NONCLAUSAL RESOLUTION SYSTEM FOR BRANCHING TEMPORAL LOGIC

Jūratė Sakalauskaitė
Institute of Mathematics and Informatics
Akademijos 4, 2600 Vilnius, Lithuania
e-mail: jurate.sakalauskaite @mlats.mii.lt

Abstract. We present a proof system for branching propositional temporal logic. The system is based on nonclausal resolution. The system is proved to be complete. The proof of completeness uses tableau construction for the logic.

1. Introduction

Temporal logic is an appropriate formalism to reason about concurrent systems. We consider here the branching propositional temporal logic BPTL, i.e. in underlying model of the logic any instant of time may split into different possible futures. Branching time logic allows to reason about different possible futures. BPTL is a subsystem of branching time logic introduced in [BPM]. The language of BPTL contains the usual propositional connectives (say $\wedge, \vee, \neg, \supset$) and temporal modalities. Time is assumed discrete and branching. In BPTL if u and v range over formulas then

- $\bigcirc\, u$ means "u is true in each next state";
- $\circledcirc\, u$ means "u is true in some next state; in other words $\circledcirc u \equiv \neg\bigcirc\neg u$;
- $\Box\, u$ means "u is always true (from now on)";
- $\Diamond\, u$ means "u is eventually true"; in other words $\Diamond u \equiv \neg\Box\neg u$.

For BPTL a Hilbert style proof system can be obtained as a subsystem of Hilbert style proof system presented in [BPM].

In this paper we present nonclausal resolution proof system for BPTL. Nonclausal resolution has the advantage over the classical clausal resolution of not requiring formulas to be in clause form. The proof of completeness uses tableau construction for BPTL extracted from [BPM]. The idea to use tableau construction to prove completeness of nonclausal resolution proof system is adopted from [AM].

In section 2 we present syntax and semantics of BPTL. In section 3 we introduce a nonclausal resolution system $\mathbb{R}$ and explore soundness issues. In section 4 we present the proof of completeness of $\mathbb{R}$. Section 5 contains some concluding remarks.

2. Syntax and semantics of BPTL

Formulas are defined as usual with the help of propositional connectives $\wedge, \vee, \supset$ and temporal modalities $\bigcirc, \circledcirc, \Box, \Diamond$.

A model for BPTL is a triple (S, P, R), where S is a set of states, P is an assignment of propositional letters to states and R is a binary relation on states such that for each $s \in S$ there is $t \in S$ $(s,t) \in R$. For a propositional letter a and a state $s \in S$, $a \in P(s)$ iff a is true in s. We extend the interpretation over the model to all formulas in BPTL as follows, where $b = (s = s_0, s_1, \ldots, s_i, \ldots)$ is an infinite path through the model such that $s_i R s_{i+1}$.

1. $s \models a$ iff $a \in P(s)$ for a atomic;
2. $s \models \neg p$ iff $s \not\models p$;
3. $s \models p \vee q$ iff $s \models p$ or $s \models q$;
4. $s \models \Box p$ iff $\forall b\ \forall t\ (t \in b$ implies $t \models p)$;
5. $s \models \bigcirc p$ iff $\forall t\ (sRt$ implies $t \models p)$;
6. $s \models \Diamond p$ iff $\exists b\ \exists t\ (t \in b$ implies $t \models p)$;
7. $s \models \circledcirc p$ iff $\exists t\ (sRt$ implies $t \models p)$.

p is satisfiable iff $s \models p$ for some model M and some state s in M. p is valid iff $s \models p$ for each model M and each state s in M.

3. The resolution system for BPTL

In this section we describe a nonclausal resolution proof system $\mathbb{R}$ for BPTL.

