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Preliminary Proceedings of the Workshop on

Geometry and Topology in Concurrency

GETCO '05

San Francisco, California, USA, August 21, 2005

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*GE*ometry and *T*opology in *CO*ncurrency

Preliminary Proceedings of GETCO 2005
Satellite Workshop of CONCUR 2005
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Foreword

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The main mathematical disciplines that have been used in theoretical computer science are discrete mathematics (especially, graph theory and ordered structures), logics (mostly proof theory for all kinds of logics, classical, intuitionistic, modal etc.) and category theory (cartesian closed categories, topoi etc.). General Topology has also been used for instance in denotational semantics, with relations to ordered structures in particular.

Recently, ideas and notions from mainstream "geometric" topology and algebraic topology have entered the scene in Concurrency Theory and Distributed Systems Theory (some of them based on older ideas). They have been applied in particular to problems dealing with coordination of multi-processor and distributed systems. Among those are techniques borrowed from algebraic and geometric topology: Simplicial techniques have led to new theoretical bounds for coordination problems. Higher dimensional automata have been modelled as cubical sets with a partial order reflecting the time flows, and their homotopy properties allow to reason about a system's global behaviour.

This workshop aims at bringing together researchers from both the mathematical (geometry, topology, algebraic topology etc.) and computer scientific side (concurrency theorists, semanticians, researchers in distributed systems etc.) with an active interest in these or related developments.

It follows the first workshop on the subject "Geometric and Topological Methods in Concurrency Theory" which has been held in Aalborg, Denmark, in June 1999. Then came GETCO'00 in Pennstate, USA, GETCO'01 in Aalborg, Denmark, all associated with CONCUR. GETCO'02 was associated with DISC'02 in Toulouse, and GETCO'03 was held jointly with CMCIM'03, associated with CONCUR in Marseille. The GETCO'04 workshop was associated with DISC, in Amsterdam, and this year GETCO'05 is associated with CONCUR in San Francisco.

The Workshop has been financially supported by BRICS (Aarhus, Denmark), which we thank very warmly. I also wish to thank the referees, the authors, and the programme committee members for their very precise and timely job. Many thanks are also due to Michael Mislove who kindly supported the workshop by letting us submit the papers through the Electronic Notes in Theoretical Computer Science.

Last but not least, I wish to thank the organizers of CONCUR 2005, Martin Abadi and Luca de Alfaro, for their cooperation regarding this workshop.

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Higher-Dimensional Automata and Other Models of Concurrency

Rob van Glabbeek [1]

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I will compare the expressiveness of several models of concurrency that could be thought of as formalisations of higher dimensional automata: cubical sets [1], presheaves over a category of bipointed sets, automata with a predicate on hypercube-shaped subgraphs, labelled step transition systems [2], and higher dimensional transition systems [3]. A series of counterexamples will illustrate the limitations of each of these models. Additionally I recall a few results [1] relating higher dimensional automata to ordinary automata, Petri nets, and various kinds of event structures.

References

[1] Glabbeek, R. J. van, *On the expressiveness of higher dimensional automata*, to appear in: Theoretical Computer Science (2005). Available at http://Boole.stanford.edu/pub/express.pdf.

[2] Glabbeek, R. J. van, *The Individual and Collective Token Interpretations of Petri Nets*, to appear in: Proceedings 16th International Conference on Concurrency Theory, CONCUR 2005, San Francisco, USA, August 2005 (M. Abadi & L. de Alfaro, eds.), LNCS 3653, Springer, pp. 323-337. Available at http://Boole.stanford.edu/pub/individual.ps.gz.

[3] Cattani, G. J. and Sassone, V., *Higher dimensional transition systems*, in: Proceedings LICS '96, Eleventh Annual IEEE Symposium on Logic in Computer Science, New Brunswick, USA, pp. 55–62. Available at ftp://ftp.cl.cam.ac.uk/users/glc25/hdts.dvi.gz.

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Using Cancellation instead of Labeling to Express van Glabbeek's EXPRESS'04 Example
(Invited Talk)

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At EXPRESS'04, R. van Glabbeek gave an 11-state labeled automaton (in his sense) witnessing that Petri nets cannot simulate automata. Viewing this automaton as a configuration structure with 3-state events (hence an HDA), we dispense with the labels by using a fourth event state of cancellation as per [1].

References

[1] Pratt, V., *Transition and Cancellation in Concurrency and Branching Time*, Mathematical Structures in Computer Science 13:4, pp. 485–529 (2003).

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Comparing topological models for concurrency

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Abstract

Several categories of models for concurrency involving topology have been put forward in each of which a notion of fundamental category is defined. One of them, the category of pospaces, is canonically included in almost all the others. Given a pospace $\overrightarrow{X}$ and $i(\overrightarrow{X})$, the image of $\overrightarrow{X}$ by the inclusion i of PoTop in some of the other category in which the fundamental category is defined, it is then natural to ask how the fundamental categories of $\overrightarrow{X}$ and $i(\overrightarrow{X})$ are related. The answer to this question is one of the purposes along of this article.

We introduce a general framework for categories in which a reasonable notion of fundamental categories can be defined.

Key words: directed paths, directed homotopy, fundamental category, models for concurrency, topologically concrete category

1 Introduction

The original motivation for studying topological models of concurrency is the notion of progress graph introduced in 1968 by *Dijkstra* in his article [3]. The idea is that a PV programm, which consists on a finite set of processes each of which performing lock and release on semaphores can be represented by a geometrical shape equiped with an order relation. Let us examine the following PV programs:

$$P_cV_c|P_cV_c \quad P_aP_bV_bV_a|P_bP_aV_aV_b \text{ and } P_cP_aP_bV_bV_aV_c|P_cP_bP_aV_aV_bV_c$$

Each of these programs have 2 processes. The letters a, b, c denote semaphores of arity 1 i.e. that each of them can be simultaneously used by, at most, 1 process. If the next instruction to be performed by the process Π is P_a then it tries to "take" the semaphore a, then either a is "free" (so Π can take it) or it is not (because it has already been taken by another process). In the first

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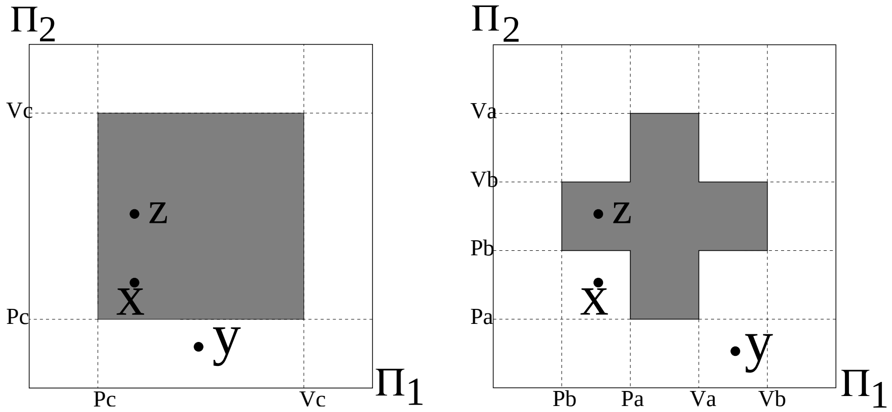
case the process Π can perform P_a and goes to the next instruction, in the other one Π has to wait till the process which holds a releases it. The process that holds a can release it performing the instruction V_a. Then the collection of states can be represented as follows



The left hand picture is associated to the first and third program while the right hand one is associated to the second. For example, consider the point x on the first figure, it represents a state in which both processes have taken a which is impossible. Such points form region of the forbidden states. On the second picture, y is a state in which Π_1 has already performed $P_bP_aV_a$ while the Π_2 has not even execute its first instruction. We would like to distinguish these shapes. A careful examination of the third program shows that it has the same behaviour as the first one, but this fact becomes immediate when we observe their geometric models provided we have a theorem such as "equivalent geometric models implies same behaviour". We have moved the analysis of PV programs to the analysis oh their geometric models. From an Euclidean point of view, the models of our example are different, but the classification up to isometry is way too strong. On the other hand, the classification up to homotopy equivalence as in classical algebraic topology is too loose since it does not distinguish these geometric shapes. Then we equip our models with the partial order induced by the one of $\mathbb{R}^2$ and observe that the second one has a local maximum while the first one does not. This remark motivates the introduction of the notions of pospaces ([16]) and directed algebraic topology ([12],[8],[6]).

Now, we briefly recall some definitions, a general reference for the topological approach to concurrency is [12]. The category of *Hausdorff* spaces is denoted Haus. A **pospace** is a pair $(X, \leq_X)$ where X is a topological space and $\leq_X$ a partial order relation on $|X|$ (the underlying set of X) whose graph is closed in $X \times X$ See [16]. Together with the increasing continuous maps, they form a category denoted PoTop. Weakening the notion of pospace asking $\leq_X$ just be reflexive we obtain the **related spaces** which also form a category, denoted RTop, together with continuous maps such that $\forall x, x' \in X$ if x and x' are related then so are $f(x)$ and $f(x')$. For technical convenience, we also require that the underlying topological space of an object of RTop be *Haus-*

dorff. Originally, I have introduced them as a technical tool to prove that PoTop is cocomplete.
A **directed space**, see [10] and [9], is a pair (X, dX) where X is a *Hausdorff* topological space and dX is a family of paths on X containing all the constant paths, stable under concatenation and satisfying $\forall\theta : [0,1] \longrightarrow [0,1]$ continuous and increasing, $\forall\gamma \in dX\ \gamma\circ\theta \in dX$.[2] Together with continuous maps f satisfying $\forall\gamma \in dX\ f\circ\gamma \in dY$, they form a category denoted dTop.
A **local pospace** is a topological space X together with an open covering V_i and a family of partial order $\leq_i$ on V_i such that $\forall i\ (V_i, \leq_i)$ is a pospace. The morphisms from $(X, V_i, \leq_i)$ to $(Y, W_j, \leq'_j)$ are the continuous maps from X to Y such that $\forall x \in X\ \forall j$ such that $f(x) \in W_j$, $\exists U \subseteq V_i$ (for some i) a neighborhood of x such that f induces a dimap from $(U, \leq_i |_U)$ to $(W_j, \leq'_j)$. Then we have a category denoted LPoTop. See [4].
Roughly speaking, the machinery we will introduce can be applied to any category whose objects are made of a (*Hausdorff*) topological space X equiped with some structure compatible with respect to the topology of X. In fact, the category of Flows introduced by *Philippe Gaucher* (see [7]) is the only one category of models for concurrency which is not topologically concrete (see definition 3.1) that I know of.

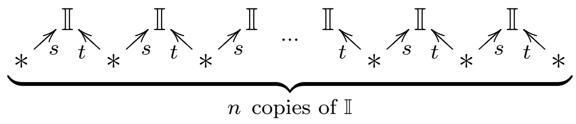
2 Category with paths

In classical algebraic topology, the unit segment $[0,1]$ plays a crucial role. This is also the case in PoTop, LPoTop, dTop or RTop provided that it is equiped with the suitable structure, that is to say a structure that makes it directed. The notion of category with paths is based on this fact. Of course, in classical algabraic topology, the idea of using $[0,1]$ as an elementary brick is not new and appears, for example, in the notion of path object, see [1] or [14].
Let C be a category with a terminal object, such an object is unique up to isomorphism, let us choose one of them and denote it $*$. A **point** of an object X of C is a morphism $p \in \mathsf{C}[*, X]$. In particular, given a point p of X and an object A of C, there is a unique morphism $f \in \mathsf{C}[A, X]$ such that $f = p\circ\zeta_A$ where ζ_A is the unique element of $\mathsf{C}[A, *]$. The morphism f we have described is called the **constant** morphism of value p from A to X. Thus, a morphism is said constant when it factorizes through the terminal object. The intention behind this definition becomes clear when it is particularized to Set. We also require and choose an object $\mathbb{I}$ of C, that will be called the **generic path** and two morphisms $s, t \in \mathsf{C}[*, \mathbb{I}]$ so that for any $\phi \in \mathsf{C}[\mathbb{I}, \mathbb{I}]$ isomorphism, we have

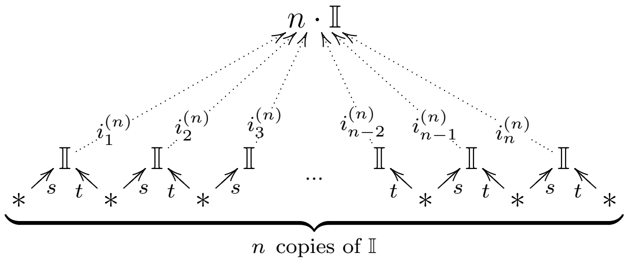
$$\Big((\phi\circ s = s \text{ and } \phi\circ t = t) \text{ or } (\phi\circ s = t \text{ and } \phi\circ t = s)\Big) \text{ i.e. } \{\phi\circ s, \phi\circ t\} = \{s, t\}$$

