= \ x \mid \lambda x.M \mid M\,M \mid \langle M, M\rangle \mid \pi_1\,M \mid \pi_2\,M$$

(where x ranges over an infinite set of variables). The *pure terms* are the usual λ-terms, i.e., terms with no occurrences of π_i or $\langle\cdot,\cdot\rangle$. The set of free variables of a term M is denoted $\mathrm{FV}(M)$. We follow practice and identify α-equivalent terms.

We use the following notation and definitions for relations on λ-terms: For any binary relation $\rhd_R$ on λ-terms, $\longrightarrow_R$ denotes the compatible closure of $\rhd_R$ as defined in Figure 1. The relation $\longrightarrow_R$ is called a *reduction relation.* The reflexive-transitive closure of $\longrightarrow_R$ is written $\longrightarrow_R^*$, and the reflexive-transitive-symmetric closure of $\longrightarrow_R$ is written $=_R$; the relation $=_R$ is a congruence in the usual sense. We write λ_R for the equational theory of λ-terms corresponding to $=_R$, i.e., λ_R is the set of formal equations "$M = N$" such that $M =_R N$.

$$\frac{M \rhd_R M'}{M \longrightarrow_R M'} \qquad \frac{M \longrightarrow_R M'}{\lambda x.M \longrightarrow_R \lambda x.M'}$$

$$\frac{M \longrightarrow_R M'}{M\,N \longrightarrow_R M'\,N} \qquad \frac{N \longrightarrow_R N'}{M\,N \longrightarrow_R M\,N'}$$

$$\frac{M \longrightarrow_R M'}{\langle M, N\rangle \longrightarrow_R \langle M', N\rangle} \qquad \frac{N \longrightarrow_R N'}{\langle M, N\rangle \longrightarrow_R \langle M, N'\rangle}$$

$$\frac{M \longrightarrow_R M'}{\pi_1\,M \longrightarrow_R \pi_1\,M'} \qquad \frac{M \longrightarrow_R M'}{\pi_2\,M \longrightarrow_R \pi_2\,M'}$$

Figure 1: The compatible closure of $\rhd_R$.

The relation $\rhd_{\beta\eta\mathrm{SP}}$ is defined by the axioms in Figure 2. This relation generates a reduction relation $\longrightarrow_{\beta\eta\pi\mathrm{SP}}$ and a congruence $=_{\beta\eta\mathrm{SP}}$. The extensional lambda calculus with surjective pairing is defined as the theory $\lambda_{\beta\eta\mathrm{SP}}$.

$$
\begin{array}{lrll}
(\beta) & (\lambda x.M)\,N & \rhd_{\beta\eta\mathrm{SP}} & M[x := N] \\
(\eta) & \lambda x.M\,x & \rhd_{\beta\eta\mathrm{SP}} & M \qquad\qquad (\text{if } x \notin \mathrm{FV}(M)) \\
(\pi_1) & \pi_1\,\langle M, N\rangle & \rhd_{\beta\eta\mathrm{SP}} & M \\
(\pi_2) & \pi_2\,\langle M, N\rangle & \rhd_{\beta\eta\mathrm{SP}} & N \\
(\textsc{sp}) & \langle \pi_1\,M, \pi_2\,M\rangle & \rhd_{\beta\eta\mathrm{SP}} & M
\end{array}
$$

Figure 2: The relation $\rhd_{\beta\eta\mathrm{SP}}$.

3 Overview of the proof

The relation $\longrightarrow_{\beta\eta\mathrm{SP}}$ is the standard reduction relation generating $=_{\beta\eta\mathrm{SP}}$. This reduction relation is however not confluent [8, p. 216]; its confluence would immediately imply the main result, namely that $\lambda_{\beta\eta\mathrm{SP}}$ is conservative over $\lambda_{\beta\eta}$.[1]

In this article we instead define a further extension $\lambda_{\beta\eta\pi}$ of $\lambda_{\beta\eta\mathrm{SP}}$ and show that $\lambda_{\beta\eta\pi}$ is conservative over $\lambda_{\beta\eta}$. Since $\lambda_{\beta\eta\pi}$ is an extension of $\lambda_{\beta\eta\mathrm{SP}}$, the main result follows.

The proof is structured in the following way:

- In Section 4 we define the extension $\lambda_{\beta\eta\pi}$ of $\lambda_{\beta\eta\mathrm{SP}}$ and show that it is generated by a confluent reduction relation $\longrightarrow_{\beta\eta\pi}$. In the relation $\longrightarrow_{\beta\eta\pi}$ the axioms (η) and (SP) are oriented as *expansion* axioms (see, e.g., the work by Jay and Ghani [6]).

- In Section 5 we show that $\lambda_{\beta\eta\pi}$ is conservative over $\lambda_{\beta\eta}$ on pure λ-terms. This result does not immediately follow from confluence of $\longrightarrow_{\beta\eta\pi}$ since $\longrightarrow_{\beta\eta\pi}$ contains (SP) oriented as an expansion axiom.

4 An extension of the theory $\lambda_{\beta\eta\mathrm{SP}}$

We first define the extension $\lambda_{\beta\eta\pi}$ of $\lambda_{\beta\eta\mathrm{SP}}$. The relation $\rhd_{\beta\eta\pi}$ is defined by the axioms in Figure 3. This relation generates the theory $\lambda_{\beta\eta\pi}$ and the reduction relation $\longrightarrow_{\beta\eta\pi}$. As discussed above, the axioms (η) and (SP) in $\longrightarrow_{\beta\eta\pi}$ are oriented as expansion axioms.

