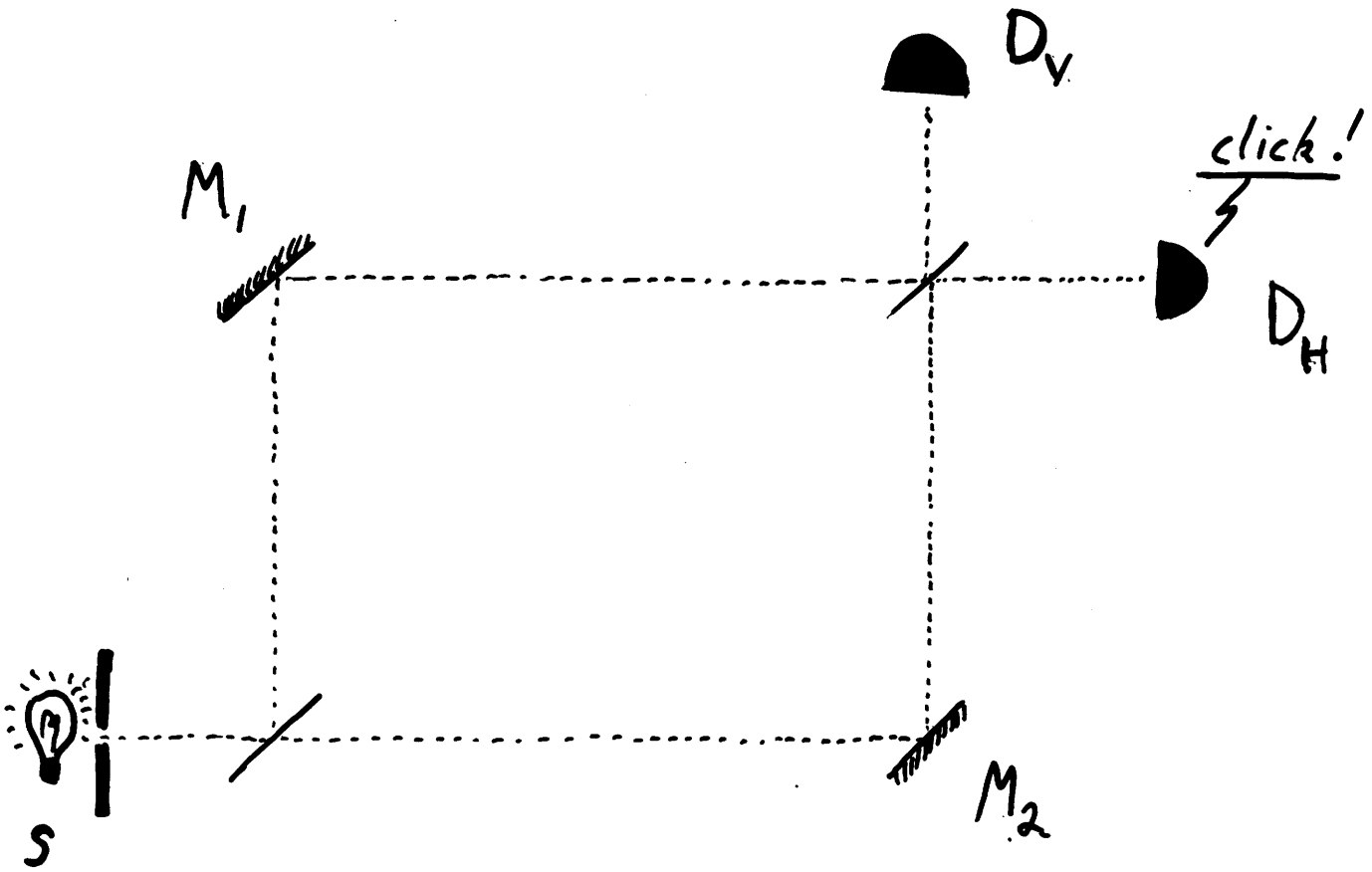
BRICS Mini-course on Quantum Computation

