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Philipp Gerhardy

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Abstract

In [J.Math.Anal.App.277(2003) 645-650], W.A.Kirk introduced the notion of asymptotic contractions and proved a fixed point theorem for such mappings. Using techniques from proof mining, we develop a variant of the notion of asymptotic contractions and prove a quantitative version of the corresponding fixed point theorem.

1 Introduction

In [3], W.A. Kirk proved a fixed-point theorem for so-called asymptotic contractions on complete metric spaces, showing that given a *continuous*[1] asymptotic contraction f for every starting point x the iteration sequence $\{f^n(x)\}$ converges to the unique fixed point of f. The proof is non-elementary, as it uses an ultrapower construction to establish the fixed point theorem. Recent alternative proofs by Jachymski and Jóźwik[2], additionally assuming that f is *uniformly* continuous, and by Arandelović [1], under the same assumptions as Kirk, are elementary and avoid ultrapowers, but neither of the three proofs provides explicit rates of convergence.

[1] In [2, 1], it is discussed that the requirement that f is continuous is a necessary condition for Kirk's fixed point theorem. By an oversight the requirement was left out in the original statement of Kirk's fixed point theorem in [3]

Using techniques from proof mining as developed e.g. in [5, 4], we first derive a suitable generalization of the notion of asymptotic contractivity and subsequently give an elementary proof of Kirk's fixed point theorem, providing an explicit rate of convergence[2] (to the unique fixed point) for sequences $\{f^n(x)\}$.

In detail, we show that:

- the rate of convergence only depends on the starting point x via a bound on the iteration sequence $\{f^n(x)\}$,
- the rate of convergence only depends on the function f via suitable moduli expressing its asymptotic contractivity,
- the continuity of f is only necessary to prove the existence of a unique fixed point, while the convergence to such a fixed point can be proved without the continuity of f.

2 Preliminaries

In [3], Kirk defines asymptotic contractions as follows:

Definition 2.1 (Kirk[3]). *A function $f : X \to X$ on a metric space (X, d) is called an asymptotic contraction with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ if ϕ, ϕ_n are continuous, $\phi(s) < s$ for all $s > 0$ and for all $x, y \in X$*

$$d(f^n(x), f^n(y)) \leq \phi_n(d(x, y))$$

and moreover $\phi_n \to \phi$ uniformly on the range of d.

What is needed to prove the fixed point theorem are not so much the moduli ϕ, ϕ_n, but instead a function η producing a witness of the inequality $\phi(s) < s$ and a modulus of convergence β for ϕ_n yielding a K s.t. for all $k \geq K$ ϕ_k

[2]Since an asymptotic contraction need not be non-expansive (cf. Example 2 in [2]), convergence need not be monotone, and hence an effective rate of convergence can at most produce a bound M s.t. $f^m(x)$ is close to the unique fixed point for some $m \leq M$. We will discuss the details later.

is close enough to ϕ and hence f^k is a contraction. For η it is sufficient to provide a witness for every interval $[l, b]$, for β it suffices to have uniform convergence on every interval $[l, b]$, in both cases with $0 < l \leq b < \infty$.

Thus, to give an elementary and effective proof of the fixed point theorem proved by Kirk, we derive the following generalized definition of asymptotic contractions:

Definition 2.2. *A function $f : X \to X$ on a metric space (X, d) is called an asymptotic contraction if for each $b > 0$ there exist moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ and the following hold:*
(1) there exists a sequence of functions $\phi_n : (0, \infty) \to (0, \infty)$ s.t. for all $x, y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$

$$b \geq d(x, y) \geq \varepsilon \to d(f^n(x), f^n(y)) \leq \phi_n(\varepsilon) \cdot d(x, y),$$

(2) for each $0 < l \leq b$ the function $\beta^b_l := \beta^b(l, \cdot)$ is a modulus of uniform convergence for ϕ_n on $[l, b]$, i.e.

$$\forall \varepsilon > 0 \; \forall s \in [l, b] \; \forall m, n \geq \beta^b_l(\varepsilon) \; (|\phi_m(s) - \phi_n(s)| \leq \varepsilon),$$

and (3) defining $\phi := \lim_{n\to\infty} \phi_n$, then for each $\varepsilon > 0$ we have that $\eta^b(\varepsilon) > 0$ and $\phi(s) + \eta^b(\varepsilon) \leq s$ for each $s \in [\varepsilon, b]$.