Remark. The theory $\lambda_{\beta\eta\pi}$ and the associated reduction relation $\longrightarrow_{\beta\eta\pi}$ have certain properties which might make them interesting in their own right:

[1] The non-confluent reduction relation considered by Klop [8] is slightly different from $\longrightarrow_{\beta\eta\mathrm{SP}}$. It is simple to construct a counter-example to confluence similar to Klop's.

$$
\begin{array}{llcll}
(\beta) & (\lambda x.M)\,N & \rhd_{\beta\eta\pi} & M[x := N] & \\
(\eta) & M & \rhd_{\beta\eta\pi} & \lambda x.M\,x & (\text{if } x \notin \mathrm{FV}(M)) \\
(\pi_1) & \pi_1\,\langle M, N\rangle & \rhd_{\beta\eta\pi} & M & \\
(\pi_2) & \pi_2\,\langle M, N\rangle & \rhd_{\beta\eta\pi} & N & \\
(\textsc{sp}) & M & \rhd_{\beta\eta\pi} & \langle \pi_1\,M, \pi_2\,M\rangle & \\
 & & & & \\
(\delta\pi) & \langle M, N\rangle\,P & \rhd_{\beta\eta\pi} & \langle M\,P, N\,P\rangle & \\
(\pi_1\lambda) & \pi_1\,(\lambda x.M) & \rhd_{\beta\eta\pi} & \lambda x.\pi_1\,M & \\
(\pi_2\lambda) & \pi_2\,(\lambda x.M) & \rhd_{\beta\eta\pi} & \lambda x.\pi_2\,M &
\end{array}
$$

Figure 3: The relation $\rhd_{\beta\eta\pi}$.

- From the point of view of semantics: The original model of $\lambda_{\beta\eta\mathrm{SP}}$ [9, 12] is also a model of $\lambda_{\beta\eta\pi}$. Indeed, let D and E be complete partial orders such that $E \cong E \times E$ and $D \cong [D \to E]$. Then $D \cong D \times D \cong [D \to D]$, and it is easy to verify that the standard interpretation[2] of λ-terms as elements of D gives rise to a model of $\lambda_{\beta\eta\pi}$.

  As an aside, if D is an arbitrary complete partial order satisfying that $D \cong D \times D \cong [D \to D]$, then the standard interpretation using these isomorphisms makes D a model of (at least) $\lambda_{\beta\eta\mathrm{SP}}$. Taking $D = E$ in the above construction now gives an alternative pair of isomorphisms, and hence an alternative interpretation of λ-terms, resulting in a model of $\lambda_{\beta\eta\pi}$.

- From the point of view of term rewriting: In the *simply-typed* lambda calculus, term constructs can be proof-theoretically classified as either *introduction forms* ($\lambda x.M$ and $\langle M, N\rangle$) or *elimination forms* ($M\,N$ and $\pi_i\,M$), using the Curry-Howard isomorphism [3]. The simply-typed counterparts of the axioms (β), (π_1), and (π_2) of Figure 3 then imply that when constructing a term bottom-up, "an introduction form followed by an elimination form is a redex." This property is preserved in the untyped reduction relation $\longrightarrow_{\beta\eta\pi}$ by virtue of the three new axioms.

In the rest of this section we prove that $\longrightarrow_{\beta\eta\pi}$ is confluent. For that purpose we describe $\longrightarrow^*_{\beta\eta\pi}$ as the union of two relations: an "extensionality-free" part $\longrightarrow^*_{\beta\pi}$ and η/SP-expansion $\longrightarrow^*_{\eta}$.

- In Section 4.1 we define the relation $\longrightarrow_{\beta\pi}$ and show that it is confluent.

- In Section 4.2 we define η/SP-expansion $\longrightarrow_{\eta}$ and show that it commutes with $\longrightarrow_{\beta\pi}$ in the following sense: If $N_1 \longleftarrow^*_{\eta} M \longrightarrow^*_{\beta\pi} N_2$, then there is a P such that $N_1 \longrightarrow^*_{\beta\pi} P \longleftarrow^*_{\eta} N_2$.

[2] See also Exercise 18.4.19 in Barendregt's book [2].

- Finally, in Section 4.3 we use the Hindley-Rosen Lemma [2, p. 64] (and the well-known fact that $\longrightarrow_\eta$ is confluent) to conclude that $\longrightarrow_{\beta\eta\pi}$ is confluent.

Earlier, van Oostrom used a similar approach to prove confluence of η-expansion (together with β-reduction) in the pure lambda calculus [10].

4.1 Confluence of an extensionality-free subrelation

In order to define the subrelation $\longrightarrow_{\beta\pi}$ of $\longrightarrow^*_{\beta\eta\pi}$ we need the auxiliary notion of π*-neutral terms*:

Definition 1. The π*-neutral terms* are generated by the following (sub)grammar:

$$A \ ::= \ \lambda x.M \mid \pi_1\, A \mid \pi_2\, A$$

In other words, the π-neutral terms are those of the form $\pi_{i_1}(\cdots(\pi_{i_n}(\lambda x.M))\cdots)$ for some $n \geq 0$.