A. Berthiaume
C.W.I.

Session I: Introduction

Interferometer:



/ : Mirror

/ : Half-Silvered Mirror

Historical Remarks:

- 1982 Benioff/Feynman
  idea of a fully QM comp.
- 1985 Deutsch
  Formalism for QTM
- 1989 Deutsch
  Formalism for Q circuits
- 1993 Yao
  QTM ≡ Q circuits
- 1995 BBC et al.
  Quantum gate results

Quantum Basics:

Basis States: all classically distinct alternative of a given property.

ex. of properties:
- position
- path
- orientation

Notation: if X is the label of an alternative, then that state is noted

$$|X\rangle$$

N.B.: $\frac{1}{2}|4\rangle \neq |2\rangle$

NO!

Quantum Basics (cont.):

amplitude: The probability of an alternative X is given by the square norm of a complex number α.

$$P(|X\rangle) = |\alpha|^2$$

State Vector: A sum of all alternatives weighted by their amplitudes.

$$|\varphi\rangle = \sum_{i \in B} \alpha_i |i\rangle$$

Normalized!

$$\sum_{i \in B} |\alpha_i|^2 = 1$$

where

$B = \{ i \mid i \text{ is a label} \}$

is a set of basis states

Quantum Basics (cont.):

All possible $\sum_{i \in B} \alpha_i |i\rangle$ form a complex vector space (a Hilbert space).

$$|\varphi\rangle = \sum_{i \in B} \alpha_i |i\rangle \quad \Rightarrow \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \in \mathcal{H}$$

$$\dim(\mathcal{H}) = \#B$$

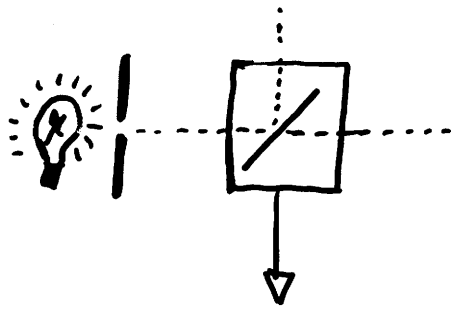
Quantum Basics (cont.):

Evolution: state vectors are transformed through unitary matrices.

$$U U^{\dagger} = I$$

$$U|\psi\rangle = |\psi'\rangle$$

ex:



$$U|H\rangle = \alpha|H\rangle + \beta|V\rangle$$

where $|\alpha|^2 = |\beta|^2 = 1/2$

Quantum Basics (cont.):

Observable: an observable O is a partition of $\mathcal{H}$ in $E_1, \dots, E_k$ subspaces such that

$$\mathcal{H} = E_1 \times E_2 \times \cdots \times E_k$$

and $\forall i \neq j \quad E_i \perp E_j$

N.B.: The property determines the partitionning.

Quantum Basics (cont.):

Observation: Let $|\varphi\rangle = \sum_{i \in B} \alpha_i |i\rangle$ in $\mathcal{H}$. We want to observe this state with

$$\mathcal{O} = \{E_1, \ldots, E_k\}$$

$$|\varphi\rangle = \sum_{i \in B} \alpha_i |i\rangle \Rightarrow |\varphi\rangle = \sum_{j=1}^{k} \beta_j |\varphi_j\rangle$$

where

$$|\varphi_j\rangle \in E_j.$$

Observing will:

1) select E_j with prob. $|\beta_j|^2$
2) $|\varphi\rangle \longmapsto |\varphi_j\rangle$
3) All we learn is "j"

Quantum Basics (cont.):

Notation: if $|\varphi\rangle = \sum_i \alpha_i |i\rangle$

then

$$(|\varphi\rangle)^\dagger \equiv \langle\varphi| = \sum_i \alpha_i^* \langle i|$$

Projection: (inner product)

Let $|\varphi\rangle = \sum_i \alpha_i |i\rangle$ and

$|\psi\rangle = \sum_i \beta_i |i\rangle$ then

$$|\varphi\rangle \cdot |\psi\rangle = \langle\psi|\varphi\rangle$$

$$= \sum_{i,j} \beta_i^* \alpha_j \underbrace{\langle i|j\rangle}_{\begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{o.w.} \end{cases}}$$