[2] in the original definition it is not required that the underling topological space be *Hausdorff*.

and so that $\forall n \in \mathbb{N}$, the following diagram



has a colimit in C. This colimit, unique up to isomorphism in C, is denoted $n \cdot \mathbb{I}$ together with

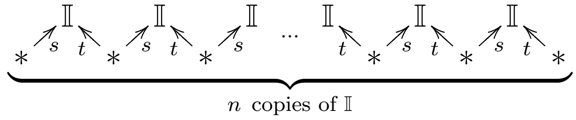


Note that, in the preceeding diagram, the arrows from $*$ have been omitted, indeed, they have to make the diagram commutative so they are implicitly determined. The stability of $\{s, t\}$ under any automorphism of $\mathbb{I}$ is the categorical way to say that s and t are the extremities of $\mathbb{I}$. Besides, an automorphism that exchanges s and t can be thought of as a "time reversal". Notice that for any isomorphism $\phi \in \mathsf{C}[A, \mathbb{I}]$, A together with $\phi^{-1} \circ s$ and $\phi^{-1} \circ t$ can be taken as a generic object. Indeed, let $\psi \in \mathsf{C}[A, A]$ be an isomorphism, since $\phi \circ \psi \circ \phi^{-1}$ is an automorphism of $\mathbb{I}$, we have $\{\phi \circ \psi \circ \phi^{-1} \circ s, \phi \circ \psi \circ \phi^{-1} \circ t\} = \{s, t\}$ i.e. $\{\psi \circ \phi^{-1} \circ s, \psi \circ \phi^{-1} \circ t\} = \{\phi^{-1} \circ s, \phi^{-1} \circ t\}$. Moreover, given isomorphisms $\phi_1, \phi_2 \in \mathsf{C}[A, \mathbb{I}]$, $\phi_1^{-1} \circ \phi_2$ is an automorphism of A, hence $\{(\phi_1^{-1} \circ \phi_2) \circ \phi_2^{-1} \circ s, (\phi_1^{-1} \circ \phi_2) \circ \phi_2^{-1} \circ t\} = \{\phi_2^{-1} \circ s, \phi_2^{-1} \circ t\}$, so $\{\phi_1^{-1} \circ s, \phi_1^{-1} \circ t\} = \{\phi_2^{-1} \circ s, \phi_2^{-1} \circ t\}$. Hence, up to a "time reversal", s and t are entirely determined by the choice of $\mathbb{I}$.

In fact, we cannot take any object of C as a generic path. For example, in Top, the Euclidean circle S^1 cannot be taken as a generic path object. Indeed, for any points x and y of S^1, there is an automorphism of S^1, for example a rotation, that respectively sends x and y onto x' and y' so that $\{x, y\} \cap \{x', y'\} = \emptyset$. On the other hand, any automorphism of [0,1] induces a 1-1 mapping from $\{0, 1\}$ to $\{0, 1\}$. The reason is that $\{0, 1\}$ is the boundary of $[0, 1]$.

We say that $\mathbb{I}$ provides a notion of **direction** to C when $\phi \circ s = s$ and $\phi \circ t = t$ for any automorphism ϕ of $\mathbb{I}$, otherwise, we say $\mathbb{I}$ provides a notion of **connection** to C.

The second hypothesis enables us to define a concatenation which is strict instead of up to isomorphism. To this end, we choose for each $n \in \mathbb{N}$ a cocone $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ that represents the colimit of the diagram



In the rest of the paper, we will refer to the preceeding diagram as V_n. Moreover, for $n := 0$ we can suppose that $0 \cdot \mathbb{I} := *$ and for $n := 1$ that $1 \cdot \mathbb{I} := \mathbb{I}$ and $i_1^1 := id_{\mathbb{I}}$. By induction over $n \in \mathbb{N}$ choose the cocones $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ so that if $n \cdot \mathbb{I} \cong p \cdot \mathbb{I}$ in C then $n \cdot \mathbb{I} = p \cdot \mathbb{I}$.

Lemma 2.1 (Monoid of paths) *Setting for all $n, p \in \mathbb{N}$ $(n \cdot \mathbb{I}) + (p \cdot \mathbb{I}) := (n+p) \cdot \mathbb{I}$, we turn $\{n \cdot \mathbb{I} | n \in \mathbb{N}\}$ into a commutative monoid whose unit is $0 \cdot \mathbb{I}$ i.e. $*$. Further, we have a morphism of monoids from $(\mathbb{N}, +, 0)$ onto $(\{n \cdot \mathbb{I}/n \in \mathbb{N}\}, +, *)$. Furthermore, if there are $n, p \in \mathbb{N}$ such that $n \cdot \mathbb{I} \neq p \cdot \mathbb{I}$ then $\{n \cdot \mathbb{I}/n \in \mathbb{N}\}$ is finite. Otherwise it is isomorphic to $\mathbb{N}$.*

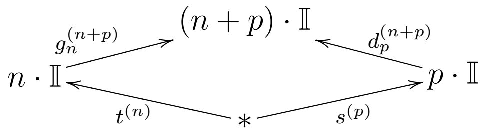
Proof. Left to the reader. □

We can always take $\mathbb{I} := *$ as a generic path, making the monoid of paths trivial. For any $n \in \mathbb{N}$, we define $s^{(n)} := i_1^{(n)} \circ s$ and $t^{(n)} := i_n^{(n)} \circ t$. Then, using the universal property of colimits, for any pair of integers (n, p) we uniquely define $g_n^{(n+p)} \in \mathsf{C}[n \cdot \mathbb{I}, (n+p) \cdot \mathbb{I}]$ and $d_p^{(n+p)} \in \mathsf{C}[p \cdot \mathbb{I}, (n+p) \cdot \mathbb{I}]$ so that

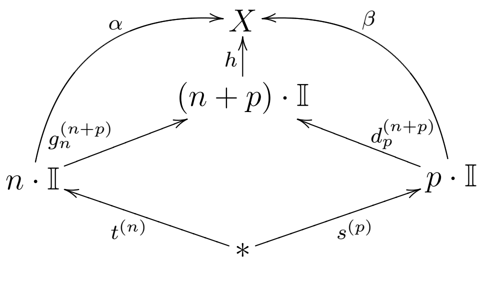
$$\begin{cases} g_n^{(n+p)} \circ i_k^{(n)} = i_k^{(n+p)} \text{ for every } k \in \{1, ..., n\} \\ d_p^{(n+p)} \circ i_k^{(p)} = i_{n+k}^{(n+p)} \text{ for every } k \in \{1, ..., p\} \end{cases}$$

In particular, $g_0^{(n)} = i_1^{(n)} \circ s = s^{(n)}$, $d_0^{(n)} = i_n^{(n)} \circ t = t^{(n)}$ and $g_n^{(n)} = d_n^{(n)} = id_{n \cdot \mathbb{I}}$. Furthermore:

Proposition 2.2 *For all $n, p \in \mathbb{N}$*



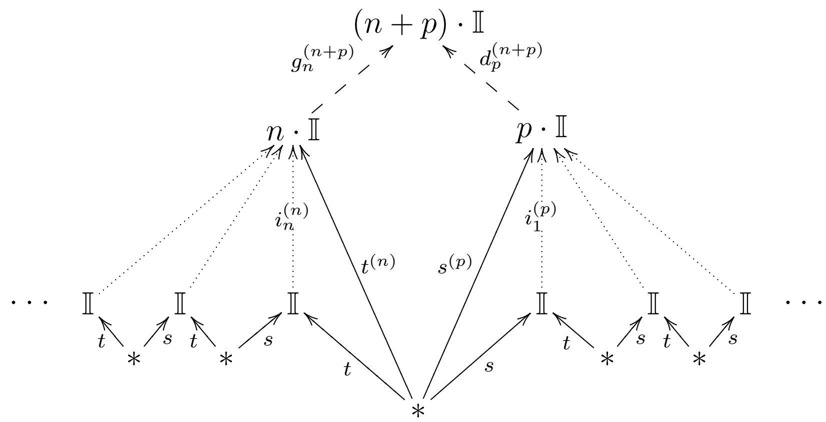
is a push-out in C. Moreover, if $\alpha \in \mathsf{C}[n \cdot \mathbb{I}, X]$ and $\beta \in \mathsf{C}[p \cdot \mathbb{I}, X]$ satisfy $\alpha \circ t^{(n)} = \beta \circ s^{(p)}$ then the unique morphism $h \in \mathsf{C}[(n+p) \cdot \mathbb{I}]$ such that



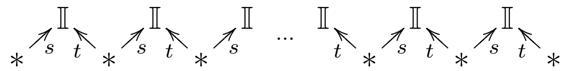
is also the unique h such that

$$\begin{cases} \alpha \circ i_k^{(n)} = h \circ i_k^{(n+p)} \text{ for every } k \in \{1, ..., n\} \\ \beta \circ i_k^{(n)} = h \circ i_{n+k}^{(n+p)} \text{ for every } k \in \{1, ..., p\} \end{cases}$$

Proof. The proof is entirely contained in the following commutative diagram



More precisely, $(\alpha\circ i_1^{(n)},...,\alpha\circ i_n^{(n)},\beta\circ i_1^{(p)},...,\beta\circ i_p^{(p)})$ is a cocone whose basis is



so we have a unique $h\in\mathsf{C}[(n+p)\cdot\mathbb{I},X]$ such that

$$\begin{cases}\alpha\circ i_k^{(n)} = h\circ i_k^{(n+p)} = h\circ g_n^{(n+p)}\circ i_k^{(n)} \text{ for every } k\in\{1,...,n\}\\ \beta\circ i_k^{(n)} = h\circ i_{n+k}^{(n+p)} = h\circ d_p^{(n+p)}\circ i_k^{(p)} \text{ for every } k\in\{1,...,p\}\end{cases}$$

Applying the uniqueness part of the universal property of colmimits $(n\cdot\mathbb{I},i_1^{(n)},...,i_n^{(n)})$ and $(p\cdot\mathbb{I},i_1^{(p)},...,i_p^{(p)})$ we have $\alpha = h\circ g_n^{(n+p)}$ and $\beta = h\circ d_p^{(n+p)}$. If $h'\in\mathsf{C}[(n+p)\cdot\mathbb{I},X]$ satisfy $\alpha = h\circ g_n^{(n+p)}$ and $\beta = h\circ d_p^{(n+p)}$, then necessarily, applying the uniqueness part of the uniqueness of the universal property of the colimit $((n+p)\cdot\mathbb{I},i_1^{(n+p)},...,i_{n+p}^{(n+p)})$ we have $h=h'$. □

Definition 2.3 A **Category with paths** is given by:

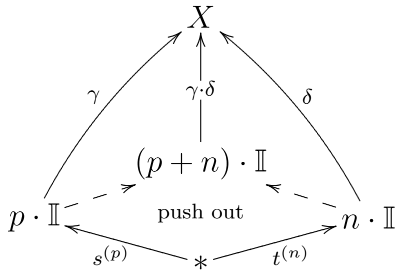
(i) a category C with a terminal object and $*$ a distinguished representative of it..

(ii) a diagram $*\overset{s}{\underset{t}{\rightrightarrows}}\mathbb{I}$ such that for any isomorphism $\phi\in\mathsf{C}[\mathbb{I},\mathbb{I}]$ we have $\{\phi\circ s,\phi\circ t\} = \{s,t\}$.

(iii) For all $n\in\mathbb{N}$, V_n has a colimit in C and we have a distinguished colimiting cocone $(n\cdot\mathbb{I},i_1^{(n)},...,i_n^{(n)})$ the colimit of the diagram V_n so that $0\cdot\mathbb{I}=*$, $1\cdot\mathbb{I}=\mathbb{I}$, $i_1^{(1)}=id_{\mathbb{I}}$ and that $\forall n,p\in\mathbb{N}$ if $n\cdot\mathbb{I}\cong p\cdot\mathbb{I}$ in C then $n\cdot\mathbb{I}=p\cdot\mathbb{I}$.

