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BRICS NS-01-7 Cousot et al. (eds.): GETCO '01 Preliminary Proceedings

Preliminary Proceedings of the Workshop on

Geometry and Topology in Concurrency Theory

GETCO '01

Aalborg, Denmark, August 25, 2001

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BRICS Notes Series NS-01-7

ISSN 0909-3206 August 2001

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GEometry and Topology in COncurrency theory

Preliminary Proceedings of GETCO'2001
Satellite Workshop of Concur'2001
Aalborg, Denmark, August 25, 2001

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Preliminary print under the patronage of
BRICS: Basic Research in Computer Science
Centre of the National Danish Research Foundation

Programme

9:00 - 9:10	Opening
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10:30 - 10:50	COFFEE BREAK
10:50 - 12:10	*Tutorial:* Topology and Directed Topology by Lisbeth Fajstrup and Martin Raussen
12:15 - 12:30	*Discussion*
12:30 - 14:00	LUNCH
14:00 - 14:35	Investigating the Algebraic Structure of Dihomotopy Types by Philippe Gaucher
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Foreword

The main mathematical disciplines that have been used in theoretical computer science are discrete mathematics (especially, graph theory and ordered structures), logics (mostly proof theory for all kinds of logics, classical, intuitionistic, modal etc.) and category theory (cartesian closed categories, topoi etc.). General Topology has also been used for instance in denotational semantics, with relations to ordered structures in particular.

Recently, ideas and notions from mainstream "geometric" topology and algebraic topology have entered the scene in Concurrency Theory and Distributed Systems Theory (some of them based on older ideas). They have been applied in particular to problems dealing with coordination of multi-processor and distributed systems. Among those are techniques borrowed from algebraic and geometric topology: Simplicial techniques have led to new theoretical bounds for coordination problems. Higher dimensional automata have been modelled as cubical complexes with a partial order reflecting the time flows, and their homotopy properties allow to reason about a system's global behaviour.

This workshop aims at bringing together researchers from both the mathematical (geometry, topology, algebraic topology etc.) and computer scientific side (concurrency theorists, semanticians, researchers in distributed systems etc.) with an active interest in these or related developments.

It follows two workshops on the subject "Geometric and Topological Methods in Concurrency Theory" which have been held in Aalborg, Denmark, in June 1999 and at Penn State University as a satellite to CONCUR 2000.

The Workshop has been financially supported by the Basic Research Institute in Computer Science (Aarhus, Denmark), and I thank this institution for this, and more specifically Uffe Engberg. I also wish to thank the referees, the authors and the programme committee members for their very precise and timely job. Many thanks are also due to Michael Mislove who kindly supported the workshop by letting us submit the papers through the Electronic Notes in Theoretical Computer Science. Last but not least, I wish to thank the Concur organizers, Anna Ingolfsdottir, Luca Aceto and Arne Skou, and the Workshop coordinator, Hans Hüttel, for making this possible.

Eric Goubault, the 5'th of July 2001.

(Di)topology with applications to concurrency. A tutorial

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1 Introduction

1.1 Topology in concurrency?

From a general perspective, concurrency theory is using many mathematical tools. Predominant are the use of graph theory (often labeled directed graphs) and of logics. Topology has also played a role. Many people talk about the topology of networks meaning nothing else than the graph determined by the connections in the netwok. General (or set-theoretic) topology has been applied in, e.g., fixed point theory, and systematically in connection with lattice theory in domain theory (work of D. Scott and al.; see [7] for a classical reference).

We shall proceed in a different direction: We want to give evidence for that also classical *algebraic topology* (with roots in mainly *geometric* problems) has a capacity of modelling concurrent processes and interesting phenomena attached to them – after a "twist".

1.2 Example: Progress graphs

The first "algebraic topological" seems to be that of a progress graph and has appeared in operating systems theory, in particular for describing the problem of "deadly embrace"[1] in "multiprogramming systems". Progress graphs are introduced in [1], but attributed there to E. W. Dijkstra. In fact they also appeared slightly earlier (for editorial reasons it seems) in [11].

The basic idea is to give a description of what can happen when several processes are modifying shared ressources. Given a shared resource a, we see it as its associated semaphore that rules its behaviour with respect to processes. For instance, if a is an ordinary shared variable, it is customary to use its semaphore to ensure that only one process at a time can write on it (this is mutual exclusion). Then, given n deterministic

[1]as E. W. Dijkstra originally put it in [2], now more usually called deadlock.

sequential processes $Q_1 \dots, Q_n$, abstracted as a sequence of locks and unlocks on shared objects, $Q_i = R_1 a_i^1.R_2 a_i^2 \dots R_n^i a_i^{n_i}$ (R_k being P or V)[2], there is a natural way to understand the possible behaviours of their concurrent execution, by associating to each process a coordinate line in $\mathbf{R}^n$. The state of the system corresponds to a point in $\mathbf{R}^n$, whose ith coordinate describes the state (or "local time") of the ith processor.

1.2.1 Example

Consider a system with finitely many processes running altogether. We assume that each process starts at (local time) 0 and finishes at (local time) 1; the P and V actions correspond to sequences of real numbers between 0 and 1, which reflect the order of the P's and V's. The initial state is $(0, ..., 0)$ and the final state is $(1, ..., 1)$. An example consisting of the two processes $T_1 = P_a.P_b.V_b.V_a$ and $T_2 = P_b.P_a.V_a.V_b$ gives rise to the two dimensional progress graph of Fig. 1.2.1.