3.1 Preliminaries

The sequence of formulas $S_0, \ldots, S_n$ is a *derivation* of S_n from S_0 if, for all i, S_{i+1} is obtained from S_i by rules of the system. We refer to the S_i as a derivation step. In the special case, where $S_n = false$, the sequence $S_0, \ldots, S_n$ is a *refutation* of S_0. A *proof* of the formula w is a refutation of $\neg w$. We write $\vdash w$ to denote that formula w has a proof within $\mathbb{R}$.

The resolution system in this paper consists of *simplification rules* and *deduction rules*. We also use the *recursive-call mechanism* which enables to apply rules to subformulas of a proof step.

We define the set of *conjuncts* of a formula u as follows: u is a conjunct of u, and if u is of the form $u_1 \wedge u_2$, then conjuncts of u_1 and the conjuncts of u_2 are conjuncts of u.

- *Simplification rules*

These rules simplify formulas, or put them in forms where the other rules can be applied. They are all of the form

$$u_1, \ldots, u_m \Rightarrow v.$$

The rule $u_1, \ldots, u_m \Rightarrow v$ can be applied to a formula S_i, if the formulas $u_1, \ldots, u_m$ occur as conjuncts in S_i. Then in order to obtain corresponding derivation step we delete one occurrence of each u_i and add the derived formula v to the conjunction.

- *Deduction rules.*

The deduction rules have a form

$$u_1, \ldots, u_m \longmapsto D.$$

The rule $u_1, \ldots, u_m \longmapsto v$ can be applied to a formula S_i, if the formulas $u_1, \ldots, u_m$ occur as conjuncts in S_i. Then, in order to obtain corresponding derivation step, add the derived formula v to the conjunction.

• *Polarity.*

An occurrence of a subformula has *positive polarity* in a formula if it is embedded in the scope of an even number of explicit or implicit $\neg$'s.

• *The recursive-call mechanism.*

The recursive-call mechanism for BPTL is: if there is a derivation of v from u then we may take S_{i+1} to be S_i, with a positive occurrence of u replaced with v.

Usually, we do not mention the recursive-call mechanism in our proofs. Instead we mention the simplification and deduction rules that enables us to apply.

3.2 Soundness For our proof notion to be meaningful we require that rules be sound, i.e., that they maintain satisfiability: if S_0 is satisfiable, then S_n is satisfiable.

The following lemma is helpful in proving soundness of the system.

Lemma 3.1. If $u \supset v$ is valid and w_1 is the result of replacing in w one positive occurrence of u with v, then $w \supset w_1$ is valid.

Proof. By induction on complexity of w.

Each of simplification rules and deduction rules of $\mathbb{R}$ has the property that

$$(u_1 \wedge \ldots \wedge u_m) \supset v$$

is valid.

This guarantees the soundness of $\mathbb{R}$:

Theorem 3.2(soundness) Assume that $(u_1 \wedge \ldots \wedge u_m) \supset v$ is valid for all simplification rules and all deduction rules. If there is a derivation of S_n from S_0 and S_0 is satisfiable, then S_n is satisfiable.

Proof. By induction on the depth of recursive calls in the derivation.

3.3 rules of the system $\mathbb{R}$

3.3.1 Simplification rules

• *True-false reduction rules*

These rules include

$$\Box \text{ true} \Rightarrow \text{true} \qquad \Diamond \text{ true} \Rightarrow \text{true}$$
$$\Box \text{ false} \Rightarrow \text{false} \qquad \Diamond \text{ false} \Rightarrow \text{false}$$
$$\bigcirc \text{ true} \Rightarrow \text{true} \qquad \circledcirc \text{ true} \Rightarrow \text{true}$$
$$\bigcirc \text{ false} \Rightarrow \text{false} \qquad \circledcirc \text{ false} \Rightarrow \text{false}$$

and the regular true-false reduction rules in propositional logic such as false $\wedge u \Rightarrow$false, etc. .

