General Algebras

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Abstract

We study categories of algebras (for a given signature) constructed from arbitrary categories C with specific finite products. We show how a left adjoint $Set \longrightarrow C$ may be utilized to construct an initial algebra satisfying a given set of equations. Finally, we give a condition under which satisfaction of equations is independent of the set of variables considered.

1 Introduction

This paper reports work in progress. It is concerned with specifications of abstract data types and in particular with the *semantics* of such specifications. Here we will only consider *one-sorted equational* specifications or *presentations*, having two parts:

- the names (called *operation symbols*) of the *operations* involved, together with their arities
- *equations* describing the *properties* of the operations. The two sides of an equation are *terms* constructed from the operation symbols and usually some *variables*.

As for semantics, a presentation is usually interpreted in so-called *total algebras*. Such an algebra comprises a *carrier* set AU_Σ, and for each n-ary operation symbol a function $AU_\Sigma{}^n \longrightarrow AU_\Sigma$. Terms and equations may then be interpreted, and only those algebras *satisfying* the equations are considered. These algebras are the objects of a *category*[1]. Of particular interest are those algebras which are *initial* in this category. The initial algebras identify only those terms whose equality follows from the equations.

However, also other semantic models have been suggested: partial algebras ([13]), ordered/monotonic algebras ([10]), continuous algebras ([6]), and topological algebras ([9]).

As a vehicle for bridging the gap between specification of abstract data types and of systems of parallel processes, one might also consider total algebras

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[1] This paper assumes a fair knowledge of category theory. See e.g. [7].

whose carrier elements are models of parallel computation, e.g. partial orders representing the causal order of the events performed by the system.

The *theories* of the various kinds of models are fairly similar: In each case, similar theorems are proved in similar ways. To simplify and clarify matters and to facilitate the consideration of *novel* models, it might be useful to construct a *general* theory subsuming (as many as possible of) the others as special cases.

In [1] we presented our first steps towards this goal. We made a construction based on the following parameters:

- a Cartesian category C
- finite product preserving functors $U : C \longrightarrow Set$ and $F : Set \longrightarrow C$ such that $F \dashv U$ and[2] $F \circ U = I_{Set}$.

However, we felt that this last condition was rather *ad hoc* and too restrictive. In this paper we refine our approach in such a way that the abovementioned restriction becomes unnecessary.

In [1] we demonstrated that our approach subsumes the theories of total and ordered algebras (in the sense that it permits the construction of an initial algebra satisfying a given set of equations). For some speculations on the success or failure of our theory in some of the other cases mentioned above, we refer the reader to the same paper. The same goes for a more detailed introduction.

The rest of the paper is organized as follows: In section 2 we review the theory of total algebras. We present our general theory in section 3. In section 4 we make a few concluding remarks.

In order to structure the next two sections, we consider the successive introduction of the two parts of a presentation: operation symbols and equations.

Omitted proofs are entirely straightforward.

1.1 Notational and Other Conventions

If C is a category, its object class is denoted by C_0 and the (hom-)class of arrows $A \longrightarrow B$ by $C(A, B)$. A finite (product) mediating arrow is denoted by $\langle f_0, \ldots, f_{n-1} \rangle$.

A natural number is the set of the smaller ones, i.e. $n = \{0, \ldots, n-1\}$. In particular, the number 0 is the empty set.

A function space $Set(A, B)$ may also be denoted by ${}^A B$, and may alternatively be regarded as an 'A-fold product of B-s'. If A is a natural number n, we will usually use the notation B^n. If $b \in B^n$ and $i \in n$, we usually (but not always) write b_i for ib.

2 Total Algebras

In this section we briefly review the relevant parts of the theory of total algebras. A fuller account is given in e.g. [5]. We also give some lemmas needed later.

[2] Please note that we write composition in an arbitrary category in *diagrammatic* order, i.e. the composite arrow $\xrightarrow{f}\xrightarrow{g}$ is denoted by $f \circ g$. In accordance with this, application of a 'function' (e.g. a functor) f to an 'argument' x is denoted by xf. This application is left associative.

2.1 Signatures

A (*one-sorted*) *signature* Σ is an assignment of a set Σ_n (of *n-ary operation symbols*) to each natural number n. A 0-ary operation symbol is called a *constant symbol*.

If Σ and Σ' are signatures, Σ' is said to *include* Σ, denoted by $\Sigma \subseteq \Sigma'$, if $\Sigma_n \subseteq \Sigma'_n$ for all n.

A *signature chain* is an $\mathbf{N}$-indexed sequence $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ of signatures such that each $\Sigma_n \subseteq \Sigma_{n+1}$. (A *set chain* is an $\mathbf{N}$-indexed sequence $V_0, V_1, V_2, \ldots$ of sets such that each $V_n \subseteq V_{n+1}$.)