The relation $\rhd_{\beta\pi}$ is defined by the axioms in Figure 4. This relation generates the reduction relation $\longrightarrow_{\beta\pi}$.

$$\begin{array}{llcll}
(\beta) & (\lambda x.M)\, N & \rhd_{\beta\pi} & M[x := N] & \\
(\pi_1) & \pi_1\, \langle M, N\rangle & \rhd_{\beta\pi} & M & \\
(\pi_2) & \pi_2\, \langle M, N\rangle & \rhd_{\beta\pi} & N & \\
(\delta\pi) & \langle M, N\rangle\, P & \rhd_{\beta\pi} & \langle M\, P, N\, P\rangle & \\
(\pi_1\lambda) & \pi_1\, (\lambda x.M) & \rhd_{\beta\pi} & \lambda x.\pi_1\, M & \\
(\pi_2\lambda) & \pi_2\, (\lambda x.M) & \rhd_{\beta\pi} & \lambda x.\pi_2\, M & \\
(\pi_1\, \nu) & (\pi_1\, M)\, N & \rhd_{\beta\pi} & \pi_1\, (M\, N) & (\text{if } M \text{ is } \pi\text{-neutral}) \\
(\pi_2\, \nu) & (\pi_2\, M)\, N & \rhd_{\beta\pi} & \pi_2\, (M\, N) & (\text{if } M \text{ is } \pi\text{-neutral})
\end{array}$$

Figure 4: The relation $\rhd_{\beta\pi}$.

Note that $\longrightarrow_{\beta\pi}$ does not contain the "extensionality" axioms (η) and (SP). On the other hand, the new axioms ($\pi_1\nu$) and ($\pi_2\nu$) of $\longrightarrow_{\beta\pi}$ are derivable in $\longrightarrow^*_{\beta\eta\pi}$, using η-expansion:

$$(\pi_i\, M)\, N \longrightarrow_{\beta\eta\pi} (\pi_i\, (\lambda x.M\, x))\, N \longrightarrow_{\beta\eta\pi} (\lambda x.\pi_i\, (M\, x))\, N \longrightarrow_{\beta\eta\pi} \pi_i\, (M\, N)$$

Therefore, $\longrightarrow_{\beta\pi} \subseteq \longrightarrow^*_{\beta\eta\pi}$.

The key property of π-neutral terms is that if a term M is π-neutral, then no substitution instance of M can $\beta\pi$-reduce to a term of the form $\langle P, Q\rangle$:

Proposition 2.

(i) If M is π-neutral and $M \longrightarrow_{\beta\pi} M'$, then M' is π-neutral.

(ii) If M is π-neutral and N is an arbitrary term, then $M[x := N]$ is π-neutral.

(iii) No term of the form $\langle P, Q\rangle$ is π-neutral.

We now prove that $\longrightarrow_{\beta\pi}$ is confluent. The proof follows the Tait/Martin-Löf proof of confluence of β-reduction in the pure lambda calculus [2, p. 60]: First we define a "parallel" [14] reduction relation $\Longrightarrow_{\beta\pi}$, shown in Figure 5.

$$\frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N'}{(\lambda x.M)\, N \Longrightarrow_{\beta\pi} M'[x := N']}$$

$$\frac{M \Longrightarrow_{\beta\pi} M'}{\pi_1\,\langle M, N\rangle \Longrightarrow_{\beta\pi} M'} \qquad \frac{N \Longrightarrow_{\beta\pi} N'}{\pi_2\,\langle M, N\rangle \Longrightarrow_{\beta\pi} N'}$$

$$\frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N' \qquad P \Longrightarrow_{\beta\pi} P'}{\langle M, N\rangle\, P \Longrightarrow_{\beta\pi} \langle M'\,P', N'\,P'\rangle}$$

$$\frac{M \Longrightarrow_{\beta\pi} M'}{\pi_1\,(\lambda x.M) \Longrightarrow_{\beta\pi} \lambda x.\pi_1\, M'} \qquad \frac{M \Longrightarrow_{\beta\pi} M'}{\pi_2\,(\lambda x.M) \Longrightarrow_{\beta\pi} \lambda x.\pi_2\, M'}$$

$$\frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N'}{(\pi_1\, M)\, N \Longrightarrow_{\beta\pi} \pi_1\,(M'\,N')} \quad (M\ \pi\text{-neutral})$$

$$\frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N'}{(\pi_2\, M)\, N \Longrightarrow_{\beta\pi} \pi_2\,(M'\,N')} \quad (M\ \pi\text{-neutral})$$

$$\frac{}{M \Longrightarrow_{\beta\pi} M} \qquad \frac{M \Longrightarrow_{\beta\pi} M'}{\lambda x.M \Longrightarrow_{\beta\pi} \lambda x.M'}$$

$$\frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N'}{M\,N \Longrightarrow_{\beta\pi} M'\,N'} \qquad \frac{M \Longrightarrow_{\beta\pi} M' \qquad N \Longrightarrow_{\beta\pi} N'}{\langle M, N\rangle \Longrightarrow_{\beta\pi} \langle M', N'\rangle}$$

$$\frac{M \Longrightarrow_{\beta\pi} M'}{\pi_1\, M \Longrightarrow_{\beta\pi} \pi_1\, M'} \qquad \frac{M \Longrightarrow_{\beta\pi} M'}{\pi_2\, M \Longrightarrow_{\beta\pi} \pi_2\, M'}$$

Figure 5: Parallel $\beta\pi$-reduction $\Longrightarrow_{\beta\pi}$.

Proposition 3.

(i) $\longrightarrow^*_{\beta\pi} = \Longrightarrow^*_{\beta\pi}$.

(ii) *If* $M \Longrightarrow_{\beta\pi} M'$ *and* $N \Longrightarrow_{\beta\pi} N'$, *then* $M[x := N] \Longrightarrow_{\beta\pi} M'[x := N']$.

(iii) *If* $M \longrightarrow^*_{\beta\pi} M'$ *and* $N \longrightarrow^*_{\beta\pi} N'$, *then* $M[x := N] \longrightarrow^*_{\beta\pi} M'[x := N']$.