• *Weakening rules*

$$u, v \Rightarrow u, \quad u, v \Rightarrow v.$$

- *Negation rules*

$$\neg\Box u \Rightarrow \Diamond\neg u, \qquad \neg\Diamond u \Rightarrow \Box\neg u,$$

$$\neg\bigcirc u \Rightarrow \odot\neg u, \qquad \neg\odot u \Rightarrow \bigcirc\neg u.$$

$$\neg(u \vee v) \Rightarrow (\neg u \wedge \neg v),$$

$$\neg(u \wedge v) \Rightarrow (\neg u \vee \neg v),$$

$$\neg\neg u \Rightarrow u.$$

Similar rules are added for the other connectives.

3.3.2 Deduction rules

•*Resolution rule*

To state this rule , we introduce some substitution notation. We write $u < v >$ to indicate that the formula v occurs in u, and then $u < w >$ denotes the result of replacing *exactly one* occurrence of v with w in u. We write $u < v_1, \ldots, v_n >$ to indicate that each of $v_1, \ldots, v_n$ occurs in u ,and then $u < w >$ denotes the result of sequentially replacing exactly one occurrence of each of $v_1, \ldots, v_n$ with w in u.

The nonclausal resolution rule for propositional logic is

$$A < u >, B < u > \longmapsto A < \text{true} > \vee B < \text{false} > .$$

That is, if the formulas $A < u, \ldots, u >$ and $B < u, \ldots, u >$ have a common subformula u, then we can derive the resolvent $A < \text{true} > \vee B < \text{false} >$. This is obtained by replacing certain occurrences of u in $A < u, \ldots, u >$ with true and certain occurrences of u in $B < u, \ldots, u >$ with false, and taking the disjunction of the results. (Here "certain occurrences" means "one or more occurrences").

As noted in [AM] this rule does not carry over to linear propositional temporal logic .It does not carry to branching propositional temporal logic as well. The reason is that while u occurs in both A and B, it need not denote the same truth value in all its occurrences; intuitively, each occurrence of u may refer to different instants of time.

The resolution rule is sound in $\mathbb{R}$ under the following *restriction:* the occurrences of u in A or B that are substituted by true or false, respectively, are all in the scope of the same number of $\bigcirc$'s and $\odot$'s and not in the scope of any $\Box$ or $\Diamond$ in A or B; and all these occurrences of u may be except one are in the scope of only $\bigcirc$'s, each of which occurs with positive polarity.

For example, consider the formulas

$$A: \ \bigcirc \odot \Box p \wedge \Diamond\Box p \quad \text{and}$$

$$B: \ \bigcirc\bigcirc\Box p \wedge \odot\bigcirc\Box p.$$

Taking u to be $\Box p$ the rule allows us to derive the resolvent

$$(\bigcirc \circledcirc \text{ true} \wedge \Diamond\Box p) \vee (\bigcirc\bigcirc \text{false} \wedge \circledcirc\bigcirc\Box p).$$

We only substituted true or false for those occurrences in the scope of $\bigcirc\circledcirc$ and $\bigcirc\bigcirc$. We cannot substitute the other occurrence of $\Box p$ in A, since it is in the scope of $\Diamond$. Also, we cannot substitute the other occurrence of $\Box p$ in B, since we substituted $\Box p$ in $\bigcirc \circledcirc \Box p$ in A, which is in the scope of $\bigcirc\circledcirc$, but not in the scope of $\bigcirc\bigcirc$.

- *Modality rules*

  These are rules to handle formulas in the scope of $\Box$, $\Diamond$, $\bigcirc$, $\circledcirc$.

  - $\Box$ rule: $\Box u \longmapsto u \wedge \bigcirc\Box u$;
  - $\Diamond$ rule: $\Diamond u \longmapsto u \vee \circledcirc\Diamond u$;
  - $\bigcirc\bigcirc$ rule: $\bigcirc u, \bigcirc v \longmapsto \bigcirc(u \wedge v)$;
  - $\circledcirc\bigcirc$ rule: $\circledcirc u, \bigcirc v \longmapsto \circledcirc(u \wedge v)$.