2.1.1 Union of Signatures

Let

- I be a set
- for each $i \in I$ Σ_i be a signature.

Then the *union* (*signature*) $\bigcup_{i \in I} \Sigma_i$ of the Σ_i is defined by

$$\left(\bigcup_{i \in I} \Sigma_i\right)_n = \bigcup_{i \in I} (\Sigma_i)_n \,.$$

Lemma 1 *Each $\Sigma_i \subseteq \bigcup_{i \in I} \Sigma_i$.*

2.1.2 Algebras and Homomorphisms

Let Σ be a signature. The category Set^{Σ} of *total* Σ*-algebras* and *total* Σ*-homomorphisms* is defined by

- A total Σ-algebra A comprises
  - a *carrier* set AU_{Σ}
  - for each $n \in \mathbf{N}$ and $f \in \Sigma_n$ a function $f_A : {AU_{\Sigma}}^n \longrightarrow AU_{\Sigma}$.
- A total Σ-homomorphism $h : A \longrightarrow B$ is a function $AU_{\Sigma} \longrightarrow BU_{\Sigma}$ such that

  (1) $$\forall n \in \mathbf{N}. \forall f \in \Sigma_n . f_A \circ h = \underbrace{h \times \cdots \times h}_{n} \circ f_B,$$

  which is equivalent to that for all $n \in \mathbf{N}$, $f \in \Sigma_n$, and $a \in {AU_{\Sigma}}^n$ $a f_A h = (a_0 h, \ldots, a_{n-1} h)\, f_B$.
- Composition is composition in *Set*.
- The identities are defined by $id_A = id_{AU_{\Sigma}}$.

Lemma 2 *Assume that Σ contains a constant symbol $c \in \Sigma_0$ and let A be a total Σ-algebra. Then AU_{Σ} is non-empty.*

Proof: The function $c_A : {AU_{\Sigma}}^0 \longrightarrow AU_{\Sigma}$, where ${AU_{\Sigma}}^0$ is a singleton, entailing that $* c_A \in AU_{\Sigma}$. ∎

2.1.3 The Forgetful Functor $Set^{\Sigma} \longrightarrow Set$

Extend the assignment $A \mapsto AU_{\Sigma}$ to a functor $U_{\Sigma} : Set^{\Sigma} \longrightarrow Set$ by $hU_{\Sigma} = h$ on arrows.

Even though $hU_{\Sigma} = h$, we will usually write hU_{Σ} when we think of it as a function.

2.1.4 Ground Terms

Let T_{Σ} be the least language (with the inclusion order) such that

$$T_{\Sigma} = \left\{ f(t_0, \ldots, t_{n-1}) \,\middle|\, n \in \mathbf{N}\ \mathbf{cand}\ f \in \Sigma_n \wedge \bigwedge_{i \in n} t_i \in T_{\Sigma} \right\}. \tag{2}$$

(When f is a constant symbol, $f()$ is written simply f, entailing that $\Sigma_0 \subseteq T_{\Sigma}$.) Each string in the language T_{Σ} is called a *ground term.*

This definition permits us to carry out proofs by *induction* on the structure of ground terms.

Lemma 3 *T_{Σ} is empty iff Σ_0 is.*

Proof: As 0 is the least language, the least language T_{Σ} such that (2) is 0 iff

$$\begin{aligned} 0 &= \left\{ f(t_0, \ldots, t_{n-1}) \,\middle|\, n \in \mathbf{N}\ \mathbf{cand}\ f \in \Sigma_n \wedge \bigwedge_{i \in n} t_i \in 0 \right\} \\ &= \Sigma_0. \end{aligned}$$

∎

Lemma 4 *The operation $\Sigma \mapsto T_{\Sigma}$ is monotone w.r.t. (signature and set) inclusion.*

2.1.4.1 Union of Ground Term Languages

Lemma 5 *Let*

- *I be a set*
- *for each $i \in I$ Σ_i be a signature.*

Then $\bigcup_{i \in I} T_{\Sigma_i} \subseteq T_{\bigcup_{i \in I} \Sigma_i}$.

Proof: Straightforward, using lemmata 4 and 1. ∎

Lemma 6 *Let $\Sigma_0 \subseteq \Sigma_1 \subseteq \cdots$ be a signature chain. Then*

$$T_{\bigcup_{n \in \mathbf{N}} \Sigma_n} = \bigcup_{n \in \mathbf{N}} T_{\Sigma_n}.$$

Proof: We have to prove that $T_{\bigcup_{n\in\mathbf{N}}\Sigma_n} \subseteq \bigcup_{n\in\mathbf{N}} T_{\Sigma_n}$. We will prove that $t \in T_{\bigcup_{n\in\mathbf{N}}\Sigma_n} \Rightarrow t \in \bigcup_{n\in\mathbf{N}} T_{\Sigma_n}$ by induction on t. We have to prove that $f(t_0,\ldots,t_{m-1}) \in T_{\bigcup_{n\in\mathbf{N}}\Sigma_n} \Rightarrow f(t_0,\ldots,t_{m-1}) \in \bigcup_{n\in\mathbf{N}} T_{\Sigma_n}$. Therefore, assume the antecedent, which is equivalent to that