Proof. Standard [2, p. 60]. Part (iii) follows from the first two parts and will be used in the next section. □

Proposition 4. *The relation* $\Longrightarrow_{\beta\pi}$ *satisfies the diamond property: If* $M \Longrightarrow_{\beta\pi} N_1$ *and* $M \Longrightarrow_{\beta\pi} N_2$, *then there is a* P *such that* $N_1 \Longrightarrow_{\beta\pi} P$ *and* $N_2 \Longrightarrow_{\beta\pi} P$.

Proof. By induction on the derivations of $M \Longrightarrow_{\beta\pi} N_1$ and $M \Longrightarrow_{\beta\pi} N_2$ according to the rules in Figure 5. Many of the cases are well-known from the proof of confluence of β-reduction. We show the interesting new cases:

- $(\lambda x.\pi_i\, M_1)\, N_1 \Longleftarrow_{\beta\pi} (\pi_i\, (\lambda x.M))\, N \Longrightarrow_{\beta\pi} \pi_i\, ((\lambda x.M_2)\, N_2)$, where $M \Longrightarrow_{\beta\pi} M_1, M_2$ and $N \Longrightarrow_{\beta\pi} N_1, N_2$.

  By induction hypothesis, there are M_3 and N_3 such that $M_1, M_2 \Longrightarrow_{\beta\pi} M_3$ and $N_1, N_2 \Longrightarrow_{\beta\pi} N_3$. Then $\pi_i\, M_1 \Longrightarrow_{\beta\pi} \pi_i\, M_3$, hence $(\lambda x.\pi_i\, M_1)\, N_1 \Longrightarrow_{\beta\pi} \pi_i\, (M_3[x := N_3])$. Also, $\pi_i\, ((\lambda x.M_2)\, N_2) \Longrightarrow_{\beta\pi} \pi_i\, (M_3[x := N_3])$.

- $(\pi_i\, M_1)\, N_1 \Longleftarrow_{\beta\pi} (\pi_i\, M)\, N \Longrightarrow_{\beta\pi} \pi_i\, (M_2\, N_2)$, where M is π-neutral, $M \Longrightarrow_{\beta\pi} M_1, M_2$, and $N \Longrightarrow_{\beta\pi} N_1, N_2$.

  By induction hypothesis, there are M_3 and N_3 such that $M_1, M_2 \Longrightarrow_{\beta\pi} M_3$ and $N_1, N_2 \Longrightarrow_{\beta\pi} N_3$. By Proposition 2(i) and 3(i), M_1 is π-neutral. Therefore, $(\pi_i\, M_1)\, N_1 \Longrightarrow_{\beta\pi} \pi_i\, (M_3\, N_3)$ and $\pi_i\, (M_2\, N_2) \Longrightarrow_{\beta\pi} \pi_i\, (M_3\, N_3)$. □

Corollary 5. *The relation* $\longrightarrow_{\beta\pi}$ *is confluent.*

Remark. Without the restriction to π-neutral terms in two of the rules, $\Longrightarrow_{\beta\pi}$ would *not* satisfy the diamond property: Then we would have $(\pi_1\, \langle x, y\rangle)\, z \Longrightarrow_{\beta\pi} x\, z$ and $(\pi_1\, \langle x, y\rangle)\, z \Longrightarrow_{\beta\pi} \pi_1\, (\langle x, y\rangle\, z)$, but *not* $\pi_1\, (\langle x, y\rangle\, z) \Longrightarrow_{\beta\pi} x\, z$.

4.2 Eta-expansion commutes with $\longrightarrow_{\beta\pi}$

We define $\rhd_\eta$ by the axioms in Figure 6. This relation generates the η/SP-expansion relation $\longrightarrow_\eta$.

The purpose of this section is to show that $\longrightarrow_{\beta\pi}$ commutes with $\longrightarrow_\eta$: If $N_1 \longleftarrow^*_\eta M \longrightarrow^*_{\beta\pi} N_2$, then there is a P such that $N_1 \longrightarrow^*_{\beta\pi} P \longleftarrow^*_\eta N_2$. In order to prove this result we define "parallel" η/SP-expansion $\Longrightarrow_\eta$ [6, 14], shown in Figure 7.

$$
\begin{array}{llll}
(\eta) & M & \rhd_\eta & \lambda x.M\,x \qquad\qquad (\text{if } x \notin \mathrm{FV}(M)) \\
(\textsc{sp}) & M & \rhd_\eta & \langle \pi_1\,M, \pi_2\,M\rangle
\end{array}
$$

Figure 6: The relation $\rhd_\eta$.

$$
\frac{M \Longrightarrow_\eta M'}{M \Longrightarrow_\eta \lambda x.M'\,x}\ (x \notin \mathrm{FV}(M)) \qquad \frac{M \Longrightarrow_\eta M'}{M \Longrightarrow_\eta \langle \pi_1\,M', \pi_2\,M'\rangle}
$$

$$
\frac{}{M \Longrightarrow_\eta M} \qquad \frac{M \Longrightarrow_\eta M'}{\lambda x.M \Longrightarrow_\eta \lambda x.M'}
$$

$$
\frac{M \Longrightarrow_\eta M' \qquad N \Longrightarrow_\eta N'}{M\,N \Longrightarrow_\eta M'\,N'} \qquad \frac{M \Longrightarrow_\eta M' \qquad N \Longrightarrow_\eta N'}{\langle M, N\rangle \Longrightarrow_\eta \langle M', N'\rangle}
$$

$$
\frac{M \Longrightarrow_\eta M'}{\pi_1\,M \Longrightarrow_\eta \pi_1\,M'} \qquad \frac{M \Longrightarrow_\eta M'}{\pi_2\,M \Longrightarrow_\eta \pi_2\,M'}
$$

Figure 7: Parallel η/SP-expansion $\Longrightarrow_\eta$.