- *Induction rule*

if $\vdash \neg(w \wedge u)$, then

$$w, \Diamond u \longmapsto \Diamond\big(\neg u \wedge \circledcirc(u \wedge \neg w)\big).$$

- *Distribution rule*

$$u, v_1 \vee \ldots \vee v_n \longmapsto (u \wedge v_1) \vee \ldots \vee (u \wedge v_n).$$

One can write similar rules for the other propositional connectives, e.g. $\supset$.

3.4 An example. Let us prove the validity of the formula

$$(p \wedge \Box(p \supset \bigcirc p)) \supset \Box p.$$

In other words , we refute

$$\neg(\neg(p \wedge \Box(p \supset \bigcirc p)) \vee \Box p.$$

The negation rules yield

$$(p \wedge \Box(p \supset \bigcirc p)) \wedge \Diamond\neg p.$$

Since $\vdash \neg(p \wedge \neg p)$, we apply the induction rule and weakening to derive

$$\Diamond(p \wedge \circledcirc\neg p) \wedge \Box(p \supset \bigcirc p).$$

Since $\vdash \neg(\Box(p \supset \bigcirc p) \wedge (p \wedge \circledcirc\neg p))$ (as we check later), as above we derive

$$\Diamond(\neg(p \wedge \circledcirc\neg p) \wedge \circledcirc((p \wedge \circledcirc\neg p) \wedge \neg\Box(p \supset \bigcirc p)) \wedge \Box(p \supset \bigcirc p).$$

By weakening this yields

$$\Diamond \circledcirc (\neg\Box(p \supset \bigcirc p)) \wedge \Box(p \supset \bigcirc p).$$

Since $\vdash \neg(\circledcirc(\neg\Box(p \supset \bigcirc p)) \wedge \Box(p \supset \bigcirc p))$ (as we check later), as above we derive

$$\Diamond(\neg(\circledcirc(\neg\Box(p \supset \bigcirc p))) \wedge \circledcirc \circledcirc \neg\Box(p \supset \bigcirc p)).$$

By $\Box$ rule and weakening this yields

$$\Diamond(\neg(\circledcirc(\neg\bigcirc\Box(p \supset \bigcirc p))) \wedge \circledcirc \circledcirc \neg\Box(p \supset \bigcirc p)).$$

By negation rule and weakening this yields

$$\Diamond(\bigcirc\bigcirc\Box(p \supset \bigcirc p) \wedge \circledcirc \circledcirc \neg\Box(p \supset \bigcirc p)).$$

We apply the resolution rule (taking $A = \bigcirc\bigcirc\Box(p \supset \bigcirc p)))$, $B = \circledcirc\circledcirc\neg\Box(p \supset \bigcirc p)$ and $u = \Box(p \supset \bigcirc p)$ to derive

$$\Diamond\bigcirc\bigcirc false \vee \circledcirc \circledcirc \neg true,$$

and then by simplification and weakening

$$\Diamond false.$$

And then

$$false.$$

We still have to show a) $\vdash \Box(p \supset \bigcirc p) \wedge (p \wedge \circledcirc\neg p)$ and b) $\vdash \neg(\circledcirc(\neg\Box(p \supset \bigcirc p)) \wedge \Box(p \supset \bigcirc p))$ to justify the application of the induction rule in this proof. We show a). b) is left to the reader. We apply $\Box$ rule and weakening to derive

$$(p \supset \bigcirc p) \wedge (p \wedge \circledcirc\neg p).$$

By distribution rule and weakening this yields

$$(\neg p \wedge p \wedge \circledcirc\neg p) \vee (\bigcirc p \wedge p \wedge \circledcirc\neg p).$$

We apply twice the resolution rule to derive

$$\neg true \vee (false \wedge \circledcirc\neg p) \vee (\bigcirc false \vee (p \wedge \circledcirc\neg true)).$$

Negation rules and simplifications yield

$$false.$$

4.Completeness of $\mathbb{R}$

In this section we prove that the resolution system R is complete.