- $$\begin{aligned} f &\in \left(\bigcup_{n\in\mathbf{N}} \Sigma_n\right)_m \\ &= \bigcup_{n\in\mathbf{N}} (\Sigma_n)_m, \end{aligned}$$

  which is equivalent to that $\exists n \in \mathbf{N}. f \in (\Sigma_n)_m$. Therefore, let $n \in \mathbf{N}$ be such that $f \in (\Sigma_n)_m$.

- $\bigwedge_{i\in m} t_i \in T_{\bigcup_{n\in\mathbf{N}}\Sigma_n}$, where $t_i \in T_{\bigcup_{n\in\mathbf{N}}\Sigma_n}$ entails that $t_i \in \bigcup_{n\in\mathbf{N}} T_{\Sigma_n}$, which is equivalent to that $\exists n \in \mathbf{N}. t_i \in T_{\Sigma_n}$. Therefore, let for each $i \in m$ $n_i \in \mathbf{N}$ be such that $t_i \in T_{\Sigma_{n_i}}$.

We have to prove the consequent, which is equivalent to that there is an $n \in \mathbf{N}$ such that $f(t_0,\ldots,t_{m-1}) \in T_{\Sigma_n}$.

We claim that $f(t_0,\ldots,t_{m-1}) \in T_{\Sigma_{\max\{n,n_0,\ldots,n_{m-1}\}}}$, which is straightforward to prove, using lemma 4. ∎

2.1.5 Extensions

Let Σ be a signature such that Σ_0 is a 'special' set, disjoint from those in Set_0.

If V is a set, define a signature $\Sigma(V)$ (the *V-extension of* Σ) by

- $\Sigma(V)_0 = \Sigma_0 \cup V$
- $\Sigma(V)_n = \Sigma_n$ otherwise.

Lemma 7 *The extension operation $V \mapsto \Sigma(V)$*

1. *is monotone w.r.t. (set and signature) inclusion*
2. *preserves chains, i.e. takes set chains to signature chains*
3. *preserves/distributes over (arbitrary) unions: $\Sigma\left(\bigcup_{i\in I} V_i\right) = \bigcup_{i\in I} \Sigma(V_i)$.*

2.1.6 The Free Algebra Functor

The functor U_Σ has a left adjoint $F_\Sigma : Set \longrightarrow Set^\Sigma$ whose object part is defined by: Let V be a *Set*-object. The Set^Σ-object VF_Σ is defined by

- The set $VF_\Sigma U_\Sigma = T_{\Sigma(V)}$.
- For each $n \in \mathbf{N}$ and $f \in \Sigma_n$ the function $f_{VF_\Sigma} : VF_\Sigma U_\Sigma{}^n \longrightarrow VF_\Sigma U_\Sigma$, i.e. $T_{\Sigma(V)}{}^n \longrightarrow T_{\Sigma(V)}$, is defined by $\mathbf{t} f_{VF_\Sigma} = f(t_0,\ldots,t_{n-1})$ (the *string*).

Each string in the language $VF_\Sigma U_\Sigma = T_{\Sigma(V)}$ is called a V*-term*.

In particular,

$$\begin{aligned} V &\subseteq \Sigma_0 \cup V \\ &= \Sigma(V)_0 \\ &\subseteq T_{\Sigma(V)}, \end{aligned}$$

so V-elements ('-variables') are V-terms. Let η_V denote the inclusion function $V \longrightarrow VF_\Sigma U_\Sigma$. Even though each $v\eta_V = v$, we usually write $v\eta_V$ when we think of it as a term.

Each V-term is on exactly one of two forms:

- $v\eta_V$
- $f(t_0, \ldots, t_{n-1})$, where n is a natural number, f is an n-ary operation symbol, and $t_0, \ldots, t_{n-1}$ are V-terms.

This allows for inductive definitions and proofs.

If A is a total Σ-algebra, a function $h : VF_\Sigma U_\Sigma \longrightarrow AU_\Sigma$ is a total Σ-homomorphism $VF_\Sigma \longrightarrow A$ iff for all $n \in \mathbf{N}$, $f \in \Sigma_n$, and $t \in VF_\Sigma U_\Sigma{}^n$ $[f(t_0, \ldots, t_{n-1})]\, h = (t_0 h, \ldots, t_{n-1} h)\, f_A$.

The functor F_Σ being a left adjoint of U_Σ means that each VF_Σ is free over V w.r.t. U_Σ. The universal arrow is the inclusion η_V.