When the context is clear, we will just refer to the structure of category with paths of C as C, letting implicit the rest of the data. However, the distinguished terminal and cocones are part of the structure. Given an object X of C, we can define a path on X as an element (γ,n) of $\bigcup_{n\in\mathbb{N}}\mathsf{C}[n\cdot\mathbb{I},X]\times\{n\}$

and the source and the target of $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$ respectively as $\gamma \circ s^{(n)}$ and $\gamma \circ t^{(n)}$. Refering to the definition of constant morphism, any point is a constant path, this remark enable us to treat paths defined on $0 \cdot \mathbb{I}$ as any other. In fact, the constant paths i.e. those that are defined on $0 \cdot \mathbb{I} = *$ will be the identities of the category of paths of X that we will defined later. Given $(\gamma, p) \in \mathsf{C}[p \cdot \mathbb{I}, X] \times \{p\}$ and $(\delta, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$ two paths on X so that $src(\gamma) = tgt(\delta)$ we define the **concatenation** of (δ, n) followed by (γ, p), denoted $(\gamma \cdot \delta, n + p)$, by means of the universal property of the push-out depicted on the figure below. An immediate corollary of Proposition 2.2 is that the concatenation we have just defined is "strictly" associative, i.e. not only up to isomorphism.



Remark 2.4 Let $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$, we have $\gamma = (\gamma \circ i_n^{(n)}) \cdot \ldots \cdot (\gamma \circ i_1^{(n)})$. The second component $\{n\}$ cannot be omitted, indeed, by definition of a category with paths, if $\mathbb{I} + \mathbb{I} \cong \mathbb{I}$ then $\forall n \in \mathbb{N}$ we have $n \cdot \mathbb{I} = 1 \cdot \mathbb{I} = \mathbb{I}$. But, as in the notion of *Moore* paths, we wish to have, with each path, an information about how many "elementary" paths it is made of. In some categories, as RTop, if $n \neq p$, we have $n \cdot \mathbb{I} \ncong p \cdot \mathbb{I}$, so the source of γ as a morphism of C i.e. $n \cdot \mathbb{I}$ contains this information. In most of the others cases, as in Top, we have $\mathbb{I} + \mathbb{I} \cong \mathbb{I}$, so this information has to be kept as a "extra data". Once again, the advantage is that we have a strict concatenation. For the sake of simplicity, we will consider that $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ really means $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$.

Now we can describe the category of paths on X denoted $\Gamma(X)$. The objects of $\Gamma(X)$ are the points of X i.e. $Ob(\Gamma(X)) = \mathsf{C}[*, X]$. Then, given two points x and y of X, $\Gamma(X)[x, y] := \left\{\gamma \in \bigcup_{n \in \mathbb{N}} \mathsf{C}[n \cdot \mathbb{I}, X] \middle/ src(\gamma) = x \text{ et } tgt(\gamma) = y\right\}$. The concatenation is defined as above and we check that we have a category whose identities are the points $x : * \to X$ which can be seen as a path on X since $* = 0 \cdot \mathbb{I}$. The preceeding construction is functorial

Proposition 2.5 *There is a functor $\Gamma : \mathsf{C} \longrightarrow \mathsf{Cat}$ which associates to any object X of C its category of paths $\Gamma(X)$. In particular, if $f \in \mathsf{C}[X, Y]$ then we have a functor $\Gamma(f) : \Gamma(X) \longrightarrow \Gamma(Y)$ given by:*

(i) *For all point x of X, $\Big(\Gamma(f)\Big)(x) := f \circ x$.*

(ii) *For all* $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ *is a path from* x_1 *to* x_2*, i.e.* $\gamma \in \Big(\Gamma(X)\Big)[x_1, x_2]$, $\Big(\Gamma(f)\Big)(\gamma) := f \circ \gamma$.

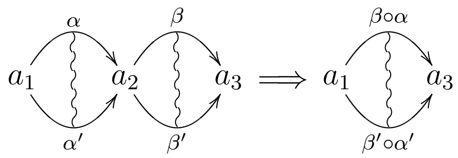
Proof. By proposition 2.2 we have

$$f \circ (\gamma \cdot \delta) = (f \circ \gamma) \cdot (f \circ \delta)$$

which proves that $\Gamma(f)$ is actually a functor from $\Gamma(X)$ to $\Gamma(Y)$.
□

Remark 2.6 If the category with paths $\Big(\mathsf{C}, * \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{I}\Big)$ has an automorphism ϕ of $\mathbb{I}$ such that $\phi \circ s = t$ and $\phi \circ t = s$ (i.e. a **time reversal**) then for all points x_1, x_2 of an object X of C, $\gamma \in \Gamma(X)[x_1, x_2] \longmapsto \gamma \circ \phi \in \Gamma(X)[x_2, x_1]$ is a bijection. It suffices to note that $\gamma \circ \phi \circ s^{(n)} = \gamma \circ t^{(n)}$ and $\gamma \circ \phi \circ t^{(n)} = \gamma \circ s^{(n)}$ and that the inverse mapping is $\gamma \in \Gamma(X)[x_2, x_1] \longmapsto \gamma \circ \phi^{-1} \in \Gamma(X)[x_1, x_2]$.

Given a category $\mathcal{A}$, a **congruence** on $\mathcal{A}$ is family of equivalence relations $\sim_{a_1,a_2}$ on $\mathcal{A}[a_1, a_2]$ where $(a_1, a_2) \in Ob(\mathcal{A}) \times Ob(\mathcal{A})$ such that



Now, we wish to have axioms that have to be satisfied by any reasonable notion of homotopy. To any object X of C, one associates a congruence over $\Gamma(X)$ denoted $\sim_X$, so we have a mapping $\Big(X \in Ob(\mathsf{C}) \longmapsto \sim_X$ a congruence over $\Gamma(X)\Big)$. Finally, we ask the **Homotopy Congruence Property** or **HCP** be satisfied, which means that

$$\forall X, Y \in Ob(\mathsf{C}) \; \forall f \in \mathsf{C}[X, Y] \; \forall x_1, x_2 \in Ob(\Gamma(X)) \; \forall \gamma, \delta \in \Gamma(X)[x_1, x_2]$$
$$\gamma \sim_X \delta \implies f \circ \gamma \sim_Y f \circ \delta.$$

and

$$\forall X \in Ob(\mathsf{C}) \; \forall x \in Ob(\Gamma(X)) \; \forall \gamma, \delta \in \Gamma(X)[x, x],$$
if γ and δ are constant with the same value x then $\gamma \sim_X \delta$

Let us make clear the meaning of the second axiom, by definition of a constant morphism, γ is constant with value p implies that $p \in \mathsf{C}[*, X]$ and then p can be seen as a path since $0 \cdot \mathbb{I} \cong *$. This is the reason why we would like to identify any point p with any constant path with value p. Note that, in this case, $x_1 = x_2 = x$.
Then given an object X of C, we define $\overrightarrow{\pi_1}(\overrightarrow{X}) := \Gamma(X)/_{\sim_X}$ thus defining

the **fundamental category** of X. By HCP, the mapping $X \in Ob(\mathsf{C}) \longmapsto \overrightarrow{\pi_1}(\overrightarrow{X}) \in Ob(\mathsf{Cat})$ induces a functor form C to Cat. Indeed, the HCP makes the following definition sound: Any object X of C is sent to the quotient $\overrightarrow{\pi_1}(X) := \Gamma(X)/_{\sim_X}$. Any morphism $f \in \mathsf{C}[X,Y]$ is sent to the functor $\overrightarrow{\pi_1}(f) \in \mathsf{Cat}\Big[\Gamma(X)/_{\sim_X}, \Gamma(X)/_{\sim_X}\Big]$ which sends any point $(* \to X)$ to the point $f \circ (* \to X)$ and any $\sim_X$-equivalence class $[\, n \cdot \mathbb{I} \xrightarrow{\alpha} X\,]_{\sim_X}$ to the $\sim_Y$-equivalence class $\Big[f \circ (n \cdot \mathbb{I} \xrightarrow{\alpha} X)\Big]_{\sim_Y}$.
While the definition of Γ can been made under very weak hypothesis, the HCP is an extremly strong requirement since it provides a "simultaneous choice" of a congruence for each object of C. In all the "concrete" cases (we will give a formal meaning to "concrete" later) these congruences come from a canonical idea of directed homotopy. A mapping $\Big(X \in Ob(\mathsf{C}) \longmapsto \sim_X$ a congruence on $\Gamma(X)\Big)$ which satisfies the HCP is called a **notion of homotopy** over C.

Remark 2.7 Suppose that $\mathbb{I} \neq *$. Let α be a constant path with value x, γ be a path whose source is x and δ be a path whose end is x then we have

$$\left.\begin{array}{l} x \sim_X \alpha \\ \gamma \sim_X \gamma \end{array}\right\} \implies \gamma = \gamma \cdot x \sim_X \gamma \cdot \alpha \quad \text{and} \quad \begin{array}{l} x \sim_X \alpha \\ \delta \sim_X \delta \end{array} \implies \delta = x \cdot \delta \sim_X \alpha \cdot \delta$$

In other words, the paths $\alpha \in \mathsf{C}[0 \cdot \mathbb{I}, X]$ can be ignored. As a consequence, we can give another definition of the fundamental category of X taking X as the set of objects and $\overrightarrow{\pi}_1(X)[x,y]$ the set of $\sim_X$-equivalence classes $[\gamma]_{\sim_X}$ where $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ with $\underline{n \neq 0}$. This will be useful when we deal with concrete categories.

Proposition 2.8 (Constant paths) *Given an object X of C, a point x of X i.e. $x \in \mathsf{C}[*, X]$ and $n \in \mathbb{N}$, we set c_x^n for the unique morphism of $\mathsf{C}[n \cdot \mathbb{I}, X]$ constant with value x. If $f \in \mathsf{C}[X,Y]$ then $f \circ c_x^n = c_{f \circ x}^n$, if $n, p \in \mathbb{N}$ and x a point of X then $c_x^n \cdot c_x^p = c_x^{n+p}$ and $c_x^{(0)} = id_x$ in $\Gamma(X)$. The relation on paths of X defined by $\alpha \sim_X \beta$ iff there exists a finite sequence $x_n, ..., x_0$ of points of X, where $n \in \mathbb{N}$ and $1 \leq n$, a finite sequence $\gamma_n, ..., \gamma_1$ of paths on X so that for all $k \in \{1, ..., n\}$ the source and the target of γ_k are respectively x_{k-1} and x_k and*

$$\begin{cases} \alpha = t_n \cdot \gamma_n \cdot ...t_1 \cdot \gamma_1 \cdot t_0 \\ \beta = t'_n \cdot \gamma_n \cdot ...t'_1 \cdot \gamma_1 \cdot t'_0 \end{cases}$$

where t_k and t'_k are constant with value x_k for $k \in \{0, ..., n\}$, satisfies the HCP.

The notion of homotopy provided by Proposition 2.8 amounts to "remove the pauses". Note that if all the constant paths $t_0, \ldots, t_n$ are defined on $0 \cdot \mathbb{I} = *$

then $\alpha = t_n \cdot \gamma_n \cdot ... \cdot t_1 \cdot \gamma_1 \cdot t_0 = \gamma_n \cdot ... \cdot \gamma_1$

Proposition 2.9 (Reparametrization) *Given an object X of C, the relation over paths of X defined by $\alpha \sim_X \beta$ iff there exists a finite sequence $x_n, ..., x_0$ of points of X, where $n \in \mathbb{N}$ and $1 \leq n$, a finite sequence $\gamma_n, ..., \gamma_1$ of paths on X so that for all $k \in \{1, ..., n\}$ the begin and the end of γ_k are respectively x_{k-1} and x_k and*

$$\begin{cases} \alpha = t_n \cdot \gamma_n \cdot ... \cdot t_1 \cdot \gamma_1 \cdot t_0 \\ \beta = t'_n \cdot \gamma'_n \cdot ... \cdot t'_1 \cdot \gamma'_1 \cdot t'_0 \end{cases}$$

where for all $k \in \{0, ..., n\}$

(i) *t_k and t'_k are constant with value x_k*

(ii) *$\gamma'_k = \gamma_k \circ \phi_k$ where ϕ_k is an automorphism.*

satisfies the HCP.

Proposition 2.10 (Lattice of notions of homotopy) *The collection of notions of homotopy over the category with paths C whose generic path is $* \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{I}$ is a complete lattice ordered by inclusion. Its least element is the notion of homotopy described in Proposition 2.8 and its greatest one identifies two paths exactly when they have the same source and the same target.*

The two extreme notions of homotopy given by proposition 2.10 are not very interesting and they do not really reflect what we have in mind when we think of homotopy. Up to some additional hypothesis about C, we are able to give many non trivial examples.