Theorem 4.1(completeness) Every valid BPTL formula is provable in the resolution system $\mathbb{R}$.

Proof. Our strategy as in [AM] is to show that any tableau proof can be transformed into a resolution proof. In other words, if the tableau decision procedure finds $\neg u$ of BPTL unsatisfiable, then $\neg u$ has a refutation. If n is a node of the tableau then U_n denotes the set of formulas at n. We prove a more general fact:

for every node n in the tableau for u ,if n is eliminated, then $\wedge U_n$ has the refutation.

The proof is by complete induction on the stage at which nodes are eliminated by the tableau decision procedure.(In other words ,the rank of a node is the stage at which it is deleted).

We recall the definition of a tableau construction for u and the tableau decision procedure ([BPM]). Formulas are classified as α-formulas and β -formulas

α	α_1	α_2
p∧q	p	q
□p	p	○□p
β	β_1	β_2
p∨q	p	q
$\Diamond p$	p	⊙◊p.

Tableau construction and decision procedure. Let u be BPTL formula. Let n_0 be the root of T and let $U_{n_0} = \{\neg u, \Box \circledcirc \text{true}\}$. T, the tableau for $\neg u$, is constructed inductively by applying the following rules to nodes which are leaves of T.

R_α: if $\alpha \in U_n$, then create a son n_1 of n and define

$$U_{n_1} = U_n \cup \{\alpha_1, \alpha_2\}.$$

R_β: if $\beta \in U_n$, then create two sons $n_1,\ n_2$ of n and define

$$U_{n_i} = U_n \cup \{\beta_i\}, \quad i = 1, 2.$$

$R_\circledcirc$: let $V_n = \{\circledcirc p_1, \ldots, \circledcirc p_k, \bigcirc q_1, \ldots \bigcirc q_m\}$ be the set of next time formulas in parent n. Create k sons $n_i,\ i = 1, \ldots, k$ of n and define

$$U_{n_i} = \{p_i, q_1, \ldots, q_m\}.$$

Notation. If the $R_\alpha/R_\beta/R_\circledcirc$-rule was applied at node n, the n is called $\alpha/\beta/\circledcirc$-node. Root and sons of $\circledcirc$-node are called *pre-states*. $\circledcirc$-node is called a *state*.

The construction of T is made finite by introducing the following two termination rules:

1. if $p \in U_n$ and $\neg p \in U_n$, then this node is called closed and is not expanded further.

2. if a node m is about to be created as a son of an $\circledcirc$-node n and there is an ancestor n' of n such that n' is an immediate descendant of a $\circledcirc$-node and $U_{n'} = U_m$, then do not create m but instead connect n to n' with "feedback" edge.

A node of T is eliminated in the following three cases:

1) it contains a proposition and its negation;
2) it is α- or β-node and all of its descendants have been eliminated; it is $\circledcirc$-node and one of its descendants have been eliminated;
3) if the node is a pre-state and contains $\Diamond v$ and on no path from the node do v occurs.

$\neg u$ is found unsatisfiable if and only if all nodes of the tableau have been eliminated.

Consider a node n. Assume that if the node m has already been eliminated then $\wedge U_m$ has a refutation, and suppose that n is eliminated. To prove completeness we show that U_n is refuted in R. We analyse three cases; they correspond to the three possible reasons why n can be eliminated.

Lemma 4.1. 1) If U_n contains p and $\neg p$ then $\wedge U_n$ can be refuted in R; 2) if n is $\alpha/\beta/\circledcirc$-node and $\wedge U_{n_1}$ is refuted in $\mathbb{R}/\wedge U_{n_1}$ and $\wedge U_{n_2}$ are refuted in $\mathbb{R}/\wedge U_{n_i}$ for some i is refuted in $\mathbb{R}$ then $\wedge U_n$ can be refuted in $\mathbb{R}$.