As a special case, $0F_\Sigma$ is free over the initial *Set*-object 0, i.e. is itself initial (in Set^Σ).

VF_Σ being free over V w.r.t. U_Σ with universal arrow η_V means that for each Set^Σ-object A and *Set*-arrow $f : V \longrightarrow AU_\Sigma$ there exists a unique Set^Σ-arrow (denoted by) $f^\# : VF_\Sigma \longrightarrow A$ such that $\eta_V \circ f^\# U_\Sigma = f$.

2.1.6.1 Term Languages The term language $VF_\Sigma U_\Sigma = T_{\Sigma(V)}$ is empty iff $\Sigma(V)_0 = \Sigma_0 \cup V$ is, i.e. iff both Σ_0 and V are.

Corollary 1 *Let V and V' be sets such that $V \subseteq V'$. Then $VF_\Sigma U_\Sigma \subseteq V'F_\Sigma U_\Sigma$.*

Proof: Straightforward, using lemmata 4 and 7, part 1.. ∎

2.1.6.1.1 Union of Term Languages

Corollary 2 *Let*

- *I be a set*
- *for each $i \in I$ V_i be a set.*

Then $\bigcup_{i\in I} V_i F_\Sigma U_\Sigma \subseteq \left(\bigcup_{i\in I} V_i\right) F_\Sigma U_\Sigma$.

Proof: Straightforward, using corollary 1. ∎

Lemma 8 *Let $V_0 \subseteq V_1 \subseteq \cdots$ be a set chain. Then*

$$\left(\bigcup_{n\in\mathbf{N}} V_n\right) F_\Sigma U_\Sigma = \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma.$$

Proof: Straightforward, using lemmata 6 and 7, parts 2. and 3.. ∎

2.2 Presentations

2.2.1 Equations

If V is a set, a V-*equation* is a pair of V-terms, i.e. an element of $VF_\Sigma U_\Sigma{}^2$. An *equation* is a V-equation for *some* set V.

2.2.1.1 Union of Equation Sets

Corollary 3 *Let*

- *I be a set*
- *for each $i \in I$ V_i be a set.*

Then $\bigcup_{i\in I} V_i F_\Sigma U_\Sigma{}^2 \subseteq \left(\bigcup_{i\in I} V_i\right) F_\Sigma U_\Sigma{}^2$.

Proof: Straightforward, using corollary 1. ∎

Lemma 9 *Let $V_0 \subseteq V_1 \subseteq \cdots$ be a set chain. Then* $\left(\bigcup_{n\in\mathbf{N}} V_n\right) F_\Sigma U_\Sigma{}^2 = \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma{}^2$.

Proof: By corollary 3 and lemma 8 we have to prove that $\left(\bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma\right)^2 \subseteq \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma{}^2$. Therefore, let $(t,u) \in \left(\bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma\right)^2$, which is equivalent to that $t, u \in \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma$, where $t \in \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma$ iff $\exists n \in \mathbf{N}.t \in V_n F_\Sigma U_\Sigma$. Therefore, let $n \in \mathbf{N}$ be such that $t \in V_n F_\Sigma U_\Sigma$. Similarly, let $m \in \mathbf{N}$ be such that $u \in V_m F_\Sigma U_\Sigma$.

We have to prove that $(t,u) \in \bigcup_{n\in\mathbf{N}} V_n F_\Sigma U_\Sigma{}^2$, which is equivalent to that $\exists n \in \mathbf{N}. (t,u) \in V_n F_\Sigma U_\Sigma{}^2$. We claim that $(t,u) \in V_{\max\{m,n\}} F_\Sigma U_\Sigma{}^2$, which is straightforward to prove, using corollary 1. ∎

2.2.2 Satisfaction

Let A be a total Σ-algebra.

2.2.2.1 Satisfaction of a Single Equation What does it mean for A to *satisfy* an equation? The standard answer is as follows: No matter which values (in AU_Σ) are assigned to the variables of the equation, its two sides *evaluate* to the same value. Generalizing a bit, we might also consider variables *not* occurring in the equation. (These should not matter, and indeed they do not. We will return to this issue in paragraph 3.2.1.4.)

Therefore, let V be a set (of variables) and (t,u) be a V-equation. An assignment of values in AU_Σ to the variables in V is simply a function $f : V \longrightarrow AU_\Sigma$, and the evaluation of the V-terms t and u is provided by its Σ-homomorphic extension $f^\# U_\Sigma$. Thus, A V-*satisfies* (t,u), denoted by $A \models_V t = u$, if $\forall f \in Set(V, AU_\Sigma) .t\left(f^\# U_\Sigma\right) = u\left(f^\# U_\Sigma\right)$, or, equivalently, if $\forall h \in Set^\Sigma(VF_\Sigma, A) .t(hU_\Sigma) = u(hU_\Sigma)$.