3 Topologically concrete catgeories

Almost all the interesting models of concurrency involving topology have objects which are built over topological spaces. We take advantage of the fact to define notions of homotopy that look like the usual one. Inspired by the usual definition of concrete category (see [15] or [17]) we have

Definition 3.1 A **topologically concrete category** is a category C equiped with a faithful functor U whose codomain is a reflective sub-category of Top and which has a left adjoint denoted F. If T is the codomain of U, we say that C is topologically concrete over T.

We recall that, in particular, Haus is a reflective sub-category of Top and thus, PoTop, RTop, LPoTop are examples of topologically concrete categories over Haus. dTop is topologically concrete over Top or Haus depending on the definition of objects of dTop we have chosen. Our aim is to equip a topologically concrete category with a suitable structure of category with paths.

Definition 3.2 [Compatibility] Let C be a topologiquement concrete category with $U \dashv F$. We also suppose that C is equiped with a structure of category with paths $\mathbb{I}'$, s', t' whose distinguished cocones are $(n \cdot \mathbb{I}', i_1'^{(n)}, ..., i_n'^{(n)})$ for $n \in \mathbb{N}$. Finally, suppose that U preserves the structures of a category with paths i.e. T is also equiped with a structure of a category with paths $\mathbb{I}$, s, t whose distinguished cocones are $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ for $n \in \mathbb{N}$ and that

(i) $\forall n \in \mathbb{N}\ U(n \cdot \mathbb{I}') = n \cdot \mathbb{I}$ so in particular $U(\mathbb{I}') = \mathbb{I}$ and $U(*') = *$.

(ii) $U(s') = s$ and $U(t') = t$.

(iii) $\forall n \in \mathbb{N}\ \forall k \in \{1, ..., n\}\ U(i_k'^{(n)}) = i_k^{(n)}$.

We summarize this data by saying that C is a **topologically concrete category with paths** or **TCCP** for short.

Lemma 3.3 $\forall n \in \mathbb{N}\ \forall k \in \{1, ..., n\}\ U(s'^{(n)}) = s^{(n)}$ *and* $U(t'^{(n)}) = t^{(n)}$.

Proof. It suffices to remark that, by definition, $s'^{(n)} = i_1'^{(n)} \circ s'$ and $t'^{(n)} = i_n'^{(n)} \circ t'$. The result follows since $U(s') = s$, $U(t') = t$, $U(i_1'^{(n)}) = i_1^{(n)}$ and $U(i_n'^{(n)}) = i_n^{(n)}$. □

Remark that since U has a left adjoint, U preserve (up to isomorphism) the terminal object of C which is the limit of the empty functor. However, the hypothesis $U(*') = *$ is stronger since it forces this preservation to be strict.

Definition 3.4 Let C be a TCCP (over T), an object D of C is called **domain for dihomotopy** (in C) if $U(D) = [0,1] \times [0,1]$ (Cartesian product in T) and if $\forall (x,y) \leq (x',y') \in [0,1] \times [0,1]\ \exists n \in \mathbb{N}\ \exists \gamma \in \mathsf{C}[n \cdot \mathbb{I}, D]$ such that

$$\Big(U(\gamma)\Big)(0) = (x,y) \text{ and } \Big(U(\gamma)\Big)(1) = (x',y').$$

The we chose a collection $\mathcal{D}$ of domains for dihomotopy whose elements are, by definition, the **acceptable domains for dihomotopy**.

Definition 3.5 Let C be a TCCP (over T). Let X be an object of C. Let $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ and $\delta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ with $n, p \in \mathbb{N}$ i.e. two paths on the object X of C. We call **concrete dihomotopy** in C from γ to δ a morphism $H \in \mathsf{C}[D, X]$, where D is an acceptable domain for dihomotopy in C, such that $U(H)$ be a classical homotopy from $U(\gamma)$ to $U(\delta)$ (with fixed end points).

In fact, definition 3.5 amounts to restrict the collection of homotopies to those which are in the image of U. This limitation is very strong. Also remark that, in general, paths whose domain is $0 \cdot \mathbb{I}$ cannot be "concretely" homotopic to paths whose domain is $n \cdot \mathbb{I}$ for some $n \neq \mathbb{N}$. This is a pathology removed by the remark 2.7.

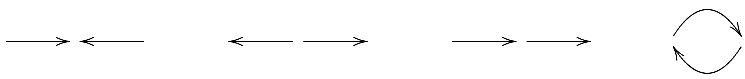
Lemma 3.6 *Let* C *be a TCCP over* T. *If there is a concrete dihomotopy from* γ *to* δ *then* γ *and* δ *have the same source and the same target.*

Proof. Let $\alpha \in \mathsf{C}[n \cdot \mathbb{I}', X]$ and $\beta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ be, then we have $U(\alpha) \in \mathsf{T}[n \cdot \mathbb{I}, U(X)]$ and $U(\beta) \in \mathsf{T}[p \cdot \mathbb{I}, U(X)]$. By hypothesis, we have a classical homotopy from $U(\alpha)$ to $U(\beta)$, hence $(U(\alpha))(0) = (U(\beta))(0)$ i.e. $(U(\alpha)) \circ t^{(n)} = (U(\beta)) \circ s^{(p)}$ or $U(\alpha \circ t'^{(n)}) = U(\alpha \circ s'^{(p)})$ by lemma 3.3. Then, since U is faithful, we have $\alpha \circ t'^{(n)} = \alpha \circ s'^{(p)}$, in other words α and β have the same source. □

Lemma 3.7 *Let C be a TCCP, then U preserves the constant morphisms and U is the functor associated to C with the notation of Definition 3.1.*

Proof. Let $f \in \mathsf{C}[X, Y]$ be constant (i.e. which factorizes in C through the terminal object). So we can write f as $f = f' \circ \zeta_X$ where ζ_X is the unique morphism from X to $*$. As $U(*) = *'$, we have $U(f) = U(f') \circ U(\zeta_X)$ hence $U(f)$ is constant. □

Next result requires the notion of **zigzag** which is defined as follows: Given a graph (V, A, s, t) where V and A are the sets of vertices and arrows of the graph and $\forall a \in A$ $s(a)$ and $t(a)$ are the source and target of a . A zigzag between to vertices v_1 and v_2 is a finite sequence $a_1, \ldots, a_{n-1}$ of arrows such that $v_1 \in \{s(a_1), t(a_1)\}$, $v_2 \in \{s(a_{n-1}), t(a_{n-1})\}$ and $\forall k \in \{1, \ldots, n-1\}$ $\{s(a_k), t(a_k)\} \cap \{s(a_{k+1}), t(a_{k+1})\} \neq \emptyset$. Given two consecutive arrows of a zigzag w_k, w_{k+1} we have one of the four following cases



Clearly, the relation $\{(v_1, v_2) \in V \times V |$ there is a zigzag between v_1 and v_2 $\}$ is an equivalence relation on V. In what follows, the vertices of the graph are paths and the arrows are the concrete dihomotopies.

Proposition 3.8 *Let C be a TCCP. We write dihomotopy for concrete dihomotopy in C. We suppose that for every object X of C we have the following properties*

(i) *(Left identities) For all $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ where $n \neq 0$ and $\alpha \in \mathsf{C}[p \cdot \mathbb{I}', X]$ such that α is constant with value $\gamma \circ t^{(n)}$ there is a zigzag of dihomotopies between $\alpha \cdot \gamma$ and γ.*

(ii) *(Right identities) For all $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ where $n \neq 0$ and $\alpha \in \mathsf{C}[p \cdot \mathbb{I}', X]$ such that α is constant with value $\gamma \circ s^{(n)}$ there is a zigzag of dihomotopies between $\gamma \cdot \alpha$ and γ.*

(iii) *(Congruence) If there is a dihomotopy from α to α', another one from β to β' such that the source of β is the target of α then there is a zigzag of dihomotopies between $\beta \cdot \alpha$ and $\beta' \cdot \alpha'$.*

(iv) *(Compatibility) If $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ and $\delta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ for $n, p \neq 0$ and $U(\gamma) = U(\delta)$ then such that there is a zigzag of dihomotopies between γ and δ.*

Then the transitive closure of

$$\left\{(\gamma,\delta)\Big/ \textit{ there is a dihomotopy from } \gamma \textit{ to } \delta \textit{ or from } \delta \textit{ to } \gamma \right\}$$

i.e. the relation

$$\left\{(\gamma,\delta)\Big/ \textit{ there is a zigzag of dihomotopies between } \gamma \textit{ and } \delta \right\}$$

defines a notion of dihomotopy over C *i.e. the family* $(\sim_X)_{X\in Ob(\mathsf{C})}$ *satisfies the HCP.*

Proof. Every $\sim_X$ is obviously an equivalence relation. The axiom (iii) implies that it is a congruence. The first part of the HCP is satisfied since every $f \in \mathsf{C}[X,Y]$ and every zigzag of dihomotopies $w_1,\ldots,w_n$ induce a zigzag of dihomotopies $f\circ w_1,\ldots,f\circ w_n$. The axioms (i) and (ii) give the second one.□

Remark 3.9 The axiom (iv) is not required by the proof of Proposition 3.8, it is just a "reasonable" requirement.

It remains to check that the machinery we have developed (proposition 3.8) applies to Top, PoTop, dTop, LPoTop etc.

4 Applications

We give several examples of a category with paths, some of them are concrete but not all. First, we notice that for any category with a terminal object $*$, we have a structure of category with paths setting $\mathbb{I} := *$. Of course, in this case, we also have $s = t$ and for any object X of C, Γ_X is just the discrete category whose objects are the points of X. This structure will be refered to as the trivial one.

4.1 Set

Up to isomorphism, the only non trivial generic path of Set is $\{0,1\}$. Indeed, if P is a set containing at least 3 elements, for all $\{s,t\} \subseteq P$ we have a bijection ϕ from P to P such that $\phi(\{s,t\}) \neq \{s,t\}$. It follows that any generic path on Set has at most two elements. Let us suppose that $\mathbb{I} := \{0,1\}$. Clearly, $n\cdot\mathbb{I} = \{0,\ldots,n\}$. Let X be a set and $a,b \in X$, the paths from a to b are the sequences $x \in X^{\{0,\ldots,n\}}$ such that $x_0 = a$ and $x_n = b$. Concatenation of $x \in X^{\{0,\ldots,n\}}$ followed by $y \in X^{\{0,\ldots,p\}}$ is $z \in X^{\{0,\ldots,n+p-1\}}$ where $z_k = x_k$ if $0 \le k \le n$ and $z_k = y_{k-n-1}$ if $n < n+p-2$. For example $(3,4,5)\cdot(1,2,3) = (1,2,3,4,5)$, 3 is not repeated. The next assertion shows the strength of the HCP: the only notions of homotopies are the extreme ones. It means that if $\sim_X$ is a notion of homotopy then we have either

(i) for every set X and $\forall x \in X^{\{0,\ldots,n\}}$ $\forall y \in X^{\{0,\ldots,p\}}$ $x \sim_X y$ iff $\forall i \in \{0,\ldots,n\}$ $\forall j \in \{0,\ldots,p\}$ $x_i = y_j$

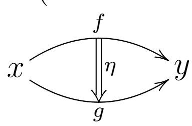
or

(ii) for every set X and $\forall x \in X^{\{0,\ldots,n\}}$ $\forall y \in X^{\{0,\ldots,p\}}$ $x \sim_X y$

4.2 Cat

We choose the generic path $\mathbb{I} := (0 \to 1)$ which can be seen as the poset $\{0 < 1\}$. A point of an object X of Cat is just an object of X. A path on X is just a morphism of X. The only automorphism of $\mathbb{I}$ is the identity hence there is no time reversal. Further, $\Gamma(X)$ is the free category generated by the underlying graph of X. In other words, if U is the forgetful functor from Cat to Grph and F its left adjoint then $\Gamma(X) := F \circ U(X)$. For any small category X, and any paths (i.e. composable sequence of X) $\alpha_n, \ldots, \alpha_0$ and $\beta_p, \ldots, \beta_0$ with the same source and target, put $\alpha_n, \ldots, \alpha_0 \sim_X \beta_p, \ldots, \beta_0$ iff their composites agree in X. This provides a notion of homotopy and the fundamental category of X (i.e. $\Gamma(X)/_{\sim_X}$) is X. Up to ismorphism, $n \cdot \mathbb{I}$ is the poset $\{0 < ... < n\}$. Note that if the generic path is $\{0 \leftrightarrow 1\}$ i.e. the equivalence relation on $\{0,1\}$ that identifies 0 and 1 then we have a time reversal.