All the relevant information is contained in the moduli η and β and we do not need to refer to ϕ, ϕ_n at all, as the following proposition shows:

Proposition 2.3. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η^b, β^b be given. Then for every $\varepsilon > 0$ there is a $K(\eta^b, \beta^b, \varepsilon)$ s.t. for all $k \geq K$, where $K = \beta^b_\varepsilon(\frac{\eta^b(\varepsilon)}{2})$,*

$$b \geq d(x, y) \geq \varepsilon \to d(f^k(x), f^k(y)) \leq (1 - \frac{\eta^b(\varepsilon)}{2}) \cdot d(x, y).$$

Proof: Let $K = \beta^b_\varepsilon(\frac{\eta^b(\varepsilon)}{2})$, let a suitable sequence ϕ_n be given and let $\phi := \lim_{n\to\infty} \phi_n$. By the definition of η^b we have that $\phi(s) + \eta^b(\varepsilon) \leq 1$ for $s \in [\varepsilon, b]$. By the definition of β^b the function ϕ_k is at least $\frac{\eta^b(\varepsilon)}{2}$-close to ϕ for all $k \geq K$ and for all $s \in [\varepsilon, b]$ and hence also $\phi_k(s) \leq 1 - \frac{\eta^b(\varepsilon)}{2}$. $\square$

Remark 2.4. *Requiring moduli η^b and β^b for each individual $b > 0$ instead of one pair of moduli for all $b > 0$ is no restriction. In the proof given in [3], it is assumed that some iteration sequence of the asymptotic contraction f is bounded, which allows to prove that every iteration sequence is bounded. As we will see, to prove the fixed point theorem it suffices to have moduli η^b and β^b for the corresponding b-bounded subsets of (X, d).*

Next, we show that Kirk's notion of asymptotic contractivity implies the more general notion in Definition 2.2.

Definition 2.5. *Let the function ϕ, a sequence of functions ϕ_n and $b > 0$ be given. Define:*

$$\begin{array}{ll} \phi'(s) := \frac{\phi(s)}{s} \text{ for } s \in (0, \infty), & \phi'_n(s) := \frac{\phi_n(s)}{s} \text{ for } s \in (0, \infty), \\ \phi''(s) := \max\limits_{t \in [s,b]} \phi'(t) \text{ for } s \in (0, b], & \phi''_n(s) := \max\limits_{t \in [s,b]} \phi'_n(t) \text{ for } s \in (0, b]. \end{array}$$

Proposition 2.6. *Let $\phi : [0, \infty) \to [0, \infty)$ be continuous, let ϕ satisfy $\phi(s) < s$ for $s > 0$ and let $\phi_n : [0, \infty) \to [0, \infty)$ be a sequence of continuous functions converging uniformly to ϕ. Then*

- *ϕ' and ϕ'_n are continuous on $(0, \infty)$, $\phi'(s) < 1$ for all $s \in (0, \infty)$ and the sequence ϕ'_n converges uniformly to ϕ' on $[l, \infty)$ for each $l > 0$,*
- *ϕ'' and ϕ''_n are continuous on $(0, b]$, $\phi''(s) < 1$ for all $s \in (0, b]$ and the sequence ϕ''_n converges uniformly to ϕ'' on $[l, b]$ for each $0 < l \leq b < \infty$.*