2.2.2.2 Satisfaction of a Set of Equations Let E be a set of equations. The total Σ-algebra A is said to satisfy E if it satisfies every equation in E.

The pair (Σ, E) is called a *presentation*. A *total* (Σ, E)*-algebra* is a total Σ-algebra satisfying E.

The total (Σ, E)-algebras determine a full subcategory $Set^{(\Sigma,E)}$ of Set^{Σ}.

2.2.2.2.1 An Initial Algebra Let $\equiv_E$ denote the congruential closure of the ground instances of the equations in E. Then the quotient total Σ-algebra $0F_{\Sigma}/\equiv_E$ is an initial total (Σ, E)-algebra.

3 General Algebras

In the total case, the rôle played by the category *Set* is to provide sets, serving as carriers of algebras, and functions, serving as components of algebras and as homomorphisms. The key idea of this paper is that this rôle may be taken over by some *other* category. (The idea itself is not new ([12, page 221]), but we believe our treatment of initiality and of the independence of satisfaction of the set of variables considered, to be so.)

However, not *any* category will do. We have to be able to form the *finite powers* $AU_{\Sigma}{}^{n}$. To allow headroom for the generalization to *several sorts*, we at once generalize this to finite *products* $B_0 \times \cdots \times B_{n-1}$. A category with all finite products is said to be *Cartesian*.

However, the mere *existence* of products is not quite enough. For each tuple of objects we need a *specific* product (object and projections). The assignment of a product to each object tuple is usually called a *finite product structure* ([3, page 196]), but for the sake of brevity, we will introduce the term *Cartesian structure*. (Of course, any Cartesian category may be equipped with a Cartesian structure. The point is that we have to *decide* on one in particular.)

A functor F from one category with Cartesian structure to another is usually said to be *strict finite product preserving* (also [3, page 196]) if it preserves the structure, i.e. if it takes the product (object and projections) of each tuple $B_0, \ldots, B_{n-1}$ to the product of $B_0F, \ldots, B_{n-1}F$. For the sake of brevity, we will call such a functor *Cartesian*. Cartesian functors also preserve (finite) mediating and product arrows, natural transformations τ from one Cartesian functor to another satisfy $\tau_{B_0\times\cdots\times B_{n-1}} = \tau_{B_0} \times \cdots \times \tau_{B_{n-1}}$, and the $^{\#}$ operation of an adjunction between Cartesian functors preserves (finite) product arrows.

The categories with Cartesian structure and the Cartesian functors constitute a category *CartStruct*.

3.1 Signatures

3.1.1 Algebras and Homomorphisms

Let C be a category with a Cartesian structure. Define the category C^{Σ} of Σ-*algebras* and Σ-*homomorphisms* by

- A Σ-algebra A comprises

- a *carrier* C-object AU_Σ
- for each $n \in \mathbf{N}$ and $f \in \Sigma_n$ a C-arrow $f_A : {AU_\Sigma}^n \longrightarrow AU_\Sigma$.

- A Σ-homomorphism $h : A \longrightarrow B$ is a C-arrow $AU_\Sigma \longrightarrow BU_\Sigma$ such that (1).

- Composition is composition in C.

- The identities are defined by $id_A = id_{AU_\Sigma}$.

3.1.2 Initiality

In the total case the initial algebra was obtained as the image of an initial object under a left adjoint $\longrightarrow Set^\Sigma$, viz. the left adjoint $Set \longrightarrow Set^\Sigma$ of the forgetful functor $Set^\Sigma \longrightarrow Set$. This suggests that we should look for a left adjoint $\longrightarrow C^\Sigma$, e.g. a left adjoint $C \longrightarrow C^\Sigma$ of the forgetful functor $C^\Sigma \longrightarrow C$. Unfortunately, we have been unable to define such a left adjoint in general. (Moreover, C does not necessarily possess an initial object.)

However, as Set^Σ has an initial object, a left adjoint $Set^\Sigma \longrightarrow C^\Sigma$ would be useful. But, are there any functors between C^Σ and Set^Σ, or more generally, D^Σ, where D is some category with a Cartesian structure? The operation of *lifting* will enable us to construct such functors from Cartesian functors between C and D.

3.1.3 Lifting

Extend the assignment $C \mapsto C^\Sigma$ to a functor $-^\Sigma : \mathit{CartStruct} \longrightarrow \mathit{Cat}$ by: If C and D are *CartStruct*-objects and F is a *CartStruct*-arrow $C \longrightarrow D$, the *Cat*-arrow, i.e. the *lifted* functor, $F^\Sigma : C^\Sigma \longrightarrow D^\Sigma$ is defined by

- If A is a C^Σ-object, the D^Σ-object AF^Σ is defined by
  - The D-object $AF^\Sigma U_\Sigma = AU_\Sigma F$.
  - For each $n \in \mathbf{N}$ and $f \in \Sigma_n$ the D-arrow $f_{AF^\Sigma} = f_A F$.