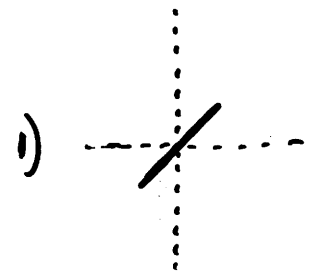
$$= \sum_i \beta_i^* \alpha_i$$

Working out the interferometer:

Basis States: $\mathcal{B} = \{ |H\rangle, |V\rangle \}$

Initial State: $|s\rangle = |H\rangle$

Transformations:

1) 

$$|H\rangle \rightarrow \tfrac{1}{\sqrt{2}}\left(|H\rangle + i|V\rangle\right)$$

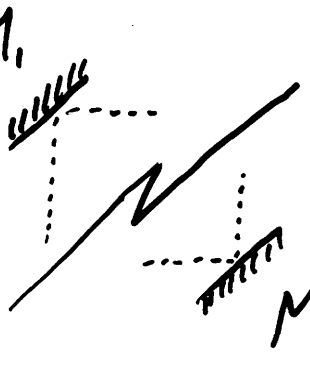
$$|V\rangle \rightarrow \tfrac{1}{\sqrt{2}}\left(i|H\rangle + |V\rangle\right)$$

if $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then

$$M_{1/2} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Note: $M_{1/2} M_{1/2}^{\dagger} = I$

Working out ...

2) 

$|H\rangle \longrightarrow i|V\rangle$

$|V\rangle \longrightarrow i|H\rangle$

so $M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Note: $MM^{\dagger} = I$

Working out... :

$$|H\rangle \longrightarrow \tfrac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$$

$$\longrightarrow \tfrac{1}{\sqrt{2}}(-|H\rangle + i|V\rangle)$$

$$\longrightarrow \tfrac{1}{\sqrt{2}}\left[-\tfrac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) + \tfrac{i}{\sqrt{2}}(i|H\rangle + |V\rangle)\right]$$

$$= \tfrac{1}{2}(-|H\rangle - i|V\rangle - |H\rangle + i|V\rangle)$$

$$= -|H\rangle$$

$$M_{\frac{1}{2}} M\, M_{\frac{1}{2}} |H\rangle = M_{\frac{1}{2}} M \underline{M_{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= M_{\frac{1}{2}} \underline{M \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}}$$

$$= M_{\frac{1}{2}} \begin{pmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Quantum Computation Basics:

Qubit: a quantum state of the form

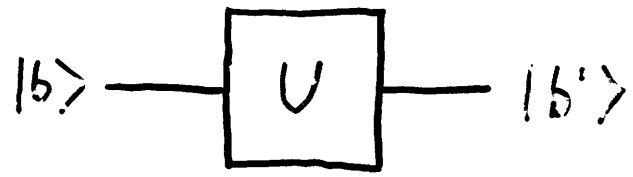
$$|b\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$

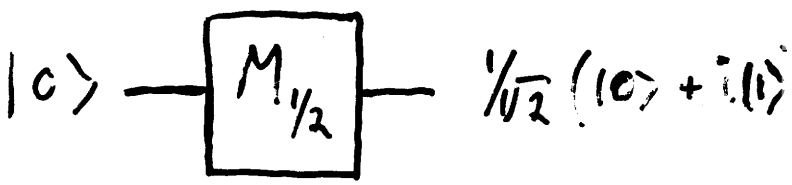
Qugate: (for 1 qubit) is a 2x2 (or gate) unitary matrix U.