Proposition 6.

(i) $\longrightarrow_\eta^* = \Longrightarrow_\eta^*$.

(ii) $\longrightarrow_\eta$ *is confluent.*

(iii) If $M \Longrightarrow_\eta M'$ *and* $N \Longrightarrow_\eta N'$, *then* $M[x := N] \Longrightarrow_\eta M'[x := N']$.

Proof. Standard [6]. The confluence of $\longrightarrow_\eta$ follows from the diamond property of $\Longrightarrow_\eta$. □

We now aim to prove that if $N_1 \Longleftarrow_\eta M \longrightarrow_{\beta\pi} N_2$, then there is a P such that $N_1 \longrightarrow_{\beta\pi}^* P \Longleftarrow_\eta N_2$. For most of the different cases (according to the axioms and congruence rules generating $\longrightarrow_{\beta\pi}$) this property can be shown using the following two lemmas.

Lemma 7. *If* $\lambda x.M \Longrightarrow_\eta N$, *then*

(i) there is a P *such that* $N\,x \longrightarrow_{\beta\pi}^* P \Longleftarrow_\eta M$, *and*

(ii) there is a Q *such that for* $i \in \{1,2\}$: $\pi_i\,N \longrightarrow_{\beta\pi}^* \lambda x.\pi_i\,Q$ *and* $M \Longrightarrow_\eta Q$.

Proof. By induction on the definition of $\lambda x.M \Longrightarrow_\eta N$. □

Lemma 8. *If* $\langle M_1, M_2\rangle \Longrightarrow_\eta N$, *then*

(i) for $i \in \{1,2\}$ *there is a* P_i *such that* $\pi_i\,N \longrightarrow_{\beta\pi}^* P_i \Longleftarrow_\eta M_i$, *and*

(ii) there are Q_1, Q_2 *such that* $N\,x \longrightarrow_{\beta\pi}^* \langle Q_1\,x, Q_2\,x\rangle$ *and also* $M_1 \Longrightarrow_\eta Q_1$ *and* $M_2 \Longrightarrow_\eta Q_2$.

Proof. By induction on the definition of $\langle M_1, M_2\rangle \Longrightarrow_\eta N$. □

The most complicated case is $N \Longleftarrow_\eta (\pi_i\,M_1)\,M_2 \longrightarrow_{\beta\pi} \pi_i\,(M_1\,M_2)$ (where M_1 is π-neutral). Here we use two additional lemmas. In the proof of Lemma 10 we need to perform induction on the *height* of derivations of "$M \Longrightarrow_\eta N$", considering these derivations as finite trees constructed according to the rules in Figure 7.

Lemma 9. *If* $M \Longrightarrow_\eta N$ *and* M *is* π*-neutral, then there is a* π*-neutral* P *such that*

(i) for $i \in \{1,2\}$, $\pi_i\,N \longrightarrow_{\beta\pi}^* \pi_i\,P$,

(ii) $M \Longrightarrow_\eta P$, *and*

(iii) for any given derivation of $M \Longrightarrow_\eta N$ *of height* n, *one can find a derivation of* $M \Longrightarrow_\eta P$ *of height no greater than* n.

Proof. By induction on the definition of $M \Longrightarrow_\eta N$. Since M is π-neutral there are only a few cases to consider.

Case 1: N is π-neutral. Then we choose $P = N$.

Case 2: $N = \pi_i\, N'$ and $M = \pi_i\, M'$ where $M' \Longrightarrow_\eta N'$ and M' is π-neutral. By the induction hypothesis there is a π-neutral P' such that $N = \pi_i\, N' \longrightarrow^*_{\beta\pi} \pi_i\, P'$ and $M' \Longrightarrow_\eta P'$. Now choose $P = \pi_i\, P'$.

Case 3: $N = \langle \pi_1\, N', \pi_2\, N' \rangle$ where $M \Longrightarrow_\eta N'$. By the induction hypothesis there is a π-neutral P' such that $\pi_1\, N' \longrightarrow^*_{\beta\pi} \pi_1\, P'$, $\quad \pi_2\, N' \longrightarrow^*_{\beta\pi} \pi_2\, P'$, and $M \Longrightarrow_\eta P'$. Then $\pi_1\, N \longrightarrow_{\beta\pi} \pi_1\, N' \longrightarrow^*_{\beta\pi} \pi_1\, P'$, and similarly $\pi_2\, N \longrightarrow^*_{\beta\pi} \pi_2\, P'$. Now choose $P = P'$.

It is easy to verify that if the given derivation of $M \Longrightarrow_\eta N$ has height n, then the above construction gives a derivation of $M \Longrightarrow_\eta P$ of height no greater than n. □

Lemma 10. *If $\pi_i\, M \Longrightarrow_\eta N$ and M is π-neutral, then there is a P such that $N\, x \longrightarrow^*_{\beta\pi} P \Longleftarrow_\eta \pi_i\, (M\, x)$.*