- $(h : A \longrightarrow B) \longmapsto hF$.

Theorem 1 *Let*

- *C and D be categories with Cartesian structure*

- *$F : C \longrightarrow D$ and $U : D \longrightarrow C$ be Cartesian functors*

such that $F \dashv U$. Then $F^\Sigma \dashv U^\Sigma$. ('Lifting preserves adjunctions.')

Proof: Let $A \in {C^\Sigma}_0$. We have to prove that AF^Σ is free over A w.r.t. U^Σ. A universal arrow from A to U^Σ is an element of $C^\Sigma\left(A, AF^\Sigma U^\Sigma\right) \subseteq C\left(AU_\Sigma, AF^\Sigma U^\Sigma U_\Sigma\right)$, where

$$\begin{aligned} AF^\Sigma U^\Sigma U_\Sigma &= AF^\Sigma U_\Sigma U \\ &= AU_\Sigma FU. \end{aligned}$$

Let $\eta : I_C \stackrel{\cdot}{\longrightarrow} F \circ U$ be the unit of the adjunction. Then η_{AU_Σ} is a C-arrow $AU_\Sigma \longrightarrow AU_\Sigma FU$. By naturality of η it is in fact a C^Σ-arrow $A \longrightarrow AF^\Sigma U^\Sigma$.

We claim that AF^Σ is free over A w.r.t. U^Σ with universal arrow η_{AU_Σ}, which is fairly straightforward to prove. ∎

3.1.4 An Initial Algebra

Let $F : Set \longrightarrow C$ and $U : C \longrightarrow Set$ be Cartesian functors such that $F \dashv U$. Then also $F^\Sigma \dashv U^\Sigma$, where $F^\Sigma : Set^\Sigma \longrightarrow C^\Sigma$.

Corollary 4 $0F_\Sigma F^\Sigma$ *is initial in* C^Σ.

3.2 Presentations

3.2.1 Satisfaction

3.2.1.1 Satisfaction of a Single Equation

3.2.1.1.1 Satisfaction w.r.t. a Set of Variables It seems difficult to generalize the definition of satisfaction of an equation by a total algebra given in subparagraph 2.2.2.1. In [1] we defined a C^Σ-object A to satisfy a set E of equations if AU^Σ did, leading to the restriction $F \circ U = I_{Set}$ mentioned in the introduction.

However, the concept of satisfaction of an equation by a total algebra A may be formulated in another way that is easily generalizable. [11] attributes the idea to Lawvere. In our opinion, the idea may best be explained by appealing to denotational semantics of simple applicative languages, where the denotation of a term is a function from environments to values, where an environment is a function from variables to values. In our case, the set of variables is V and the set of values is AU_Σ, making the set of environments ${}^V AU_\Sigma$ and the set of possible term denotations $Set\left({}^V AU_\Sigma, AU_\Sigma\right)$. The 'semantic function' $h_{AV} : VF_\Sigma U_\Sigma \longrightarrow Set\left({}^V AU_\Sigma, AU_\Sigma\right)$ is then defined in the following way: The function $th_{AV} : {}^V AU_\Sigma \longrightarrow AU_\Sigma$ is defined by $g\left(th_{AV}\right) = t\left(g^\# U_\Sigma\right)$.

Corollary 5 *A total* Σ*-algebra* A V*-satisfies a* V*-equation* (t, u) *iff* $th_{AV} = uh_{AV}$.

However, we need a *compositional* definition of h_{AV}. Such a definition is provided by the following two corollaries:

Corollary 6 *Let* $v \in V$ *and* $g : V \longrightarrow AU_\Sigma$. *Then* $g\left(v\eta_V h_{AV}\right) = vg$.

Corollary 7 $\left[f\left(t_0, \ldots, t_{n-1}\right)\right] h_{AV} = \left\langle t_0 h_{AV}, \ldots, t_{n-1} h_{AV}\right\rangle \circ f_A$

The intended generalization may seem to require the existence of arbitrary (non-finite) products ${}^V AU_\Sigma$. However, this is not the case, as an equation contains a finite number of variables only. However, this finite number may be arbitrarily large, so we need an infinite set of variables altogether. On the other hand, a countable number is sufficient, as the variables of distinct equations are completely unrelated, although they may have the same names.

To simplify matters, we identify variables with the natural numbers. Furthermore, the only sets of variables (natural numbers) we will consider, are the natural numbers themselves and the set $\mathbf{N}$ of (all) natural numbers.

Corollary 8 $\mathbf{N}F_\Sigma U_\Sigma{}^2 = \bigcup_{n\in\mathbf{N}} nF_\Sigma U_\Sigma{}^2$

Proof: Straightforward, using lemma 9. ∎

Thus, an $\mathbf{N}$-equation is an n-equation for some natural number n. Moreover, there is a *least* such number.