4.3 2-Cat

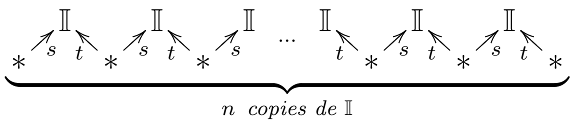
Let us be loose about what a small 2-category is and just say that it is a small category with 2-arrows between arrows (the usual ones) with the same source and target. The idea is pictured by 

Given a small 2-category X, we set $\Gamma_2(X) := \Gamma(UX)$ where Γ is the functor defined in the example of Cat and UX the underlying small category of X (there is an obvious forgetful functor from 2-Cat to Cat). The the congruence $\sim_X$ over $\Gamma(X)$ is generated by the relation that identifies two morphisms $\alpha_n, \ldots, \alpha_0$ and $\beta_p, \ldots, \beta_0$ of $\Gamma(X)$ when there is a 2-morphism from the composite of α to the one of β. Note that if for all morphisms α, β of X, there is a 2-arrow from α to β iff there is a 2-arrow from β to α, then the relation $\sim_X$ is an equivalence relation and we do not need to say "generated by".

4.4 Top *and* Haus

We choose the generic path $\mathbb{I} := [0,1]$, s and t send $* := \{0\}$ to 0 respectively 1. The map $t \in [0,1] \longmapsto (1-t) \in [0,1]$ is time reversal.
In order to obtain a category with paths, we set $0 \cdot \mathbb{I} = \{0\}$, $1 \cdot \mathbb{I} = [0,1]$ and for $n \in \mathbb{N}$ $n \geq 2$, $n \cdot \mathbb{I} := [0,1]$. We also set for $n \in \mathbb{N} \backslash \{0\}$ and $k \in \{1, ..., n\}$ $i_k^{(n)}$: $x \in \mathbb{I} = [0,1] \longmapsto \frac{(k-1)+x}{n} \in n \cdot \mathbb{I} = [0,1]$. Then we have:

Lemma 4.1 *For all $n \in \mathbb{N}$ $(n \cdot \mathbb{I} = \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ is colimit representation of the diagram*



Moreover, given $\gamma_1 : n \cdot \mathbb{I} = [0,1] \longrightarrow X$, $\gamma_2 : m \cdot \mathbb{I} = [0,1] \longrightarrow X$ and $\gamma_3 : p \cdot \mathbb{I} = [0,1] \longrightarrow X$ so that $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$, we have $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1$ where $\cdot$ is the barycentric concatenation. The equality is strict, of course, the definition of $\cdot$ depends on n, m and p.

Proof. We check that

$\forall x \in \left[0, \frac{n}{n+m+p}\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_1\left(\frac{n+m+p}{n}x\right)$

$\forall x \in \left[\frac{n}{n+m+p}, \frac{n+m}{n+m+p}\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_2\left(\frac{n+m+p}{m}x - \frac{n}{m}\right)$

$\forall x \in \left[\frac{n+m}{n+m+p}, 1\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_3\left(\frac{n+m+p}{p}x - \frac{n+m}{p}\right)$

□

Lemma 4.1 provides a structure of category with paths over Top. It is defined for couples (n, γ) where $n \in \mathbb{N}$ and γ a continuous map from $[0,1]$ to X, in other words, with respect to $\cdot$, (n, γ) and (p, γ) are different when $n \neq p$.

Let X be an object of Top, $\Gamma(X)$ is the category of *Moore* paths of X. In particular, we have an isomorphic category setting $n \cdot \mathbb{I} := [0, n]$ and $i_k^{(n)} : x \in [0,1] \longmapsto x + k - 1 \in [0, n]$. The relation $\sim_X$ over $\Gamma(X)$ is the classical homotopy relation, we check that it provides a notion of homotopy and that the fundamental category of X is just its (classical) fundamental groupoid ([13]). The structure of category of paths that we have defined over Top also provides such a structure over Haus.

4.5 PoTop

The generic path is $\mathbb{I} := \overrightarrow{[0,1]}$ the closed unit segment with classical topology and order, s, t are defined as in the example of Top. Any automorphism ϕ of $\mathbb{I}$ satisfies $\phi(0) = 0$ and $\phi(1) = 1$, there is no time reversal. We set $0 \cdot \mathbb{I} = \{0\}$, $1 \cdot \mathbb{I} = \overrightarrow{[0,1]}$ and for $n \in \mathbb{N}$ $n \geq 2$, $n \cdot \mathbb{I} := \overrightarrow{[0,1]}$. We also set for $n \in \mathbb{N} \backslash \{0\}$ and $k \in \{1, ..., n\}$ $i_k^{(n)} : x \in \mathbb{I} = \overrightarrow{[0,1]} \longmapsto \frac{(k-1)+x}{n} \in n \cdot \mathbb{I} = \overrightarrow{[0,1]}$. Moreover, Lemma 4.1 can be adapted to PoTop without changes, providing it with structure of a category with paths.

Moreover, the forgetful functor U : PoTop $\longrightarrow$ Haus is faithful and has a left adjoint since Haus is a reflective sub-category of Top, PoTop is topologically concrete over Haus.

Let $\overrightarrow{X}$ be an object of PoTop. The only acceptable domain for dihomotopy is $\overrightarrow{[0,1]} \times \overrightarrow{[0,1]}$ (see Definition 3.4), we apply proposition 3.8 to have the notion of (concrete) dihomotopy (see definition 3.5). Then, the relation $\sim_X$ we put over $\Gamma(X)$ is the usual notion of dihomotopy. It follows that $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the usual fundamental category of $\overrightarrow{X}$ (see [12], [8], [6] or [5]).

4.6 RTop

A similar construction proves that RTop is a TCCP over Haus. We have an obvious inclusion functor i from PoTop to RTop which satifies $\overrightarrow{\pi_1}(\overrightarrow{X}) = \overrightarrow{\pi}_1(X, \leq_X)$ where the fundamental categories on both sides of the equality are respectively determined in PoTop and RTop (see [11]).

4.7 dTop

As suggested by *Marco Grandis* in [10], we take as generic path

$$\mathbb{I} := ([0,1], \{\text{continuous increasing mappings from } [0,1] \text{ to } [0,1]\})$$

and s, t as in the preceeding examples. We note that there is no time reversal. This category is concrete over Haus (assuming that the underlying topological space of a directed space has to be Hausdorff), the concrete dihomotopy from $\alpha \in dX$ to $\beta \in dX$ is a morphism of $\mathsf{dTop}[\mathbb{I} \times \mathbb{I}, (X, dX)]$ whose underlying map is a classical homotopy from α to β. Proposition 3.8 can be applied: the relation $\sim_X$ over $\Gamma(X)$ that it provides as well as the fundamental category it leads to correspond to the directed homotopy respectively the fundamental category of a directed space defined by *M.Grandis* in [10].
Any pospace $\overrightarrow{X}$ can be seen as a directed space (X, dX) where

$$dX := \mathsf{PoTop}[\overrightarrow{[0,1]}, \overrightarrow{X}] \; ;$$

this remark induces a kind of "inclusion functor" denoted i from PoTop to dTop, with the preceeding notation, we have $\overrightarrow{\pi_1}(\overrightarrow{X}) = \overrightarrow{\pi}_1(X, dX)$ where the fundamental categories on both sides of the equality are respectively determined in PoTop and dTop (see [11]). Moreover the functor i has a left adjoint, the proof of this fact use, as a technical intermediate, the category RTop. More precisely, the inclusion functor from PoTop to RTop has a left adjoint, thus PoTop is a reflective sub-category of RTop and we deduce the cocompleteness of PoTop from the one of RTop, indeed, it is a general fact that any reflective sub-category of a cocomplete category is cocomplete itself (see [2]). Besides, we also have an "inclusion" functor from RTop to dTop applying the same construction as for the "inclusion" of PoTop in RTop. This inclusion also has a left adjoint. We conclude by composing the adjunctions. All the details can be found in [11].

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T-HOMOTOPY AND REFINEMENT OF OBSERVATION (I) : INTRODUCTION

PHILIPPE GAUCHER

Abstract. This paper is the extended introduction of a series of papers about modelling T-homotopy by refinement of observation. The notion of T-homotopy equivalence is discussed. A new one is proposed and its behaviour with respect to other construction in dihomotopy theory is explained.

1. About deformations of HDA

The main feature of the two algebraic topological models of *higher dimensional automata* (or HDA) introduced in [GG03] and in [Gau03] is to provide a framework for modelling continuous deformations of HDA corresponding to subdivision or refinement of observation. Globular complexes and flows are specially designed to modelling the *weak S-homotopy equivalences* (the spatial deformations) and the *T-homotopy equivalences* (the temporal deformations). The first descriptions of spatial deformation and of temporal deformation dates back from the informal and conjectural paper [Gau00].

Let us now explain a little bit what the spatial and temporal deformations consist of before presenting the results. The computer-scientific and geometric explanations of [GG03] must of course be preferred for a deeper understanding.

In dihomotopy theory, processes running concurrently cannot be distinguished by any observation. For instance in Figure 1, each axis of coordinates represents one process and the two processes are running concurrently. The corresponding geometric shape is a full 2-cube. This example corresponds to the flow $\overrightarrow{C}_2$ defined as follows:

- Let us introduce the flow $\partial\overrightarrow{C}_2$ defined by $(\partial\overrightarrow{C}_2)^0 = \{0,1,2,3\}$, $\mathbb{P}_{0,1}\partial\overrightarrow{C}_2 = \{U\}$, $\mathbb{P}_{1,2}\partial\overrightarrow{C}_2 = \{V\}$, $\mathbb{P}_{0,3}\partial\overrightarrow{C}_2 = \{W\}$, $\mathbb{P}_{3,2}\partial\overrightarrow{C}_2 = \{X\}$. The flow $\partial\overrightarrow{C}_2$ corresponds to an empty square, where the execution paths $U * V$ and $W * X$ are *not* running concurrently.
- Then consider the pushout diagram

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^0) & \xrightarrow{q} & \partial\overrightarrow{C}_2 \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^1) & \longrightarrow & \overrightarrow{C}_2 \end{array}$$

1991 *Mathematics Subject Classification.* 55P99, 68Q85.

Key words and phrases. concurrency, homotopy, directed homotopy, model category, refinement of observation, poset, cofibration.

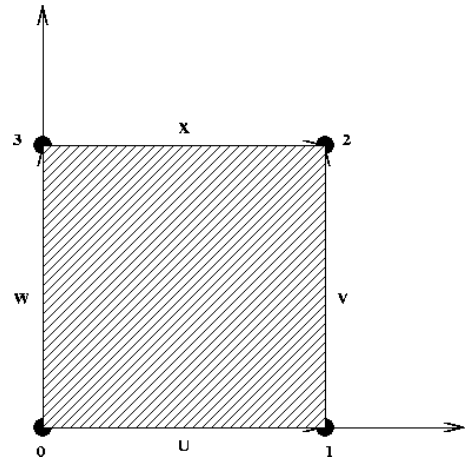

FIGURE 1. Two concurrent processes

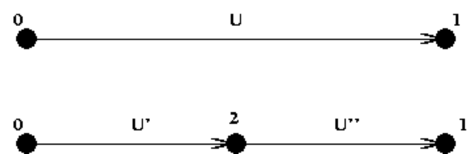

FIGURE 2. The most simple example of T-homotopy equivalence

with $q(\mathbf{S}^0) = \{U * V, W * X\}$ (the globe functor $\mathrm{Glob}(-)$ is defined below). The presence of $\mathrm{Glob}(\mathbf{D}^1)$ creates a *S-homotopy* between the execution paths $U * V$ and $W * X$, modelling this way the concurrency.

It does not matter for $\mathbb{P}_{0,2}\overrightarrow{C}_2$ to be homeomorphic to $\mathbf{D}^1$ or only homotopy equivalent to $\mathbf{D}^1$, or even only weakly homotopy equivalent to $\mathbf{D}^1$. The only thing that matters is that the topological space $\mathbb{P}_{0,2}\overrightarrow{C}_2$ be weakly contractible. Indeed, a hole like in the flow $\partial\overrightarrow{C}_2$ (the space $\mathbb{P}_{0,2}\partial\overrightarrow{C}_2$ is the discrete space $\{U * V, W * X\}$) means that the execution paths $U * V$ and $W * X$ are not running concurrently, and therefore that they are distinguishable by observation. This kind of identification is well taken into account by the notion of weak S-homotopy equivalence. This notion is introduced in [GG03] in the framework of globular complexes, in [Gau03] in the framework of flows and it is proved that these two notions are equivalent in [Gau05a].