Proof. By induction on the height of the derivation of $\pi_i\, M \Longrightarrow_\eta N$. We show the interesting case: Assume that $N = \langle \pi_1\, N', \pi_2\, N' \rangle$ where $\pi_i\, M \Longrightarrow_\eta N'$. Let the height of the given derivation of $\pi_i\, M \Longrightarrow_\eta N$ be $n+1$; the height of the subderivation $\pi_i\, M \Longrightarrow_\eta N'$ is then n. By Lemma 9 there is a π-neutral Q such that $\pi_1\, N' \longrightarrow^*_{\beta\pi} \pi_1\, Q$, $\quad \pi_2\, N' \longrightarrow^*_{\beta\pi} \pi_2\, Q$, and $\pi_i\, M \Longrightarrow_\eta Q$. Furthermore, the lemma gives a derivation of $\pi_i\, M \Longrightarrow_\eta Q$ of height no greater than n. Therefore the induction hypothesis gives a P' such that $Q\, x \longrightarrow^*_{\beta\pi} P' \Longleftarrow_\eta \pi_i\, (M\, x)$. Hence,

$$
\begin{array}{rll}
N\, x = \langle \pi_1\, N', \pi_2\, N' \rangle\, x & \longrightarrow^*_{\beta\pi} & \langle \pi_1\, Q, \pi_2\, Q \rangle\, x \\
& \longrightarrow_{\beta\pi} & \langle (\pi_1\, Q)\, x, (\pi_2\, Q)\, x \rangle \\
& \longrightarrow^*_{\beta\pi} & \langle \pi_1\, (Q\, x), \pi_2\, (Q\, x) \rangle \\
& \longrightarrow^*_{\beta\pi} & \langle \pi_1\, P', \pi_2\, P' \rangle \\
& \Longleftarrow_\eta & \pi_i\, (M\, x).
\end{array}
$$

□

We now prove the main lemma needed in the commutation proof:

Lemma 11. *If $N \Longleftarrow_\eta M \longrightarrow_{\beta\pi} M'$, then there is a P such that $N \longrightarrow^*_{\beta\pi} P \Longleftarrow_\eta M'$.*

Proof. Induction on the definition of $M \Longrightarrow_\eta N$, using Lemmas 7-10. We show some illustrative cases.

Case 1: $\langle \pi_1 N', \pi_2 N' \rangle \Longleftarrow_\eta M \longrightarrow_{\beta\pi} M'$ where $N' \Longleftarrow_\eta M$. By the induction hypothesis there is a P' such that $N' \longrightarrow^*_{\beta\pi} P' \Longleftarrow_\eta M'$. Then

$$\langle \pi_1 N', \pi_2 N' \rangle \longrightarrow^*_{\beta\pi} \langle \pi_1 P', \pi_2 P' \rangle \Longleftarrow_\eta M'$$

so we choose $P = \langle \pi_1 P', \pi_2 P' \rangle$.

Case 2: $N_1 N_2 \Longleftarrow_\eta (\lambda x.M_1) M_2 \longrightarrow_{\beta\pi} M_1[x := M_2]$ where $N_1 \Longleftarrow_\eta \lambda x.M_1$ and $N_2 \Longleftarrow_\eta M_2$. Without loss of generality, $x \notin \mathrm{FV}(N_1)$. By Lemma 7(i) there is a P' such that $N_1 x \longrightarrow^*_{\beta\pi} P' \Longleftarrow_\eta M_1$. Then by Propositions 3 and 6, $N_1 N_2 \longrightarrow^*_{\beta\pi} P'[x := N_2] \Longleftarrow_\eta M_1[x := M_2]$, so we choose $P = P'[x := N_2]$.

Case 3: $N_1 N_2 \Longleftarrow_\eta (\pi_i M_1) M_2 \longrightarrow_{\beta\pi} \pi_i (M_1 M_2)$ where M_1 is π-neutral, $N_1 \Longleftarrow_\eta \pi_i M_1$, and $N_2 \Longleftarrow_\eta M_2$. Choose $x \notin \mathrm{FV}(N_1)$. Lemma 10 gives a P' such that $N_1 x \longrightarrow^*_{\beta\pi} P' \Longleftarrow_\eta \pi_i (M_1 x)$. Then by Propositions 3 and 6, $N_1 N_2 \longrightarrow^*_{\beta\pi} P'[x := N_2] \Longleftarrow_\eta \pi_i (M_1 M_2)$, so we choose $P = P'[x := N_2]$. □

Lemma 12.

(i) If $N \Longleftarrow_\eta M \longrightarrow^*_{\beta\pi} M'$, then there is a P such that $N \longrightarrow^*_{\beta\pi} P \Longleftarrow_\eta M'$.

(ii) If $N \Longleftarrow^*_\eta M \longrightarrow^*_{\beta\pi} M'$, then there is a P such that $N \longrightarrow^*_{\beta\pi} P \Longleftarrow^*_\eta M'$.

Proof.

(i) By induction on the length of the reduction sequence $M \longrightarrow^*_{\beta\pi} M'$, using Lemma 11.

(ii) By induction on the length of the reduction sequence $M \Longrightarrow^*_\eta N$, using Part (i). □

By Proposition 6(i), $\longrightarrow^*_\eta = \Longrightarrow^*_\eta$. We therefore conclude from Lemma 12(ii) that the relations $\longrightarrow_{\beta\pi}$ and $\longrightarrow_\eta$ commute:

Proposition 13. *If $N \longleftarrow^*_\eta M \longrightarrow^*_{\beta\pi} M'$, then there is a P such that $N \longrightarrow^*_{\beta\pi} P \longleftarrow^*_\eta M'$.*

4.3 Confluence of $\longrightarrow_{\beta\eta\pi}$

We now use the results of Sections 4.1 and 4.2 to prove the main result of Section 4:

Proposition 14. *The relation $\longrightarrow_{\beta\eta\pi}$ is confluent.*

Proof. Proposition 5 states that $\longrightarrow_{\beta\pi}$ is confluent, Proposition 6(ii) states that $\longrightarrow_{\eta}$ is confluent, and Proposition 13 states that $\longrightarrow_{\beta\pi}$ commutes with $\longrightarrow_{\eta}$. By the Hindley-Rosen Lemma [5], the relation $\longrightarrow^*_{\beta\eta\pi} = \longrightarrow^*_{\beta\pi} \cup \longrightarrow^*_{\eta}$ is confluent. More specifically, by constructing the following diagram we see that the composition of $\longrightarrow^*_{\beta\pi}$ with $\longrightarrow^*_{\eta}$ satisfies the diamond property:

$$
\begin{tikzcd}
 & & M \arrow[dl, "*"', "\beta\pi"] \arrow[dr, "*", "\beta\pi"'] & & \\
 & \cdot \arrow[dl, "*"', "\eta"] \arrow[dr, dashed, "*", "\beta\pi"'] & & \cdot \arrow[dl, dashed, "*"', "\beta\pi"] \arrow[dr, "*", "\eta"'] & \\
N_1 \arrow[dr, dashed, "*", "\beta\pi"'] & & \cdot \arrow[dl, dashed, "*"', "\eta"] \arrow[dr, dashed, "*", "\eta"'] & & N_2 \arrow[dl, dashed, "*"', "\beta\pi"] \\
 & \cdot \arrow[dr, dashed, "*", "\eta"'] & & \cdot \arrow[dl, dashed, "*"', "\eta"] & \\
 & & P & &
\end{tikzcd}
$$

$\square$

Corollary 15 (Church-Rosser property). *If $M =_{\beta\eta\pi} N$, then there is a P such that $M \longrightarrow^*_{\beta\eta\pi} P$ and $N \longrightarrow^*_{\beta\eta\pi} P$.*

Proof. Follows from confluence of $\longrightarrow_{\beta\eta\pi}$ in the standard way [2, p. 54]. $\square$

Remark. Orienting the axioms (SP) and (η) of $\longrightarrow_{\beta\eta\pi}$ as *contraction* axioms does not give rise to a confluent reduction relation: With these axioms we would have $\langle y, z\rangle \longleftarrow_{\beta\eta\pi} \lambda x.(\langle y, z\rangle\, x) \longrightarrow_{\beta\eta\pi} \lambda x.\langle y\, x, z\, x\rangle$, but both $\langle y, z\rangle$ and $\lambda x.\langle y\, x, z\, x\rangle$ would be normal forms.

5 Main result

We are now almost in a position to prove the main result: Suppose M and N are pure λ-terms such that $M =_{\beta\eta\mathrm{SP}} N$. Then $M =_{\beta\eta\pi} N$, and by the Church-Rosser property (Corollary 15) there is a P such that $M \longrightarrow^*_{\beta\eta\pi} P$ and $N \longrightarrow^*_{\beta\eta\pi} P$. However, since $\longrightarrow_{\beta\eta\pi}$ contains SP-*expansion*, we cannot immediately conclude that P is a pure λ-term with $M \longrightarrow^*_{\beta\eta} P$ and $N \longrightarrow^*_{\beta\eta} P$.

Definition 16 (π-erasure). The *π-erasure* of a λ-term M is the pure λ-term $|M|$ defined inductively as follows:

$$
\begin{array}{rcl}
|x| & = & x \\
|M\, N| & = & |M|\,|N| \\
|\lambda x.M| & = & \lambda x.|M| \\
|\langle M, N\rangle| & = & |M| \\
|\pi_1\, M| & = & |M| \\
|\pi_2\, M| & = & |M|
\end{array}
$$

We could just as well have defined $|\langle M, N\rangle|$ as $|N|$, since we are only interested in $|P|$ when P is π-*symmetric*:

Definition 17. A λ-term M is π-*symmetric* if for every subterm of M of the form $\langle P, Q\rangle$, the π-erasures of P and Q are $\beta\eta$-equivalent: $|P| =_{\beta\eta} |Q|$.

In particular, every pure λ-term is π-symmetric.

Proposition 18.

(i) $|M[x := N]| = |M|[x := |N|]$

(ii) If M and N are π-symmetric, then $M[x := N]$ is π-symmetric.

Proof. By induction on M. □

Proposition 19. *If M is π-symmetric and $M \longrightarrow_{\beta\eta\pi} N$, then*

(i) $|M| =_{\beta\eta} |N|$, *and*

(ii) N is π-symmetric.

Proof. By induction on the definition of $M \longrightarrow_{\beta\eta\pi} N$, using Proposition 18. □

Now we are ready to prove that $\lambda_{\beta\eta\pi}$ is a conservative extension of $\lambda_{\beta\eta}$:

Theorem 20. *Let M, N be pure λ-terms. If $M =_{\beta\eta\pi} N$, then $M =_{\beta\eta} N$.*

Proof. Suppose M and N are pure λ-terms such that $M =_{\beta\eta\pi} N$. By the Church-Rosser property (Corollary 15) there is a P such that $M \longrightarrow^*_{\beta\eta\pi} P$ and $N \longrightarrow^*_{\beta\eta\pi} P$. Since M and N are pure, they are in particular π-symmetric; it follows from Proposition 19 that P is π-symmetric, and that $|M| =_{\beta\eta} |P| =_{\beta\eta} |N|$. Hence,

$$M = |M| =_{\beta\eta} |P| =_{\beta\eta} |N| = N.$$ □

Corollary 21. *The theory $\lambda_{\beta\eta\pi}$ is consistent.*

Proof. By Theorem 20 and consistency of $\lambda_{\beta\eta}$ [2, p. 67]. □

Finally we turn to the main result of this article:

Theorem 22. *Let M, N be pure λ-terms. If $M =_{\beta\eta\mathrm{SP}} N$, then $M =_{\beta\eta} N$.*

Proof. By Theorem 20 and the fact that $\lambda_{\beta\eta\pi}$ is an extension of $\lambda_{\beta\eta\mathrm{SP}}$. □

We have also obtained a new—syntactic—proof of consistency of $\lambda_{\beta\eta\mathrm{SP}}$:

Corollary 23. *The theory $\lambda_{\beta\eta\mathrm{SP}}$ is consistent.*

Remark. The question of conservativity was originally formulated in a slightly different setting [7]: Let D, D_1 and D_2 be three new constants, and add the following axioms to the pure $\lambda_{\beta\eta}$-calculus:

$$\begin{array}{rcl} D_1\,(D\,M\,N) & =_{\beta\eta D} & M \\ D_2\,(D\,M\,N) & =_{\beta\eta D} & N \\ D\,(D_1\,M)\,(D_2\,M) & =_{\beta\eta D} & M \end{array}$$

To see that the resulting theory $\lambda_{\beta\eta D}$ is conservative over $\lambda_{\beta\eta}$, one can simulate $\lambda_{\beta\eta D}$ in $\lambda_{\beta\eta\mathrm{SP}}$ by defining D as $\lambda x.\lambda y.\langle x, y\rangle$, D_1 as $\lambda x.\pi_1\,x$, and D_2 as $\lambda x.\pi_2\,x$.

6 Related problems

The conservativity proof presented here can be adapted to the non-extensional case settled by de Vrijer [15], i.e., a minor modification gives an alternative proof that $\lambda_{\beta\mathrm{SP}}$ is conservative over the lambda calculus λ_β. To this end, one should remove the axiom (η) from every definition and add the two $(\pi_i\nu)$ axioms to the definition of $\longrightarrow_{\beta\eta\pi}$. The electronic, formalized version of the proof allows for a straightforward verification that the modification is correct.

Another related problem posed by Klop and de Vrijer is still open: whether the reduction relation $\longrightarrow_{\beta\eta\mathrm{SP}}$ has the *unique normal-form property* [7]. The theory $\lambda_{\beta\eta\pi}$ does not seem useful in solving that problem.

Meyer asked whether *any* lambda theory can be conservatively extended with surjective pairing [4]. That problem also remains open.

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A Formalized statement of the main result

```
%%% Terms of the untyped lambda calculus with surjective pairing.

term : type.

@ : term -> term -> term.  %infix left 10 @.
lam : (term -> term) -> term.
p1 : term -> term.
p2 : term -> term.
pair : term -> term -> term.


%%% Lambda calculus with the extensionality rules eta and SP.

==SP : term -> term -> type.  %infix none 5 ==SP.

sp_beta : (lam F) @ N ==SP F N.

sp_eta : lam ([x] M @ x) ==SP M.

sp_proj1 : p1 (pair M N) ==SP M.

sp_proj2 : p2 (pair M N) ==SP N.

sp_SP : pair (p1 M) (p2 M) ==SP M.

% Congruence rules.

sp_refl : M ==SP M.

sp_sym : M ==SP N -> N ==SP M.

sp_trans : M ==SP N -> N ==SP P -> M ==SP P.

sp_c-app : M @ N ==SP M' @ N'
            <- M ==SP M'
            <- N ==SP N'.

sp_c-lam : lam F ==SP lam F'
            <- ({x} F x ==SP F' x).
```

```
sp_c-p1 : p1 M ==SP p1 M'
           <- M ==SP M'.

sp_c-p2 : p2 M ==SP p2 M'
           <- M ==SP M'.

sp_c-pair : pair M N ==SP pair M' N'
             <- M ==SP M'
             <- N ==SP N'.


%%% Pure lambda-terms, i.e., no "pair", "p1", or "p2".

pterm : type.

^ : pterm -> pterm -> pterm.  %infix left 10 ^.
lambda : (pterm -> pterm) -> pterm.

%block pvar : block {y : pterm}.


%%% Beta-eta equality on pure terms.

==be : pterm -> pterm -> type.  %infix none 5 ==be.

be_beta : (lambda F) ^ N ==be F N.

be_eta : lambda ([x] M ^ x) ==be M.

% Congruence rules.

be_refl : M ==be M.

be_sym : M ==be N -> N ==be M.

be_trans : M ==be N -> N ==be P -> M ==be P.

be_c-app : M ^ N ==be M' ^ N'
           <- M ==be M'
           <- N ==be N'.

be_c-lam : lambda F ==be lambda F'
           <- ({x} F x ==be F' x).
```

```
%%% Injecting pure terms into the general terms.

inject : pterm -> term -> type.
%mode inject +P -T.

inj_app : inject (P1 ^ P2) (M1 @ M2)
           <- inject P1 M1
           <- inject P2 M2.

inj_lam : inject (lambda P) (lam M)
           <- ({x} {y} inject x y -> inject (P x) (M y)).

%block inj : block {x : pterm} {y : term} {thm : inject x y}.

%worlds (inj) (inject _ _).
%total P (inject P _).


%%% The main theorem: ==SP is conservative over ==be.

conservative : inject M M' -> inject N N'
                           -> M' ==SP N'
                           -> M ==be N
                           -> type.
%mode conservative +I1 +I2 +E1 -E2.

% [The proof is omitted.]

%worlds () (conservative _ _ _ _).
%total I1 (conservative I1 _ _ _).

% With empty "worlds", the main theorem is actually only shown
% for closed terms. (The generalization to open terms easily
% follows by lambda-abstracting every free variable).
```

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