Please notice that $m \leq n$ iff $m \subseteq n$, in which case $mF_\Sigma U_\Sigma \subseteq nF_\Sigma U_\Sigma$ (by corollary 1), entailing that $mF_\Sigma U_\Sigma{}^2 \subseteq nF_\Sigma U_\Sigma{}^2$.

Let n be a natural number. The function $h_{An} : nF_\Sigma U_\Sigma \longrightarrow Set\,(AU_\Sigma{}^n, AU_\Sigma)$ then satisfies $v\eta_n h_{An} =$ the vth projection $AU_\Sigma{}^n \longrightarrow AU_\Sigma$.

Let A be a C^Σ-object. The function[3] $h_{An} : nF_\Sigma U_\Sigma \longrightarrow C\,(AU_\Sigma{}^n, AU_\Sigma)$ is defined by

- $v\eta_n h_{An} =$ the vth projection $AU_\Sigma{}^n \longrightarrow AU_\Sigma$
- $[f\,(t_0, \ldots, t_{m-1})]\,h_{An} = \langle t_0 h_{An}, \ldots, t_{m-1} h_{An}\rangle \circ f_A$.

The Σ-algebra A is now said to n-satisfy an n-equation (t, u), denoted by $A \models_n t = u$, if $th_{An} = uh_{An}$.

Define a function $g_{An} : n \longrightarrow C\,(AU_\Sigma{}^n, AU_\Sigma)$ by $vg_{An} = v\eta_n h_{An}$. (In the total case, $a\,(vg_{An}) = va$.)

3.2.1.1.2 Absolute Satisfaction

Whether the algebra A n-satisfies an n-equation, should ideally not depend on n. In paragraph 3.2.1.4 we will show that under a certain condition this is indeed the case, but for the moment we will content ourselves with the following: Let (t, u) be an $\mathbf{N}$-equation, and let n be the *least* natural number such that (t, u) is an n-equation. The Σ-algebra A is said to *satisfy* the equation (t, u), denoted by $A \models t = u$, if it n-satisfies it.

3.2.1.2 Satisfaction of a Set of Equations

Let E be a set of $\mathbf{N}$-equations, making (Σ, E) a presentation. The Σ-algebra A is said to satisfy E, or to be a (Σ, E)-algebra, if it satisfies every equation in E. The (Σ, E)-algebras determine a full subcategory $C^{(\Sigma,E)}$ of C^Σ.

3.2.1.3 Satisfaction and Lifting

Lemma 10 *Let*

- *C and D be categories with Cartesian structure*
- *$F : C \longrightarrow D$ be a Cartesian functor*
- *A be a C^Σ-object.*

[3] Its domain is a *set*.

Then

1. *Let t be an n-term. Then $th_{AF^\Sigma,n} = th_{An}F$.*
2. *Let $v \in n$. Then $vg_{AF^\Sigma,n} = vg_{An}F$.*
3. *An n-equation n-satisfied by A is also n-satisfied by AF^Σ.*
4. *An* $\mathbf{N}$*-equation satisfied by A is also satisfied by AF^Σ.*
5. *A set of* $\mathbf{N}$*-equations satisfied by A is also satisfied by AF^Σ.*

Proof:

1. Straightforward induction on t.
2. Straightforward, using 1..
3. Straightforward, using 1..
4. Straightforward, using 3..
5. Straightforward, using 4..

∎

3.2.1.4 More on Absolute Satisfaction We will now show that under a certain condition n-satisfaction does not depend on n.

Let m and n be natural numbers such that $m \leq n$. Then for each $i \in m$ ig_{An} is a C-arrow $AU_\Sigma{}^n \longrightarrow AU_\Sigma$, entailing that $\langle 0g_{An}, \ldots, (m-1)\, g_{An}\rangle$ is a C-arrow $AU_\Sigma{}^n \longrightarrow AU_\Sigma{}^m$.

Lemma 11 *Let t be an m-term. Then $th_{An} = \langle 0g_{An}, \ldots, (m-1)\, g_{An}\rangle \circ th_{Am}$.*

Lemma 12 *Assume that $C = Set$ and that $mF_\Sigma U_\Sigma$ is non-empty. Then $\langle 0g_{An}, \ldots, (m-1)\, g_{An}\rangle$ is epi.*

Proof: We have to prove that $\langle 0g_{An}, \ldots, (m-1)\, g_{An}\rangle : AU_\Sigma{}^n \longrightarrow AU_\Sigma{}^m$ is *surjective*. Therefore, let $a \in AU_\Sigma{}^m$. We have to prove that there is a $b \in AU_\Sigma{}^n$ such that

$$\begin{aligned} a &= b\,\langle 0g_{An}, \ldots, (m-1)\, g_{An}\rangle \\ &= (b\,[0g_{An}], \ldots, b\,[(m-1)\, g_{An}]) \\ &= [0b, \ldots, (m-1)\, b]\,, \end{aligned}$$

which is equivalent to that $\bigwedge_{i \in m} ia = ib$, determining b on $m \subseteq n$. If $i \in n \setminus m$, ib may be taken to be any element of AU_Σ. Thus, we have to prove that AU_Σ is non-empty.