Proof: Obvious. □

Remark 2.7. *The moduli η^b, β^b may equivalently be given as functions $\eta^b : \mathbb{N} \to \mathbb{N}$ and $\beta^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where real numbers are approximated from below by suitable 2^{-n}. Given $b > 0$, if ϕ and a modulus β for ϕ_n are given as computable number-theoretic functions, then η^b and β^b_l are effectively computable in b.*

Proposition 2.8. *If a function $f : X \to X$ on a metric space (X, d) is an asymptotic contraction with moduli ϕ, ϕ_n, then the function f is an asymptotic contraction with suitable moduli η^b, β^b for every $b > 0$.*

Proof: Follows from the above remarks and constructions. □

3 Main results

We are now in position to give an elementary proof of Kirk's fixed point theorem. The general idea of the proof is similar to the constructivization of Edelstein's fixed point theorem in [5]. We first derive (variants of) a modulus of uniqueness and of a modulus of asymptotic regularity. These two moduli combined give us a modulus of convergence for the iteration sequence and thereby of the convergence to the unique fixed point.

Throughout this section we assume that $f : X \to X$ is a self-mapping on a metric space (X, d). Given $x_0 \in X$ we write x_n for $f^n(x_0)$ and $\{x_n\}$ for the corresponding iteration sequence. When there is no ambiguity we will omit the superscript b from the moduli η^b, β^b.

Lemma 3.1. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η, β be given. Then for every $b \geq \varepsilon > 0$, for all $n \geq N$ and all $x, y \in X$ with $d(x, y) \leq b$*

$$d(x, f^n(x)), d(y, f^n(y)) \leq \delta \to d(x, y) \leq \varepsilon,$$

where $\delta(\eta, \varepsilon) = \frac{\eta(\varepsilon)\cdot\varepsilon}{4}$ and $N(\eta, \beta, \varepsilon) = \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$.

Proof: Let $b \geq \varepsilon > 0$ be given. Let $n \geq N$, then by Proposition 2.3

$$b \geq d(x, y) \geq \varepsilon \to d(f^n(x), f^n(y)) \leq (1 - \frac{\eta(\varepsilon)}{2}) \cdot d(x, y).$$

Let $d(x, f^n(x)), d(y, f^n(y)) \leq \delta$, with $\delta = \frac{\eta(\varepsilon)\cdot\varepsilon}{4}$ as described above and assume $d(x, y) > \varepsilon$. Then by the triangle inequality

$$\begin{aligned} d(x, y) &\leq d(x, f^n(x)) + d(f^n(x), f^n(y)) + d(y, f^n(y)) \\ &\leq \tfrac{\eta(\varepsilon)\cdot\varepsilon}{2} + (1 - \tfrac{\eta(\varepsilon)}{2}) \cdot d(x, y) \end{aligned}$$

and hence $\frac{\eta(\varepsilon)}{2} \cdot d(x, y) \leq \frac{\eta(\varepsilon)}{2} \cdot \varepsilon$ which implies $d(x, y) \leq \varepsilon$. But this contradicts the assumption $d(x, y) > \varepsilon$ and therefore $d(x, y) \leq \varepsilon$. $\square$

Lemma 3.2. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η, β be given. Then for every $\delta > 0$, for every $x_0 \in X$ s.t. $\{x_n\}$ is bounded by b and for every N there exists an $m \leq M$, s.t.*

$$d(x_m, f^N(x_m) < \delta,$$

where $M(\eta, \beta, \delta, b) = k \cdot \lceil (\frac{lg(\delta) - lg(b)}{lg(1 - \frac{\eta(\delta)}{2})}) \rceil$ with $k = \beta_\delta(\frac{\eta(\delta)}{2})$.