Proof. In case 1) $\wedge U_n = p \wedge \neg p \wedge \phi$ for some formula ϕ. We derive $false$ by resolution on p and simplifications.

We consider 2).

• Let U_{n_1} be created from U_n by α-rule. Then $p \wedge q \in U_n$ and $U_{n_1} = U_n \cup \{p, q\}$ or $\Box p \in U_n$ and $U_{n_1} = U_n \cup \{p, \bigcirc\Box p\}$. Assume, there is a refutation of $\wedge U_{n_1}$ to show that there is a refutation of $\wedge U_n$. In both cases the refutation of $\wedge U_{n_1}$ can be extended to the refutation of U_n.

• Let U_{n_1} and U_{n_2} be obtained from U_n by β-rule. Then $p \vee q \in U_n$ or $\Diamond p \in U_n$. In the first case $(\wedge U_{n_1}) \vee (\wedge U_{n_2})$ is obtained from $\wedge U_n$ by the distribution rule . In the second case $(\wedge U_1) \vee (\wedge U_{n_2})$ is obtained from $\wedge U_n$ by applying $\Diamond$-rule and next the distribution rule. Assume there is a refutation of $\wedge U_{n_1}$ and $\wedge U_{n_2}$, to show there is a refutation of $\wedge U_n$. In both cases the refutation of U_{n_1} and U_{n_2} can be extended to the refutation of U_n.

• Let U_{n_i} be created from U_n by $\circledcirc$-rule. Then $\wedge U_n = \circledcirc p_1 \wedge \ldots \wedge \circledcirc p_k \wedge \bigcirc q_1 \wedge \ldots \wedge \bigcirc q_l \wedge v_1 \wedge \ldots \wedge v_m$, where v_i's are not of the form $\bigcirc w$ or $\circledcirc w$ and $\wedge U_{n_i} = p_i \wedge q_1 \wedge \ldots \wedge q_l$. Assume there is a refutation of $\wedge U_{n_i}$ in $\mathbb{R}$. The following lines are refutation of $\wedge U_n$ in $\mathbb{R}$.

1. $\circledcirc p_1 \wedge \ldots \circledcirc p_k \wedge \bigcirc q_1 \wedge \ldots \wedge \bigcirc q_l \wedge v_1 \wedge \ldots \wedge v_m$	$\wedge U_n$
2. $\circledcirc p_i \wedge \bigcirc q_1 \wedge \ldots \wedge \bigcirc q_l$	by weakening from 1.
3. $\circledcirc p_i \wedge \bigcirc(q_1 \wedge \ldots \wedge q_l)$	by $\bigcirc\bigcirc$-rule and weakening from 2.
4. $\circledcirc(p_i \wedge q_1 \wedge \ldots \wedge q_l)$	by $\circledcirc\bigcirc$-rule and weakening from 3.
$\vdots$	
S. $\circledcirc$ false	by assumption on

U_{n_i}.

S+1. false by simplification from S.

Lemma 4.2. If a node n is a prestate which contains $\Diamond v$ and on no path from the node do v occurs, then $\wedge U_n$ can be refuted in R.

To prove lemma 4.2 we prove some auxiliary lemmas.

Let $[n]^*$ be the set of prestates accessible from n by choosing at $\circledcirc$-node the son corresponding to $\circledcirc\Diamond v$. Then $\Diamond v \in U_n$ and on no path from $u \in [n]^*$ do v occurs. Let $Y_u,\ u \in n \cup [n]^*$, denote a part of prestate u obtained by erasing $\bigcirc$'s in a state preceding u. We call Y_u *universal next time part of* u. We will want to construct universal next time parts of prestates in $[n]^*$. More precisely, we get w_i's that say what universal next time parts of prestates at depth i counting from n will be true then. We call w_i the i-th fringe. For example, if $U_n = \{\Box(\bigcirc q \vee \circledcirc q_1), \Diamond v\}$, then $w_1 = \bigcirc(q \wedge \Box(\bigcirc q \vee \circledcirc q_1)) \vee \bigcirc\Box(\bigcirc q \vee \circledcirc q_1)$, where $(q \wedge \Box(\bigcirc q \vee \circledcirc q_1), \Box(\bigcirc q \vee \circledcirc q_1)$, are universal next time parts of prestates at depth 1 in $[n]^*$