$$U|b\rangle = |b'\rangle$$

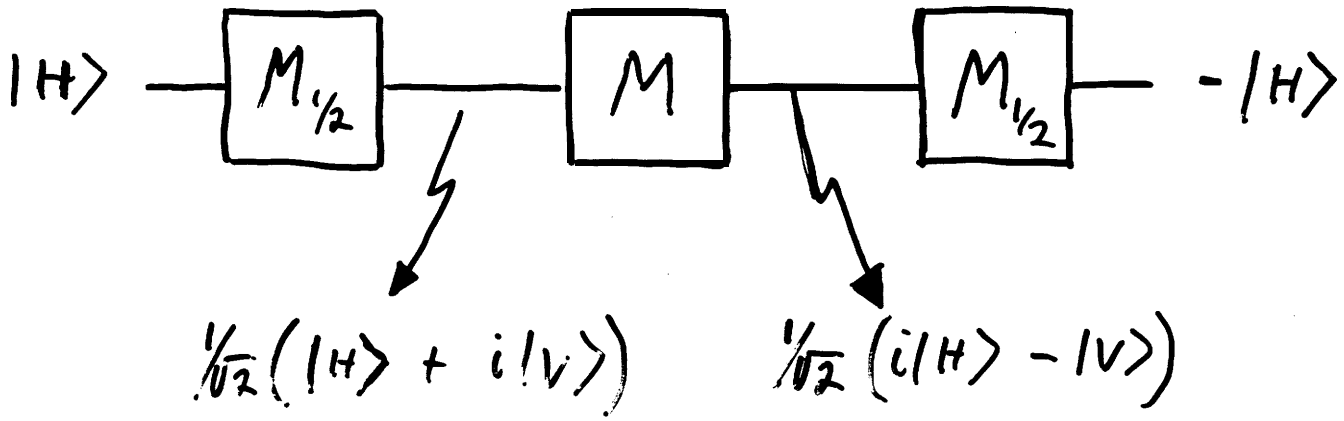
Notation:



ex: $U = M_{1/2}$

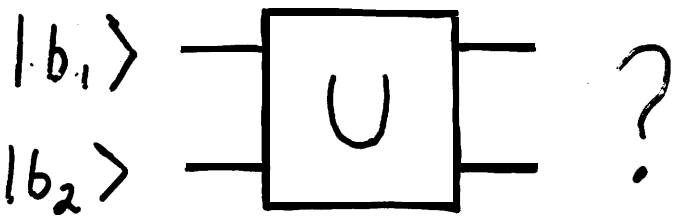


The interferometer in gates:



QC Basics (cont.):

Qugate: (for 2 qubits)



What's the joint basis?

What's the joint state?

Tensor Product: $\otimes$

When we have 2 qubits:

Joint basis: $\mathcal{B}_2 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Joint state: if $|b_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$
and $|b_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$
then

$$|b_1\rangle \otimes |b_2\rangle = |b_1\rangle|b_2\rangle = \sum_{i,j=0}^{1} \alpha_i \beta_j |ij\rangle$$

Entanglement:

Let $|\psi\rangle = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle$

If $|\psi\rangle = (\beta_1 |0\rangle + \beta_2 |1\rangle) \otimes (\gamma_1 |0\rangle + \gamma_2 |1\rangle)$

$= |b_1\rangle \otimes |b_2\rangle$

then $|b_1\rangle$ and $|b_2\rangle$ are <u>independant</u>

Otherwise $|\psi\rangle$ is <u>entangled</u>

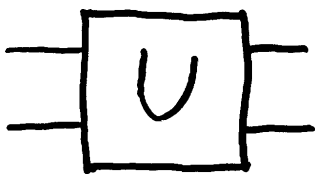
ex: $|\psi_1\rangle = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

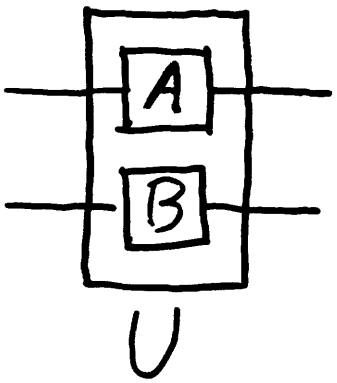
$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$ is entangled

Qugates on 2 qubits:

In general: a 4x4 unitary matrix



Special Case



then $U = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}$

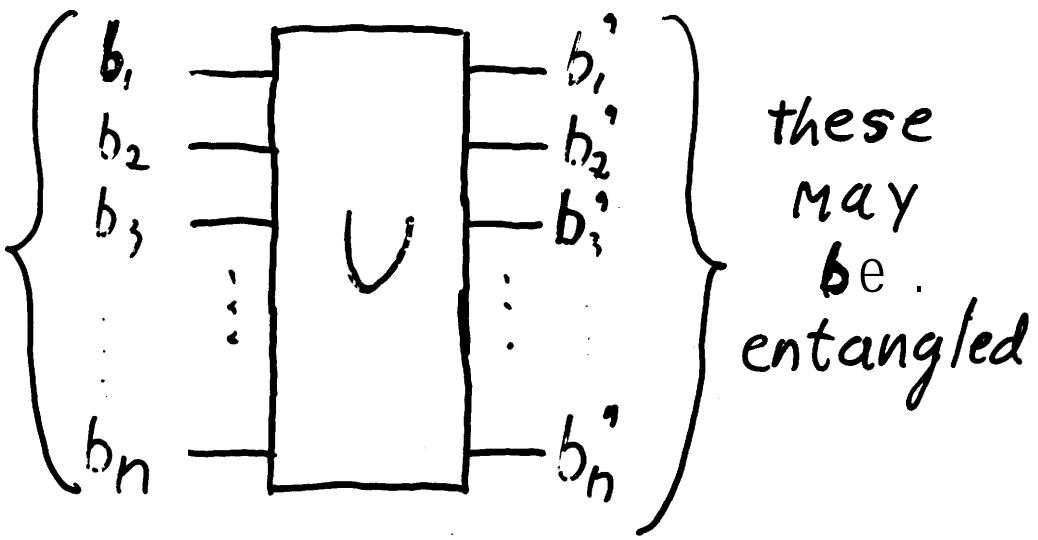
Quantum Register:

Def: an n qubit quantum register is a quamtum state of the form

$$|\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle$$

Where $|i\rangle$ is the i^{th} binary string on n bits

n-qugates: qugates on n qubit registers are $2^n \times 2^n$ unitary matrices.



Modified Deutsch-Jozsa Problem:

Let $f: \{0,1\}^n \longrightarrow \{0,1\}$ computable.

We define:

* Non-Balanced: There is a majority of 0's (or 1's)

** Non-Constant: $\exists\ x, y$ s.t. $f(x) \neq f(y)$

Problem: Given f, output $P \in \{*, **\}$ such that $P(f)$ is true

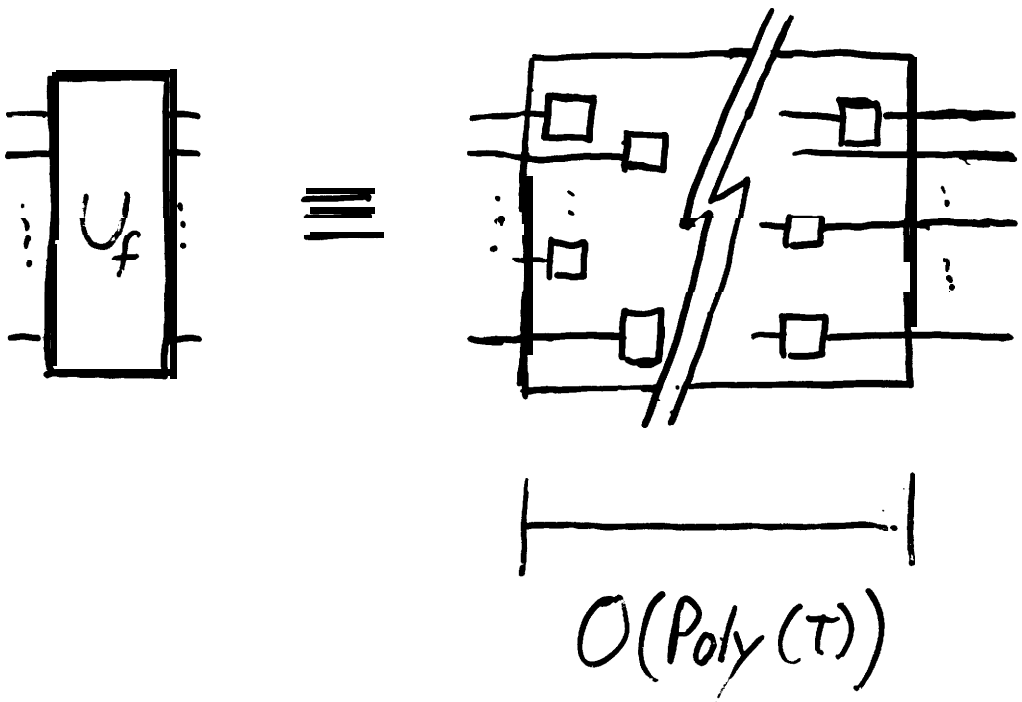
Lecerf-Bennett Theorem:

Thm: For all f computable in time $O(T)$, there exist a unitary matrix U such that

$$U_f|x, o\rangle = |x, f(x)\rangle$$

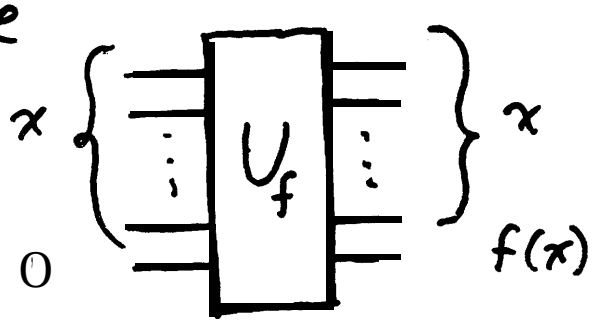
(actually: $U_f|x, b\rangle = |x, b \oplus f(x)\rangle$).

also



Idea of the solution to MDJP:

We have



But

$$U\left[\frac{1}{\sqrt{2}}(|x,0\rangle + |y,0\rangle)\right]$$

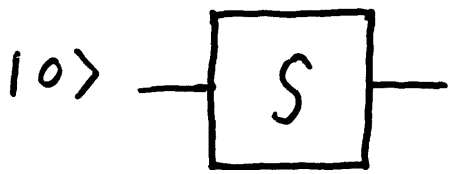
$$= \frac{1}{\sqrt{2}}(|x,f(x)\rangle + |y,f(y)\rangle)$$

So

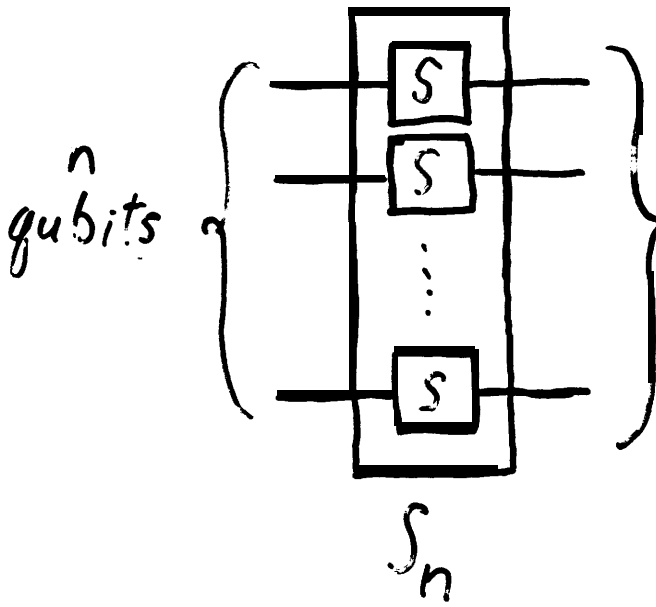
$$U\left(\frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i,0\rangle\right) = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i,f(i)\rangle$$

Superposing a Register:

Let $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then



Now, consider



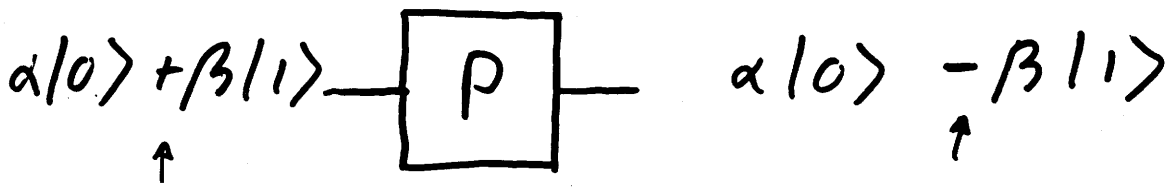
$$|\psi\rangle = \bigotimes_n (\quad)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle$$

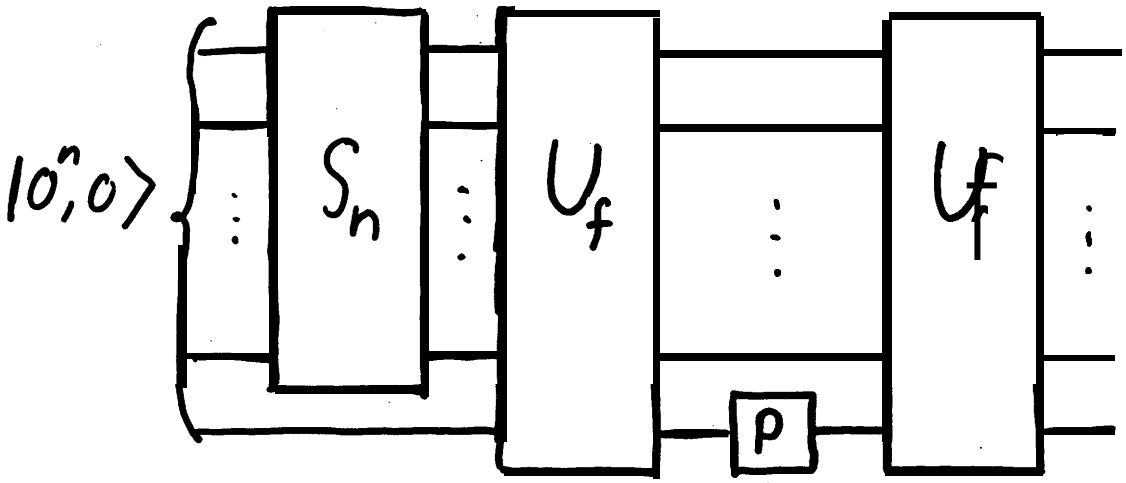
and $S_n = \bigotimes_n S$

Another Gate:

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then



The Solution:



$$|0^n, 0\rangle \longrightarrow \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i, 0\rangle$$

$$\longrightarrow \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i, f(i)\rangle$$

$$\longrightarrow \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)} |i, f(i)\rangle$$

$$\longrightarrow \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)} |i, 0\rangle$$

Solution (cont.):

Let $|\chi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i, 0\rangle$

and consider the observable $\Theta = \{E_1, E_2\}$ where

$$E_1 = [\![|\chi\rangle]\!] \quad \text{and} \quad E_2 \perp E_1$$

Claim: the gate array + Θ solve the MOJP.

Proof:

Final State: $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)} |i,0\rangle$

What is the proj. of $|\psi\rangle$ in E_1?