Proof: Let $k = \beta_\delta(\frac{\eta(\delta)}{2})$. Assume for some M_0 and all $m < M_0$ we have $d(x_{mk}, f^N(x_{mk})) \geq \delta$, then

$$d(x_{M_0 k}, f^N(x_{M_0 k})) \leq (1 - \frac{\eta(\delta)}{2})^{M_0} d(x_0, f^N(x_0)) \leq (1 - \frac{\eta(\delta)}{2})^{M_0} \cdot b$$

since by assumption $d(x_0, f^N(x_0)) \leq b$.

Solving the inequality $(1 - \frac{\eta(\delta)}{2})^{M_0} \cdot b < \delta$ w.r.t. M_0 yields the described upper bound $M = k \cdot M_0$ on an m s.t. $d(x_m, f^N(x_m)) < \delta$. $\square$

Remark 3.3. *Bounding m by M is in this context optimal. Since f^k only is a contraction with constant $(1 - \frac{\eta(\delta)}{2})$ for x, y s.t. $d(x, y) \geq \delta$, we cannot be certain, that $d(x_M, f^N(x_M)) < \delta$. If for some earlier $m < M$ we have $d(x_m, f^N(x_m)) < \delta$, then for these points we no longer have the guarantee that f^k is a contraction, and hence we have no (computable) control over the next iteration step, i.e. over $d(x_{m+k}, f^N(x_{m+k}))$.*
If the function f and the space (X, d) have a computable representation one can of course check $x_0, \ldots, x_M$ to find which one is close to the unique fixed point z.

Theorem 3.4. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η, β be given. Assume that f has a (unique) fixed point z. Then for every $\varepsilon > 0$ and every $x_0 \in X$ s.t. $\{x_n\}$ is bounded by b and $d(x_n, z) \leq b$ for all n there exists an $m \leq M$ s.t.*

$$d(x_m, z) \leq \varepsilon,$$

where $M(\eta, \beta, \varepsilon, b) = k \cdot \lceil (\frac{lg(\delta) - lg(b)}{lg(1 - \frac{\eta(\delta)}{2})}) \rceil$, $k = \beta_\delta(\frac{\eta(\delta)}{2})$, $\delta = \frac{\eta(\varepsilon) \cdot \varepsilon}{4}$.

Proof: By Lemma 3.1 for every $\varepsilon > 0$ there exist δ, N as described above s.t. if $d(x, f^N(x)), d(y, f^N(x)) \leq \delta$ then $d(x, y) \leq \varepsilon$. Any (trivially unique) fixed point z of f satisfies $d(z, f^N(z)) = 0 \leq \delta$, so if $d(x, f^N(x)) \leq \delta$ then $d(x, z) \leq \varepsilon$.

Now, by Lemma 3.2 for every δ and every N we can find an $m \leq M$ as described above s.t. $d(x_m, f^N(x_m)) < \delta$ and hence x_m is ε-close to the fixed point z. $\square$

Note, that the rate of convergence does not depend on the starting point x_0, but only on a bound b on $\{x_n\}$. Also, the rate of convergence only depends on f via the moduli η, β. Finally, if we know that a fixed point z exists, we do not need the continuity of f to prove $\{x_n\}$ converges to z.

Lemma 3.5. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η, β be given. Then for every $\delta > 0$, for every $x_0 \in X$ s.t. $\{x_n\}$ bounded by b and for every N there exists an M s.t. for all $m \geq M$*

$$d(x_m, f^N(x_m)) < \delta.$$

Proof: By Lemma 3.2 there exists an m s.t. $d(x_m, f^N(x_m)) < \delta$. Either $d(x_m, f^N(x_m)) = 0$ – then we are done – or $d(x_m, f^N(x_m)) > \varepsilon_0$ for some $\varepsilon_0 > 0$.