In dihomotopy theory, it is also required to obtain descriptions of HDA which are invariant by refinement of observation. The simplest example of refinement of observation is represented in Figure 2, in which the directed segment U is divided in two directed segments U' and U''. This kind of identification is well taken into account by the notion of T-homotopy equivalence. This notion is introduced in [GG03] in the framework of globular complexes, and in [Gau05a] in the framework of flows. The latter paper also proves that the two notions are equivalent. In the case of Figure 2, the T-homotopy equivalence is the unique morphism of flows sending U to $U' * U''$.

Each weak S-homotopy equivalence as well as each T-homotopy equivalence preserves as above the initial states and the final states of a flow. More generally, any good notion of dihomotopy equivalence must preserve the *branching and merging homology theories* introduced in [Gau05c]. This paradigm dates from the beginning of dihomotopy theory: a dihomotopy equivalence must not change the topological configuration of branching and merging areas of execution paths [Gou03]. It is also clear that any good notion of dihomotopy equivalence must preserve the *underlying homotopy type*, that is the topological space, defined only up to weak homotopy equivalence, obtained after removing the time flow. In the case of Figure 1 and Figure 2, this underlying homotopy type is the one of the point.

2. Prerequisites and notations

The initial object (resp. the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp. $\mathbf{1}$).

Let $\mathcal{C}$ be a cocomplete category. If I is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to any morphism of I is denoted by $\mathbf{inj}(I)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of I is denoted by $\mathbf{cell}(I)$. Denote by $\mathbf{cof}(I)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to any morphism of $\mathbf{inj}(I)$. It is a purely categorical fact that $\mathbf{cell}(I) \subset \mathbf{cof}(I)$. Moreover, any morphism of $\mathbf{cof}(I)$ is a retract of a morphism of $\mathbf{cell}(I)$. An element of $\mathbf{cell}(I)$ is called a *relative I-cell complex*. If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative I-cell complex, one says that X is a *I-cell complex*.

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the objects X of $\mathcal{C}$ such that the canonical morphism $\varnothing \longrightarrow X$ is an object of $\mathbf{cell}(I)$. In other terms, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing{\downarrow}\mathcal{C}) \cap \mathbf{cell}(I)$.

Possible references for *model categories* are [Hov99], [Hir03] and [DS95]. The original reference is [Qui67] but Quillen's axiomatization is not used in this paper. The axiomatization from Hovey's book is preferred. If $\mathcal{M}$ is a cofibrantly generated model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$. A cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \longrightarrow \mathbf{cell}(\mathcal{M})$.

A *partially ordered set* $(P, \leqslant)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relation $\leqslant$. A poset is *locally finite* if for any $(x, y) \in P \times P$, the set $[x, y] = \{z \in P, x \leqslant z \leqslant y\}$ is finite. A poset $(P, \leqslant)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P \subset [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. Let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element).

The category $\mathbf{Top}$ of *compactly generated topological spaces* (i.e. of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03]). For the sequel, any topological space will be supposed to be compactly generated. A *compact space* is always Hausdorff.

The time flow of a higher dimensional automaton is encoded in an object called a *flow* [Gau03]. A flow X consists of a set X^0 called the *0-skeleton* and whose elements correspond to the *states* (or *constant execution paths*) of the higher dimensional automaton. For each pair of states $(\alpha, \beta) \in X^0 \times X^0$, there is a topological space $\mathbb{P}_{\alpha,\beta}X$ whose elements correspond to the *(nonconstant) execution paths* of the higher dimensional automaton *beginning at* α and *ending at* β. If $x \in \mathbb{P}_{\alpha,\beta}X$, let $\alpha = s(x)$ and $\beta = t(x)$. For each triple $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, there exists a continuous map $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ called the *composition law* which is supposed to be associative in an obvious sense. The topological space $\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X$ is called the *path space* of X. The category of flows is denoted by **Flow**. A point α of X^0 such that there are no non-constant execution paths ending to α (resp. starting from α) is called an *initial state* (resp. a *final state*). A morphism of flows f from X to Y consists of a set map $f^0 : X^0 \longrightarrow Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ preserving the structure. A flow is therefore “almost” a small category enriched in **Top**.

The category **Flow** is equipped with the unique model structure such that [Gau03]:

- The weak equivalences are the *weak S-homotopy equivalences*, i.e. the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set $I^{gl}_+ = I^{gl} \cup \{R, C\}$ with

$$I^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}$$

where $\mathbf{D}^n$ is the n-dimensional disk, where $\mathbf{S}^{n-1}$ is the $(n-1)$-dimensional sphere, where R and C are the set maps $R : \{0, 1\} \longrightarrow \{0\}$ and $C : \varnothing \longrightarrow \{0\}$ and where for any topological space Z, the flow $\mathrm{Glob}(Z)$ is the flow defined by $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = \widehat{0}$ and $t = \widehat{1}$, and a trivial composition law. The set of generating trivial cofibrations is

$$J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0, 1]), n \geqslant 0\}.$$

3. Why adding new T-homotopy equivalences ?

It turns out that the T-homotopy equivalences, as defined in [Gau05a], are the deformations which locally act like in Figure 2 [1]. So it becomes impossible with this old definition to identify the directed segment of Figure 2 with the full 3-cube of Figure 3 by a zig-zag sequence of weak S-homotopy and of T-homotopy equivalences preserving the initial state and the final state of the 3-cube since any point of the 3-cube is related to three distinct edges (cf. Theorem 3.4). This contradicts the fact that concurrent execution paths cannot be distinguished by observation. More precisely, one has:

Proposition 3.1. *Let X and Y be two flows. There exists a unique structure of flows $X \otimes Y$ on the set $X \times Y$ such that*

1. $(X \otimes Y)^0 = X^0 \times Y^0$
2. $\mathbb{P}(X \otimes Y) = (\mathbb{P}X \times \mathbb{P}Y) \cup (X^0 \times \mathbb{P}Y) \cup (\mathbb{P}X \times Y^0)$
3. $s(x, y) = (s(x), s(y))$, $t(x, y) = (t(x), t(y))$, $(x, y) * (z, t) = (x * z, y * t)$.

Definition 3.2. *The* directed segment $\overrightarrow{I}$ *is the flow* $\mathrm{Glob}(Z)$ *with* $Z = \{u\}$.

[1] This fact was of course not known when [GG03] was being written down. The definition of T-homotopy equivalence presented in that paper was based on the notion of homeomorphism and it sounded so natural...

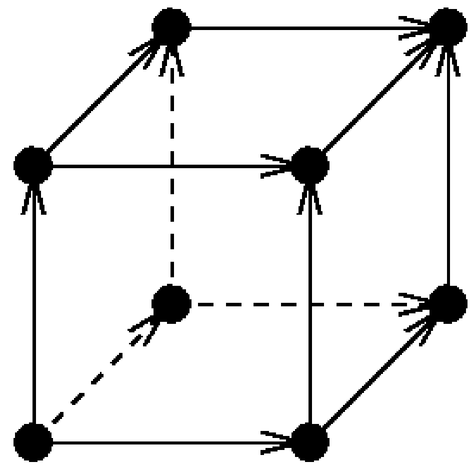

FIGURE 3. The full 3-cube

Definition 3.3. *Let $n \geqslant 1$. The* full n-cube $\overrightarrow{C}_n$ *is by definition the flow $Q(\overrightarrow{I}^{\otimes n})$, where Q is the cofibrant replacement functor.*

Notice that for $n \geqslant 2$, the flow $\overrightarrow{I}^{\otimes n}$ is not cofibrant. Indeed, the composition law contains relations. For instance, with $n = 2$, one has $(\widehat{0}, u) * (u, \widehat{1}) = (u, \widehat{0}) * (\widehat{1}, u)$

Theorem 3.4. *Let $n \geqslant 3$. There does not exist any zig-zag sequence*

$$\overrightarrow{C}_n = X_0 \xrightarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xleftarrow{f_{2n-1}} X_{2n} = \overrightarrow{I}$$

where each X_i is an object of **cell**(**Flow**) *and where each morphism f_i is either a S-homotopy equivalence* [2] *or a T-homotopy equivalence.*

We must suppose in the statement of Theorem 3.4 that each flow X_i belongs to **cell**(**Flow**) because T-homotopy is only defined between this kind of flow.

4. Full directed ball

We need to generalize the notion of subdivision of the directed segment $\overrightarrow{I}$.

Definition 4.1. *A flow X is* loopless *if for every $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.*

A flow X is loopless if and only if the transitive closure of the set $\{(\alpha, \beta) \in X^0 \times X^0$ such that $\mathbb{P}_{\alpha,\beta}X \neq \varnothing\}$ induces a partial ordering on X^0.

Definition 4.2. *A* full directed ball *is a flow $\overrightarrow{D}$ such that:*

- *the* 0*-skeleton $\overrightarrow{D}^0$ is finite*
- *$\overrightarrow{D}$ has exactly one initial state $\widehat{0}$ and one final state $\widehat{1}$ with $\widehat{0} \neq \widehat{1}$*
- *each state α of $\overrightarrow{D}^0$ is between $\widehat{0}$ and $\widehat{1}$, that is there exists an execution path from $\widehat{0}$ to α, and another execution path from α to $\widehat{1}$*

[2] Recall that a morphism between two objects of **cell**(**Flow**) is a weak S-homotopy equivalence if and only if it is a S-homotopy equivalence.

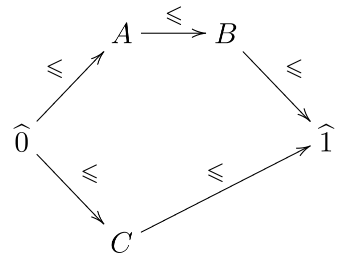

FIGURE 4. Example of finite bounded poset

- *$\overrightarrow{D}$ is loopless*
- *for any $(\alpha, \beta) \in \overrightarrow{D}^0 \times \overrightarrow{D}^0$, the topological space $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}$ is empty or weakly contractible.*

Let $\overrightarrow{D}$ be a full directed ball. Then the set $\overrightarrow{D}^0$ can be viewed as a finite bounded poset. Conversely, if P is a finite bounded poset, let us consider the *flow $F(P)$ associated to P*: it is of course defined as the unique flow (up to isomorphism) $F(P)$ such that $F(P)^0 = P$ and $\mathbb{P}_{\alpha,\beta}F(P) = \{u\}$ if $\alpha < \beta$ and $\mathbb{P}_{\alpha,\beta}F(P) = \varnothing$ otherwise. Then $F(P)$ is a full directed ball and for any full directed ball $\overrightarrow{D}$, the two flows $\overrightarrow{D}$ and $F(\overrightarrow{D}^0)$ are weakly S-homotopy equivalent.

Let $\overrightarrow{E}$ be another full directed ball. Let $f : \overrightarrow{D} \longrightarrow \overrightarrow{E}$ be a morphism of flows preserving the initial and final states. Then f induces a morphism of posets from $\overrightarrow{D}^0$ to $\overrightarrow{E}^0$ such that $f(\min \overrightarrow{D}^0) = \min \overrightarrow{E}^0$ and $f(\max \overrightarrow{D}^0) = \max \overrightarrow{E}^0$. Hence the following definition:

Definition 4.3. *Let $\mathcal{T}$ be the class of morphisms of posets $f : P_1 \longrightarrow P_2$ such that:*

1. *The posets P_1 and P_2 are finite and bounded.*
2. *The morphism of posets $f : P_1 \longrightarrow P_2$ is one-to-one; in particular, if x and y are two elements of P_1 with $x < y$, then $f(x) < f(y)$.*
3. *One has $f(\min P_1) = \min P_2$ and $f(\max P_1) = \max P_2$.*

Then a generalized T-homotopy equivalence *is a morphism of* $\mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$ *where Q is the cofibrant replacement functor of* **Flow**.

In a HDA, a n-transition, that is the concurrent execution of n processes, is represented by the full n-cube $\overrightarrow{C}_n$. The corresponding poset is the product poset $\{\widehat{0} < \widehat{1}\}^n$. In particular, the poset corresponding to the full directed ball of Figure 3 is $\{\widehat{0} < \widehat{1}\}^3 = \{\widehat{0} < \widehat{1}\} \times \{\widehat{0} < \widehat{1}\} \times \{\widehat{0} < \widehat{1}\}$.

The poset corresponding to Figure 1 is the poset $\{\widehat{0} < \widehat{1}\}^2 = \{\widehat{0} < \widehat{1}\} \times \{\widehat{0} < \widehat{1}\}$. If for instance U is subdivided in two processes as in Figure 2, the poset of the full directed ball of Figure 1 becomes equal to $\{\widehat{0} < 2 < \widehat{1}\} \times \{\widehat{0} < \widehat{1}\}$.