$$\langle \chi | \psi \rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \langle j,0| \right) \left(\frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)} |i,0\rangle \right)$$

$$= \frac{1}{2^n} \sum_{i=0}^{2^n-1} (-1)^{f(i)}$$

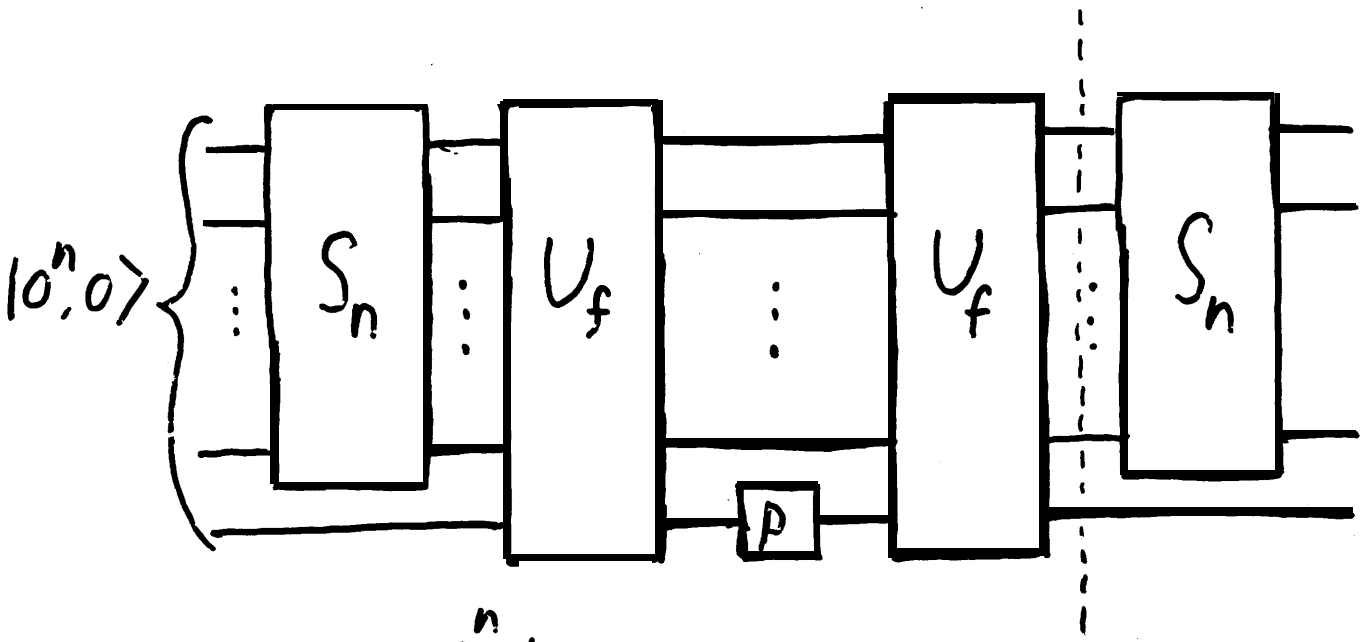
3 cases:

1) f <u>balanced</u>: $\langle \chi | \psi \rangle = 0$
(always an "E_2" answer)

2) f <u>constant</u>: $\langle \chi | \psi \rangle = \pm 1$
(always an "E_1" answer)

3) f not 1 or 2: $\langle \chi | \psi \rangle \in]0,1[$
(either answer)

Improvement on the DJ problem:



$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} (-1)^{f(i)} |i,0\rangle$$

Recall that $S_n|w\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{w\cdot z} |z\rangle$

So $$S_n|\psi\rangle = \frac{1}{2^n} \sum_{i=0}^{2^n-1} \sum_{z=0}^{2^n-1} (-1)^{f(i)} (-1)^{i\cdot z} |z\rangle$$

We now observe with the Standard basis.

Improvement (cont.):

Final State: $\frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{i=0}^{2^n-1} (-1)^{f(i)} (-1)^{i \cdot j} |j, 0\rangle$

Consider the amplitude of $|0^n, 0\rangle$:

$\forall i \quad i \cdot 0^n = 0 \rightarrow \frac{1}{2^n} \sum_{i=0}^{2^n-1} (-1)^{f(i)} = a_0$

3 cases

		observing
1) f balanced:	$a_0 = 0$ →	Never $\|0^n, 0\rangle$
2) f constant:	$a_0 = \pm 1$ →	always $\|0^n, 0\rangle$
3) f other:	$a_0 \neq 0, \pm 1$ →	maybe $\|0^n, 0\rangle$

Same argument as before