Let $K = \beta_{\varepsilon_0}(\frac{\eta(\varepsilon_0)}{2})$, then it follows by Proposition 2.3 that for all $k \geq K$

$$d(x_{m+k}), f^N(x_{m+k})) \leq (1 - \frac{\eta(\varepsilon_0)}{2})d(x_m, f^N(x_m)) < \delta.$$

Let $M = m + K$ and the result follows. $\square$

Lemma 3.6. *Let (X, d) be a metric space, let f be an asymptotic contraction and let $b > 0$ and η, β be given. If $\{x_n\}$ is bounded by b then $\{x_n\}$ is a Cauchy sequence.*

Proof: By Lemma 3.1 for every $\varepsilon > 0$ there exists a $\delta > 0$ and an N s.t. for all $x, y \in X$ if $d(x, f^N(x)), d(y, f^N(y) \leq \delta$ then $d(x, y) \leq \varepsilon$. By Lemma 3.5 for every $\delta > 0$ and every N there exists an M s.t. $d(x_m, f^N(x_m)) < \delta$ for all $m \geq M$. Then $d(x_m, x_n) \leq \varepsilon$ for all $m, n \geq M$. $\square$

Theorem 3.7. *Let (X, d) be a complete metric space, let f be a continuous asymptotic contraction and let $b > 0$ and η, β be given. If for some $x_0 \in X$ the sequence $\{x_n\}$ is bounded by b then f has a unique fixed point z, $\{x_n\}$ converges to z and for every $\varepsilon > 0$ there exists an $m \leq M$ s.t.*

$$d(x_m, z) \leq \varepsilon,$$

where M is as in Theorem 3.4.

Proof: By Lemma 3.6 every iteration sequence $\{x_n\}$ which is bounded is a Cauchy sequence. Since (X, d) is complete the limit z of $\{x_n\}$ exists. To show that z is a fixed point we show that $d(z, f(z)) \leq \varepsilon$ for every $\varepsilon > 0$. Every fixed point of f is trivially unique.

Since $\{x_n\}$ is a Cauchy sequence and z is the limit, for every $\delta > 0$ there is an M s.t. $d(x_m, z), d(x_m, f(x_m)) \leq \delta$ for all $m \geq M$. Next, since f is continuous at z for every $\varepsilon_0 > 0$ there is a $\delta_0 > 0$ s.t. for all x if $d(x, z) \leq \delta_0$ then $d(f(x), f(z)) \leq \varepsilon_0$.

Let $\varepsilon > 0$ be given, choose δ_0 for $\varepsilon_0 = \varepsilon/3$ and M s.t. $d(x_m, z), d(x_m, f(x_m)) \leq \min(\delta_0, \varepsilon/3)$ for all $m \geq M$, then

$$\begin{aligned} d(z, f(z)) &\leq d(x_m, z) + d(x_m, f(x_m)) + d(f(x_m), f(z)) \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

The rate of convergence M follows by Theorem 3.4. □

Remark 3.8. *By Remark 2.4 and Proposition 2.8 this implies Kirk's fixed point theorem for asymptotic mappings in [3].*

Definition 3.9. *A function $f : X \to X$ is called quasi-nonexpansive if*

$$\forall x, p \in X (f(p) = p \to d(f(x), p) \leq d(x, p)).$$

Corollary 3.10. *Let (X, d) be a complete metric space, let f be a continuous, quasi-nonexpansive asymptotic contraction and let $b > 0$ and η, β be given. If for some x_0 the sequence $\{x_n\}$ is bounded by b then f has a unique fixed point z, $\{x_n\}$ converges to z and for every $\varepsilon > 0$ and all $n \geq M$*

$$d(x_n, z) \leq \varepsilon,$$

where $M(\eta, \beta, \varepsilon, b)$ is as in Theorem 3.4.

Proof: By Theorem 3.7 there exists $m \leq M$ s.t. $d(x_m, z) \leq \varepsilon$ where z is the unique fixed point of f and M is given as in Theorem 3.4. The function f is quasi-nonexpansive, so for all $n \geq M \geq m$ we have that $d(x_n, z) \leq d(x_m, z)$ and hence also $d(x_n, z) \leq \varepsilon$. □

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