One has the isomorphism of flows $\overrightarrow{I}^{\otimes n} \cong F(\{\widehat{0} < \widehat{1}\}^n)$ for every $n \geqslant 1$. The flow $\overrightarrow{C}_n$ $(n \geqslant 1)$ is identified to $\overrightarrow{I}$ by the zig-zag sequence of S-homotopy and generalized T-homotopy equivalences

$$\overrightarrow{I} \xleftarrow{\simeq} Q(\overrightarrow{I}) \xrightarrow{Q(F(g_n))} Q(\overrightarrow{I}^{\otimes n}),$$

where $g_n : \{\widehat{0} < \widehat{1}\} \longrightarrow \{\widehat{0} < \widehat{1}\}^n \in \mathcal{T}$.

5. Is this new definition well-behaved ?

First of all, we must verify that each old T-homotopy equivalence as defined in [Gau05a] will be a particular case of this new definition. And indeed, one has:

Theorem 5.1. *Let X and Y be two objects of* **cell**(**Flow**). *Let $f : X \longrightarrow Y$ be a T-homotopy equivalence as defined in* [Gau05a]. *Then f can be written as a composite $X \longrightarrow Z \longrightarrow Y$ where $g : X \longrightarrow Z$ is a generalized T-homotopy equivalence and where $h : Z \longrightarrow Y$ is a weak S-homotopy equivalence.*

The two other tests consist of verifying that the branching and merging homology theories [Gau05c], as well as the underlying homotopy type functor [Gau05a] are preserved with this new definition of T-homotopy equivalence. And indeed, one has:

Theorem 5.2. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then for any $n \geqslant 0$, the morphisms of abelian groups $H_n^-(f) : H_n^-(X) \longrightarrow H_n^-(Y)$ and $H_n^+(f) : H_n^+(X) \longrightarrow H_n^+(Y)$ are isomorphisms of groups where H_n^- (resp.H_n^+) is the n-th branching (resp. merging) homology group. And the continuous map $|f| : |X| \longrightarrow |Y|$ is a weak homotopy equivalence where $|X|$ denotes the underlying homotopy type of the flow X.*

6. Conclusion

This new definition of T-homotopy equivalence seems to be well-behaved. It will hopefully have a longer lifetime than other ones that the author proposed in the past. It is already known after [Gau05b] that it is impossible to construct a model structure on **Flow** such that the weak equivalences are exactly the weak S-homotopy equivalences and the generalized T-homotopy equivalences. So new models of dihomotopy will be probably necessary to understand the T-homotopy equivalences.

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A Domain-Theoretic Approach to the Causal Structure of Spacetime
(Invited Talk)

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The causal structure of spacetime is a part of a fundamental mathematical model of the spacetimes that appear in general relativity. The causality relation defines a partial order which has been studied by Geroch, Hawking, Kronheimer, Penrose any many others. Recently Sorkin has initiated an approach to quantum gravity that is based on the causality relation as the most basic structure. This raises the question as to whether the causality relation determines other aspects of spacetime, for example, the topology. The techniques of domain theory are perfectly adapted to this study and – I hope – that the GETCO community will find this to be a fertile source of problems.

We prove that a globally hyperbolic spacetime with its causality relation is a bicontinuous poset whose interval topology is the manifold topology. From this one can show that from only a countable dense set of events and the causality relation, it is possible to reconstruct a globally hyperbolic spacetime in a purely order theoretic manner. The ultimate reason for this is that globally hyperbolic spacetimes belong to a category that is equivalent to a special category of domains called interval domains. We obtain a mathematical setting in which one can study causality independently of geometry and differentiable structure, and which also suggests that spacetime emerges from something discrete.

The talk will begin with expository material about spacetime structure and will be easily accessible to anyone who can follow the other talks at GETCO.

This is joint work with Keye Martin who really did almost everything.

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A Convenient Category of Locally Ordered Spaces

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An adaptation of homotopy theory for the "locally ordered" state spaces of concurrent machines promises a more tractable approach to static program analysis, as demonstrated by Fajstrup, Goubault, Raussen, and others. Certain "convenient" categories of spaces allow homotopy theorists to define general constructions on spaces of interest with a minimum of fuss. In this talk, we will define a notion of "locally ordered spaces" broad enough to describe the state spaces of machines found in nature, coherent enough to exhibit some interplay of order and topology, yet robust enough to form a complete, cocomplete, and Cartesian closed category. We compare our notion with others in the literature and discuss potential applications to the "directed" homotopy theory of state spaces.

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Bisimulations of Higher-Dimensional Automata Lift Directed Paths

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Abstract

We introduce notions of simulation and bisimulation up to homotopy, for higher-dimensional automata. We conjecture that our notion of bisimulation is closely related to the notion of history-preserving bisimilarity as introduced for higher-dimensional automata by van Glabbeek. We show that a mapping is a bisimulation if and only if its geometric realisation lifts directed paths up to directed homotopy, thus making the property of bisimilarity susceptible to some machinery from algebraic topology.

1 Introduction

Higher-dimensional automata are a formalism for concurrent systems, introduced by Vaughan Pratt [18] in 1991. They have the specific feature that they can express all higher-order dependencies between the processes in a concurrent system, a capacity which other popular formalisms are lacking. Rob van Glabbeek [12] has recently shown that, measured in terms of expressivity and up to a certain notion of *history-preserving bisimilarity*, higher-dimensional automata are the most general of the main formalisms for concurrent systems.

In our paper [3], we developed a framework for simulations and bisimulations of higher-dimensional automata, building on earlier work of Goubault in [13,14]. We showed that our therein defined notion of bisimulation has an expression as maps of directed topological spaces which lift directed paths. Here we follow up on a concluding remark of [3] and introduce another, weaker kind of (bi)simulation, and we show that these bisimulations are maps of directed topological spaces which lift directed paths *up to directed homotopy*.

The findings reported in this extended abstract are part of the author's Ph.D. research programme. Due to space and time limitations, some details had to be omitted here; these can be found in [4].

This is a preliminary version. The final version will be published in
Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs

2 (Pre)Cubical Sets

A *precubical set* is a graded set $A = \{A_n\}_{n\in\mathbb{N}}$ together with mappings $\delta_i^\nu : A_n \to A_{n-1}$, $i = 1, \dots, n$, $\nu = 0, 1$, satisfying the *precubical identity*

$$\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu \quad (i < j) \tag{1}$$

These are called *face maps*, and we write $a \lhd b$ (a is a *direct face* of b) if $a = \delta_i^\nu b$ for some i, ν. If $\nu = 0$, then a is a *direct lower face* of b, denoted $a \lhd_- b$, and if $\nu = 1$, then a is a *direct upper face* of b, $a \lhd_+ b$. The reflexive, transitive closures of the relations $\lhd$, $\lhd_-$, and $\lhd_+$ are denoted $\lhd^*$, $\lhd_-^*$, respectively $\lhd_+^*$.

A *cubical set* is a precubical set A together with mappings $\varepsilon_i : A_n \to A_{n+1}$, $i = 1, \dots, n+1$, such that

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad (i \le j) \qquad\qquad \delta_i^\nu \varepsilon_j = \begin{cases} \varepsilon_{j-1} \delta_i^\nu & (i < j) \\ \varepsilon_j \delta_{i-1}^\nu & (i > j) \\ \mathrm{id} & (i = j) \end{cases} \tag{2}$$

These are called *degeneracies*, and equations (1) and (2) together form the *cubical identities*.

Morphisms of (pre)cubical sets are required to commute with the structure maps, i.e. if A, B are two (pre)cubical sets, then a morphism $f : A \to B$ is a sequence of mappings $f = \{f_n : A_n \to B_n\}$ that fulfill the first, respectively both, of the equations

$$\delta_i^\nu f_n = f_{n-1} \delta_i^\nu \qquad \varepsilon_i f_n = f_{n+1} \varepsilon_i$$

This defines two categories, pCub and Cub, both of which are presheaf categories over certain small categories of *elementary cubes*, cf. [16,2], hence they are complete and cocomplete. The forgetful functor

$$\mathsf{Cub} \longrightarrow \mathsf{pCub}$$

has a left adjoint, providing us with a "free" functor in the opposite direction which we shall denote F.

We shall henceforth assume any precubical set $A = \{A_n\}$ to satisfy the following:

- $A_n \cap A_m = \emptyset$ for $n \neq m$. Hence any cube a in A has a unique *dimension* $\dim a$ for which $a \in A_{\dim a}$.
- A is *non-selflinked*: Whenever $a \lhd^* b$ in A, then $a = \delta_{j_1}^{\nu_1} \cdots \delta_{j_\ell}^{\nu_\ell} b$ for a unique sequence $(\nu_1, \dots, \nu_\ell)$ and a unique increasing sequence $(j_1, \dots, j_\ell)$.
- A is *geometric*: Whenever $a, b \in A$ have a common face, then there is a unique $c \in A$ such that $c \lhd^* a$, $c \lhd^* b$, and for all $d \in A$ such that $d \lhd^* a$ and $d \lhd^* b$, also $d \lhd^* c$.

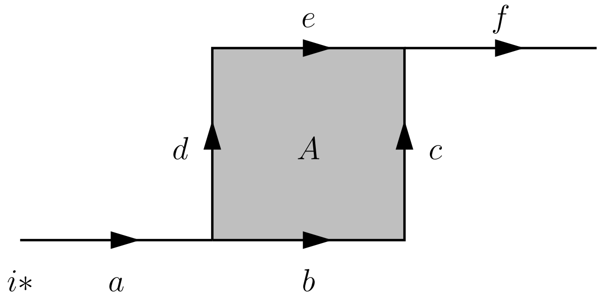


Fig. 1. Three equivalent computations: $(i*, a, b, c, f)$, $(i*, a, A, f)$, $(i*, a, d, e, f)$. Note that for sake of readability, we have omited some 1-cubes in the sequences.

A *non-selflinked* does not contain any *small loops*; any loop consists of at least two different cubes. In the geometric realisation (cf. Section 6) of a *geometric* precubical set, the intersection of two cubes, if nonempty, is a face in both cubes. Indeed, a non-selflinked precubical set A is geometric if and only if the property holds that for any $a, b \in A$ such that $[a] \cap [b] \neq \emptyset$, there exists $c \in A$ such that $[a] \cap [b] = [c]$. This is analogous to a condition one frequently sees required of *simplicial complexes*, cf. [1, Def. IV.21.1].

3 Higher-Dimensional Automata

Higher-dimensional automata are precubical sets with a specified *initial state* (a 0-dimensional cube). For labeling them, we follow [14]:

Let Σ be a finite set of labels, and let $\leq$ be a total order on Σ. Define a precubical set $!\Sigma$ as follows: $!\Sigma_0 = \{*\}$, $!\Sigma_n$ is the set of (not necessarily strictly) increasing sequences of length n of elements of Σ, and

$$\delta^{\alpha}_{i(n)}(x_1, \ldots, x_n) = (x_1, \ldots, \hat{x}_i, \ldots, x_n)$$

Then $!\Sigma$ is a precubical set. Note that different orders on Σ yield isomorphic precubical sets.

A *labeled higher-dimensional automaton* is a precubical diagram

$$* \xrightarrow{i} A \xrightarrow{\lambda} !\Sigma$$

where A is a geometric precubical set, $*$ is the precubical set with only one 0-cube, and $!\Sigma$ is constructed from a finite labeling set Σ as above. The map i determines the initial state, and λ is the *labeling map.*

Computations in higher-dimensional automata are sequences of adjacent cubes, cf. [11]. We change the notion of [11] slightly and call them *rooted cube paths*, see below. Also in [11], *homotopy* of computations is introduced, the idea being that homotopic computations are computationally equivalent. We give a precise definition below; see figure 1 for an example of three equivalent computations.

4 Cube Paths

A *cube path* in a cubical set A is a sequence $(a_1, \dots, a_n)$ of cubes in A such that $a_i \neq a_{i+1}$, and $a_i \lhd_-^* a_{i+1}$ or $a_{i+1} \lhd_+^* a_i$ for all $i = 1, \dots, n-1$. This is a generalisation of the *computation paths* defined in [3,12], in that in a cube path, adjacent cubes are not necessarily direct faces of each other. A cube path as in [12], i.e. where adjacent cubes are direct faces, will be called *full*.

A cube b in A is *reachable from* another cube a if there exists a cube path starting in a and ending in b. A *rooted* cube path in a higher-dimensional automaton $* \xrightarrow{i} A \to !\Sigma$ is a cube path $(a_1, \dots, a_n)$ in A such that $a_1 = i*$, and a cube b in A is *reachable* if it is reachable from $i*$.