Lemma 4.2.1. Given a prestate n $\mathbb{R}$ can derive formulas w_i built from the universal next time parts p_i^l of prestates from $[n]^*$ at depth i from Y_n with $\vee$ and $\bigcirc$. Furthermore, each p_i^l occurs in the scope of exactly i $\bigcirc$'s.

Proof. Put $w_o = \wedge Y_n$. We show how to obtain w_{i+1} from w_i. Let Y_t be a universal next time part of a prestate $t \in [n]^*$ in w_i at depth i. Then $U_t = Y_t \cup \Diamond v$.

Let $[t]$ be the states accessible from t by α, β nodes which include $\circledcirc\Diamond v$. We will show that we derive $\bigvee_{u\in[t]} \wedge(U_u - \{\circledcirc\Diamond v\})$ from $Y_t = \wedge(U_t - \{\Diamond v\})$ in $\mathbb{R}$. In fact applying $\Box-, \Diamond$-rules and distribution rule we derive $\bigvee_{u\in[t]^+}(\wedge U_u)$ from U_t, where $[t]^+$ is all states accessible from t by α, β nodes, i.e. $[t]^+$ is obtained from $[t]$ by adding states which include v. But these states are eliminated, so refuted by induction. Thus we derive $\bigvee_{u\in[t]}(\wedge U_u)$ from $\wedge U_t$. Hence we derive $\bigvee_{u\in[t]}(U_u - \{\circledcirc\Diamond v\})$ from $\wedge(U_t - \Diamond v)$. By weakening we drop from U_u all the formulas other than prefixed with $\bigcirc$. So we derive $\bigvee_{u\in[t]}(\bigwedge_{p\in Y_u} \bigcirc p)$. Apply $\bigcirc\bigcirc$-rule to pull $\bigcirc$'s out of conjunction. We derive $\bigvee_{u\in[t]}(\bigcirc \bigwedge_{p\in Y_u} p)$. Applying this procedure to all universal next time parts at depth i from w_i we derive w_{i+1}.

Lemma 4.2.2. Given a prestate $t \in n \cup [n]^*$ $\wedge Y_t \wedge v$ can be refuted in $\mathbb{R}$.

Proof. $U_t = Y_t \cup \{\Diamond v\}$ as mentioned above. Without loss of generality we can assume that β rule was applied to $\Diamond v$ at t to construct sons t_1, t_2. Then $U_{t_1} = \{U_t - \{\Diamond v\}\} \cup \{v\} = Y_t \cup \{v\}$. But t_1 have been eliminated; otherwise we have path with v from n. Thus U_{t_1} is refuted in $\mathbb{R}$ by induction.

Lemma 4.2.3. For any $i \geqslant 0 \vdash \neg(w_i \wedge \circledcirc^i v)$.

Proof. For $i = 0$ distribute v by the distribution rule and apply lemma 4.2.2. For $i > 0$ distribute $\circledcirc^i v$ by the distribution rule. Next use $\circledcirc\bigcirc$- rule to push v

inwards. Then we obtain conjunctions of universal next time parts of prestates in the depth i from $n \cup [n]^*$ with v. Applying lemma 4.2.2 we obtain the refutation of this conjunctions. Then applying reduction rules false $\vee A \to A$, $\circledcirc$ false $\to$ false we obtain refutation of $w_i \wedge \circledcirc^i v$.

The lemma 4.2.4 is the main lemma of completeness proof.