As $mF_\Sigma U_\Sigma$ is non-empty, m is non-empty or Σ contains a constant symbol. Thus, there are two cases:

- m is non-empty. Let $i \in m$, entailing that $ia \in AU_\Sigma$.

- Σ contains a constant symbol, entailing that AU_Σ is non-empty (by lemma 2).

$\blacksquare$

Assume that there is a *faithful* Cartesian functor $V : C \longrightarrow Set$ (this being the abovementioned condition).

Lemma 13 *Assume that $mF_\Sigma U_\Sigma$ is non-empty. Then $\langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle$ is* epi.

Proof: As the faithful V reflects epics, we may prove this by proving that $\langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle\, V$ is epi, where

$$\begin{aligned} \langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle\, V &= \langle 0g_{An}V, \ldots, (m-1)\, g_{An}V \rangle \\ &= \left\langle 0g_{AV^\Sigma,n}, \ldots, (m-1)\, g_{AV^\Sigma,n} \right\rangle , \end{aligned}$$

which is epi (by lemma 12). $\blacksquare$

Corollary 9 *Let (t, u) be an m-equation. Then $A \models_m t = u$ iff $A \models_n t = u$.*

Proof: We have to prove that

$$\begin{aligned} & th_{Am} = uh_{Am} \\ & \quad \Leftrightarrow \quad th_{An} = uh_{An} \\ & \quad \Leftrightarrow \quad \langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle \circ th_{Am} = \langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle \circ uh_{Am}, \end{aligned}$$

where $\Rightarrow$ is trivial, whereas $\Leftarrow$ follows from the fact that $\langle 0g_{An}, \ldots, (m-1)\, g_{An} \rangle$ is epi (in the presence of the m-terms t and u). $\blacksquare$

This shows that n-satisfaction is indeed independent of n (under the condition that there is a faithful Cartesian functor $C \longrightarrow Set$).

In the total case lemma 12 may be used directly in the proof of corollary 9. Moreover, V-satisfaction is independent of V (for *arbitrary* variable sets V), which may be proved in a similar fashion.

3.2.2 Initiality

Recall the initial total (Σ, E)-algebra $0F_\Sigma / \equiv_E$. Then $(0F_\Sigma / \equiv_E)\, F^\Sigma \in C^{(\Sigma,E)}{}_0$.

Theorem 2 $(0F_\Sigma / \equiv_E)\, F^\Sigma$ *is* initial *in $C^{(\Sigma,E)}$.*

Proof: Let $A \in C^{(\Sigma,E)}{}_0$, entailing that $AU^\Sigma \in Set^{(\Sigma,E)}{}_0$ (by lemma 10, part 5). We have to prove that

$$\begin{aligned} \exists_1 h &\in C^{(\Sigma,E)} \left[(0F_\Sigma / \equiv_E)\, F^\Sigma, A \right] \\ &= C^\Sigma \left[(0F_\Sigma / \equiv_E)\, F^\Sigma, A \right] , \end{aligned}$$

which is equivalent to that

$$\begin{aligned} \exists_1 h &\in Set^\Sigma \left(0F_\Sigma / \equiv_E, AU^\Sigma \right) \\ &= Set^{(\Sigma,E)} \left(0F_\Sigma / \equiv_E, AU^\Sigma \right) , \end{aligned}$$

which holds. $\blacksquare$

4 Concluding Remarks

We have made a general construction based on some *parameters*: an adjunction $F \dashv U : C \longrightarrow Set$ in *CartStruct*. From C we have constructed the category of Σ-algebras and its full subcategory of (Σ, E)-algebras. Using the functors, we have shown that the two categories have initial objects closely related to the ordinary term and quotient algebras, which we believe to be new results. The same goes for the condition under which we have found satisfaction to be independent of the set of variables considered.

We will conclude the paper by pointing out some (possible) directions for (our) future research:

- To base the approach on algebraic (Lawvere) *theories* ([8]) (instead of on presentations)
- To (try to) apply our theory to some of the cases mentioned in the introduction: topological and continuous algebras, and in particular total algebras whose carrier elements are e.g. partial orders
- Generalization to several sorts
- By considering a *category* of signatures one might possibly obtain an *institution* ([2]) depending on C. This way of constructing institutions should then be compared to others, e.g. *chartering* ([4]).

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