A cube path $(b_1, \dots, b_m)$ is a *filler* of another cube path $(a_1, \dots, a_n)$, denoted $(a_1, \dots, a_n) \preceq (b_1, \dots, b_m)$, if

- the sequence $(a_1, \dots, a_n)$ is a subsequence of $(b_1, \dots, b_m)$,
- $a_1 = b_1$ and $a_n = b_m$, and
- for each b_j there are a_i and a_k such that $a_i \lhd^* b_j \lhd^* a_k$.

Two *full* cube paths $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ are *adjacent* if $n = m$ and there is at most one $i \in \{1, \dots, n\}$ for which $a_i \neq b_i$. *Homotopy* of full cube paths in A, denoted by the symbol $\sim$, is the transitive closure of the adjacency relation. See also [12].

Two *general* cube paths $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ are homotopic if they can be filled in to full cube paths which are homotopic. Note that it does not matter in what way we fill in the cube paths; any two fillers of a cube path are homotopic. We shall denote homotopy classes of cube paths by $[a_1, \dots, a_n]$.

The *universal covering* of a pointed precubical set $i : * \to A$ consists of a pointed precubical set $\tilde{i} : * \to \tilde{A}$ and a morphism $\pi_A : \tilde{A} \to A$ defined as follows:

$$
\begin{aligned}
\tilde{A}_n &= \{[a_1, \dots, a_k] \mid (a_1, \dots, a_k) \text{ cube path in } A, a_1 = i*, a_k \in A_n\} \\
\tilde{\delta}_i^1[a_1, \dots, a_k] &= [a_1, \dots, a_k, \delta_i^1 a_k] \\
\tilde{\delta}_i^0[a_1, \dots, a_k] &= \{(b_1, \dots, b_\ell) \mid b_\ell = \delta_i^0 a_k \\
&\qquad \text{and } (b_1, \dots, b_\ell, a_k) \sim (a_1, \dots, a_k)\} \\
\tilde{i}* &= [i*] \\
\pi_A[a_1, \dots, a_k] &= a_k
\end{aligned}
$$

The morphism $\pi_A : \tilde{A} \to A$ is called the *covering map*. It requires a proof that $\tilde{A}$ is indeed a precubical set, see [4].

5 Simulations and Bisimulations

In [3], we defined notions of simulation and bisimulation for higher-dimensional automata. Here we generalise these notions, also taking into account homotopy of computations. We call them h-simulation and h-bisimulation, where the "h" stands for "homotopy."

An *h-simulation* of labeled higher-dimensional automata $\langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$, $\langle * \to B \xrightarrow{\mu} !\Xi\rangle$ consists of *cubical* morphisms $f : \langle * \to A\rangle \to \langle * \to B\rangle$, $\sigma : !\Sigma \to !\Xi$ such that that for any rooted cube path $(a_1, \dots, a_n)$ in A with $\dim a_n = 0$, there exists a rooted cube path $(b_1, \dots, b_n) \sim \big(f(a_1), \dots, f(a_n)\big)$ in B such that $\mu(b_i) = \sigma(\lambda(a_i))$ for all $i = 1, \dots, n$.

We need *cubical* morphisms above to be able to map "real" transitions to *idle* transitions, cf. [19]. So to be precise, f and σ are cubical morphisms between the cubical diagrams freely generated by the precubical diagrams $\langle * \to A \to !\Sigma\rangle$, $\langle * \to B \to !\Xi\rangle$. For *bisimulations* however, we do not want to map real to idle transitions, so in that case, we are back to precubical morphisms:

An *h-bisimulation* of labeled higher-dimensional automata $\langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$, $\langle * \to B \xrightarrow{\mu} !\Xi\rangle$ is an h-simulation (f, id), where f is a *pre*cubical morphism, and such that for any rooted cube path $(a_1, \dots, a_n)$ in A and for any cube path $(b_1, \dots, b_m)$ in B with $b_1 = f(a_n)$ and $\dim b_m = 0$, there exists a cube path $(a_n, \dots, a_\ell)$ in A such that $\big(f(a_1), \dots, f(a_\ell)\big) \sim \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big)$.

Following [17], we say that two higher-dimensional automata $\langle * \to A \to !\Sigma\rangle$, $\langle * \to B \to !\Sigma\rangle$ are *h-bisimilar* if there exists a third higher-dimensional automaton $\langle * \to C \to !\Sigma\rangle$ and a span of h-bisimulation maps

$$\langle * \to A \to !\Sigma\rangle \longleftarrow \langle * \to C \to !\Sigma\rangle \longrightarrow \langle * \to B \to !\Sigma\rangle$$

Proposition 5.1 *A precubical h-simulation* $(f, \mathrm{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle$ *is an h-bisimulation if and only if* $\tilde f : \tilde A \to \tilde B$ *has the property that for any* $\tilde a \in \tilde A$, *and for any* $\tilde b \in \tilde B_0$ *which is reachable from* $\tilde f(\tilde a)$, *there exists* $\tilde c \in \tilde A_0$ *reachable from* $\tilde a$ *such that* $\tilde b = \tilde f(\tilde c)$.

The *unfolding*, cf. [11], of a higher-dimensional automaton $\langle * \xrightarrow{i} A \xrightarrow{\lambda} !\Sigma\rangle$ is the automaton $\langle * \xrightarrow{\tilde i} \tilde A \xrightarrow{\tilde\lambda} !\tilde\Sigma\rangle$, where $\tilde A$, $!\tilde\Sigma$ are the universal coverings of $i : * \to A$ respectively $\lambda \circ i : * \to !\Sigma$, and $\tilde\lambda[a_1, \dots, a_n] = [\lambda a_1, \dots, \lambda a_n]$. This construction is analogous to the unfolding of a transition system to a synchronisation tree, cf. [19]; indeed, the universal covering of a precubical set is what should be the proper generalisation of a *tree* to higher dimensions, see [4].

Corollary 5.2 *Any higher-dimensional automaton* $\langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$ *is h-bisimilar to a relabeling of its unfolding via the h-bisimulation map*

$$(\pi_A, \mathrm{id}) : \langle * \to \tilde A \xrightarrow{\pi_{!\Sigma} \circ \tilde\lambda} !\Sigma\rangle \longrightarrow \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$$

Conjecture 5.3 *The notion of* h-bisimilarity *defined above is equivalent to the notion of* history-preserving bisimilarity *from [12].*

6 Geometric Realisation

The *geometric realisation* of a geometric precubical set A is the *directed space* [15] (indeed, a *local po-space* [10])

$$|A| = \bigsqcup_{n\in\mathbb{N}} A_n \times \vec{I}^n / \equiv$$

where $\vec{I}$ denotes the standard directed unit interval, and the equivalence relation $\equiv$ is induced by identifying

$$(\delta_i^\nu a; t_1, \dots, t_{n-1}) \equiv (a; t_1, \dots, t_{i-1}, \nu, t_i, \dots, t_{n-1})$$

for all $a \in A_n$, $n \in \mathbb{N}$, $i = 1, \dots, n$, $\nu = 0, 1$, $t_i \in I$. Geometric realisation is turned into a functor by mapping $f : A \to B \in \mathsf{pCub}$ to the *directed map* $|f| : |A| \to |B|$ defined by

$$|f|(a; t_1, \dots, t_n) = (f(a); t_1, \dots, t_n)$$

The *image* of a cube $a \in A_n$ is the closed set

$$[a] = \{(a, t_1, \dots, t_n) \mid 0 \le t_j \le 1 \text{ for all } j\} \subseteq |A|$$

the *interior image* of a is the set

$$]a[= \{(a, t_1, \dots, t_n) \mid 0 < t_j < 1 \text{ for all } j\} \subseteq |A|$$

and the *carrier* $\operatorname{carr} x$ of a point $x \in |A|$ is the unique cube $a \in A$ such that $x \in]a[$.

It can be shown, cf. [8], that the geometric realisation of the unfolding of a pointed precubical set $i : * \to A$ is the universal directed covering space of $|A|$ with respect to $[i*]$, cf. [5].

A *d-path*, or *directed path*, $p : \vec{I} \to |A|$ is a continuous function for which there exists a partition $0 = s_1 < \dots < s_n = 1$ and cubes $a_1, \dots, a_{n-1}$ in A such that $\operatorname{im} p|_{[s_i, s_{i+1}]} \subseteq a_i$ and $p(s) \le_{a_i} p(t)$ whenever $s_i \le s \le t \le s_{i+1}$, where $\le_{a_i}$ is the standard partial order on the unit cube $[a_i] \approx \vec{I}^{\dim a_i}$.

Lemma 6.1 (cf. [7]) *Given a d-path $p : \vec{I} \to |A|$ in the geometric realisation of a precubical set A, there exists a partition of the unit interval $0 = s_1 \le \dots \le s_{n+1} = 1$ and a unique cube path $(a_1, \dots, a_n)$ in A such that $\operatorname{carr} p(s_i) \in \{a_{i-1}, a_i\}$, and $\operatorname{carr} p(s) = a_i$ for $s_i < s < s_{i+1}$.*

The *carrier sequence* $\operatorname{carrs} p$ of a d-path $p : \vec{I} \to |A|$ is the unique cube path given by the lemma.

Two d-paths $p, q : \vec{I} \to |A|$ are *d-homotopic*, denoted $p \sim q$, if there exists a continuous mapping $H : I^2 \to |A|$ such that $H(s, \cdot)$ is a d-path in $|A|$ for all $s \in I$, $H(s, 0) = H(0, 0)$ and $H(s, 1) = H(0, 1)$ for all $s \in I$, and $H(0, \cdot) = p$,

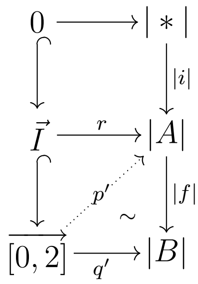
$H(1, \cdot) = q$. Technically, what we have defined here is *dihomotopy*, not *d-homotopy*, but by a result in [7], these agree in the geometric realisations of precubical sets.

Lemma 6.2 (cf. [7]) *Given d-paths* $p, q : \vec{I} \to |A|$ *such that* $\operatorname{carr} p(0) = \operatorname{carr} q(0) \in A_0$ *and* $\operatorname{carr} p(1) = \operatorname{carr} q(1) \in A_0$*, then* $p \sim q$ *if and only if* $\operatorname{carrs} p \sim \operatorname{carrs} q$*.*

Theorem 6.3 *A precubical h-simulation* $(f, \mathrm{id}) : \langle * \xrightarrow{i} A \to !\Sigma\rangle \to \langle * \xrightarrow{j} B \to !\Sigma\rangle$ *is an h-bisimulation map if and only if one of the following equivalent properties holds:*

- *For any d-path* $r : \vec{I} \to |A|$ *with* $r(0) = [i*]$ *and any d-path* $q : \vec{I} \to |B|$ *with* $q(0) = |f|(r(1))$ *and* $\operatorname{carr} q(1) \in B_0$*, there exists a d-path* $p : \vec{I} \to |A|$ *such that* $p(0) = r(1)$ *and* $|f| \circ (r * p) \sim (|f| \circ r) * q$*.*
- *For any d-path* $r : \vec{I} \to |\tilde{A}|$ *with* $r(0) = [\tilde{i}*]$ *and any d-path* $q : \vec{I} \to |\tilde{B}|$ *with* $q(0) = |\tilde{f}|(r(1))$ *and* $\operatorname{carr} q(1) \in \tilde{B}_0$*, there exists a d-path* $p : \vec{I} \to |\tilde{A}|$ *such that* $p(0) = r(1)$ *and* $|\tilde{f}|(p(1)) = q(1)$*.*

Note that equivalence of the two above properties also follows from $|\tilde{A}|$ being the universal directed covering space of A. Also, if we let $q' = (f \circ r) * q$ and $p' = r * p$ be the concatenations, we can encode the first property in the diagram



where the lower triangle commutes up to d-homotopy. That is, $|f|$ lifts d-paths up to d-homotopy.

7 Conclusion and Future Work

We have in this article introduced a notion of h-bisimulation for higher-dimensional automata which appears to be closely related to van Glabbeek's [12] *history-preserving bisimilarity*. We have shown that h-bisimulation has an interpretation as a dipath-lifting property of morphisms, making the problem of deciding bisimilarity susceptible to some machinery from algebraic topology.

In topological language, a dipath-lifting morphism is a weak kind of *homotopy fibration*, hinting that fibrations (well-studied in algebraic topology) could have applications, as well. This also suggests that a general theory of directed fibrations should be developed.

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