Lemma 4.2.4. If for all $i \vdash \urcorner(w_i \wedge \circledcirc^i v)$, then $\vdash \urcorner \wedge U_n$.

Proof. $\wedge U_n = \wedge Y_n \wedge \Diamond v$ by assumption. Assume $\vdash \urcorner(w_i \wedge \circledcirc^i v)$, for all$i$. We put $w_0 = \wedge Y_n$. From w_0 and $\Diamond v$, by the induction rule and weakening we derive

$$\Diamond \circledcirc (v \wedge \urcorner w_0)$$

since $\vdash \urcorner(w_0 \wedge v)$ by assumption.
By lemma 4.2.1 we can derive w_1. Furthermore, $\vdash \urcorner(w_1 \wedge \circledcirc v)$ by assumption. Thus $\vdash \urcorner(w_1 \wedge \circledcirc(v \wedge \urcorner w_0))$. This by the induction rule yields

$$\Diamond \circledcirc (\circledcirc(v \wedge \urcorner w_0) \wedge \urcorner w_1).$$

In general, we can get all w_i's and we have that $\vdash \urcorner(w_i \wedge \circledcirc^i v)$, by assumption. Successive applications of the induction rule will give

$$\Diamond \circledcirc (\circledcirc(\ldots \circledcirc (v \wedge \urcorner w_0) \wedge \ldots) \wedge \urcorner w_{m-1}) \wedge \urcorner w_m)$$

for any m.

We weaken this to

$$\Diamond \circledcirc (\circledcirc(\ldots \circledcirc (\urcorner w_0) \wedge \ldots) \wedge \urcorner w_{m-1}) \wedge \urcorner w_m)$$

Call this formula Ψ_m and define also Ω_m by $\Psi_m = \Diamond \Omega_m$.

The finite model property tells us that there are only finitely many universal next time parts generated from w_0. Thus for some s we have all universal next time parts generated from w_0 in $\mathbb{R}$. We check that $\vdash \urcorner(w_0 \wedge \Omega_s)$. Take $s+1$-th fringe w_{s+1} of w_0. Each universal next time part Y in w_{s+1} is in the scope of $s+1$ of $\bigcirc$'s and it is denied in Ω_s. Furthermore $\urcorner Y$ is in the scope of $s+1$ of $\circledcirc$'s in Ω_s. Applying the resolution rule we obtain false derived from $w_0 \wedge \Omega_s$, i.e. $w_o \wedge \Omega_s$ has the refutation in $\mathbb{R}$. Thus we can apply the induction rule (and weakening) once more and get

$$\Diamond(\urcorner \Omega_s \wedge \circledcirc \Omega_s).$$

Call this formula ϕ. ϕ says that at each point in depth $s+1$ we are in one of universal next time prestates reachable from w_0 and that there are the next instant we are in none of them. Of course, this cannot be the case; in fact, we can refute $\urcorner \Omega_s \wedge \circledcirc \Omega_s$: we derive the first fringe of all universal next time parts of prestates in $\urcorner \Omega_s$ and check that universal next time parts of prestates in these fringes were already in Ω_s. Thus by the resolution rule we derive $\Diamond$ false and hence false.

Proof of lemma 4.2. From lemma 4.2.3 and lemma 4.2.4 we obtain that $\wedge U_n$ can be refuted in R.

5. Concluding remarks

We have presented a nonclausal resolution approach to theorem proving in BPTL with temporal modalities $\bigcirc, \circledcirc, \Box, \Diamond$. We show that presented system is complete. We expect to generalize this approach to get nonclausal resolution system for branching temporal logic with modality *Until*.

References

[AM] (1990) Abadi M. and Z.Manna, Nonclausal deduction in first order temporal logic, Journal of ACM 37, 2, 279-317.

[BPM] (1983) Ben-Ari M., A.Pnueli and Z.Manna, The temporal logic of branching time, Acta Informatica 20, 207–226.