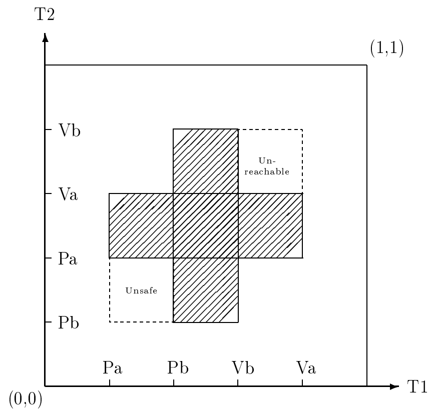

Figure 1: Example of a progress graph

The shaded area represents states which are not allowed in any execution path, since they correspond to mutual exclusion. Such states constitute the *forbidden region*. An execution path in an n-dimensional progress graph in the unit square in $\mathbf{R}^n$ is a path from the initial state $(0, ..., 0)$ to the final state $(1, ..., 1)$ avoiding the forbidden region and increasing in each coordinate – time cannot run backwards. We call these paths directed paths or *dipaths*. This entails that paths reaching the states in the dashed square underneath the forbidden region, marked "unsafe" are deemed to deadlock, i.e., they cannot possibly reach the allowed terminal state which is $(1, 1)$ in dimension 2. Similarly, by reversing the direction of time, the states in the square above the forbidden region, marked "unreachable", cannot be reached from the initial state, which is $(0, 0)$ here. Also notice

[2] Using E.W. Dijkstra's notation P and V [2] for respectively acquiring and releasing a lock on a semaphore.

that all terminating paths above the forbidden region are "equivalent" in some sense, given that they are all characterized by the fact that T_2 gets a and b before T_1 (as far as resources are concerned, we call this a schedule). Similarly, all paths below the forbidden region are characterized by the fact that T_1 gets a and b before T_2 does.

1.3 Directed homotopy

In this picture, one can already recognize many ingredients that are at the center of the main problem of algebraic topology, namely the classification of shapes modulo "elastic deformation". As a matter of fact, the actual coordinates that are chosen for representing the times at which Ps and Vs occur are unimportant, and these can be "stretched" (preserving the order on the axes) in any manner, so the properties (deadlocks, schedules etc.) are invariant under some notion of deformation. A deformation (e.g. of paths) is called a *homotopy* in topology. Since directions (partial orders) are essential, we have to insist on that those are preserved under deformations. We call such an order preserving deformation of paths a directed homotopy or dihomotopy. Already for 2-dimensional progress graphs, this yields a concept different from the classical one: Consider for instance the two homeomorphic shapes (deformable into each other by an elastic deformation) with two holes in Fig. 2 and Fig. 3. In Fig. 2, there are four essentially different dipaths up to dihomotopy (i.e. four schedules corresponding to all possibilities of accesses of resources a and b) whereas in Fig. 3, there are only three dipaths up to dihomotopy.

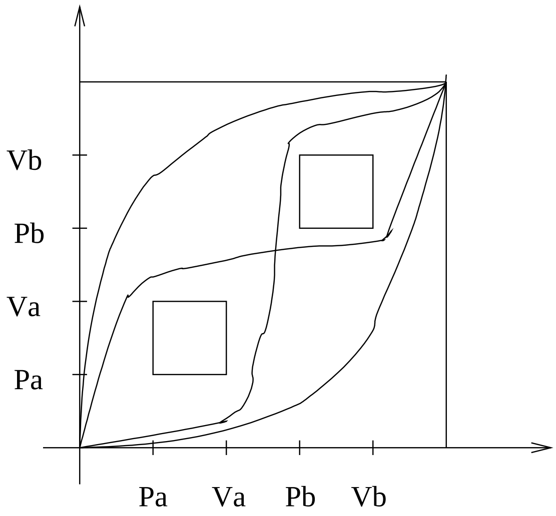

Figure 2: $P_a.V_a.P_b.V_b|P_a.V_a.P_b.V_b$

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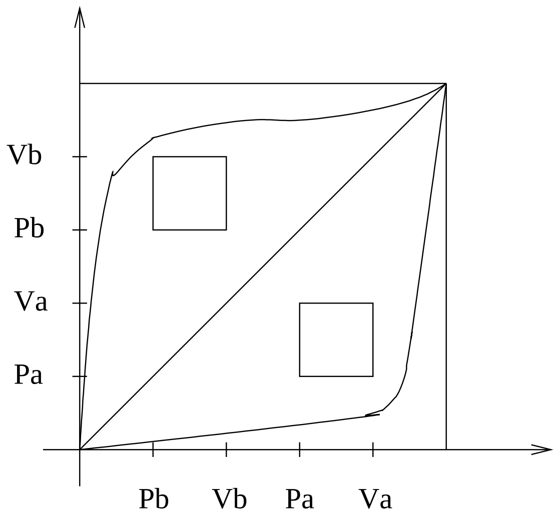

Figure 3: $P_b.V_b.P_a.V_a|P_a.V_a.P_b.V_b$

2 A short Tutorial in Topology

In this chapter, we touch upon central notions, methods and results from algebraic topology that have been applied or modified with applications in concurrency in mind – or where there should be a potential to do so in future work. Of course, these pages cannot replace a book; proofs are mainly omitted. There are lots of books at all levels on Algebraic Topology on the market; a nice one [9] is available on the internet.

2.1 Topological Spaces

Topological spaces are generalizations of metric spaces. They model *"nearness"* in more abstract situations. An axiomatic formulation makes use of *open subsets*. In a metric space X with distance function d, a subset $U \subseteq X$ is open if, for every $x \in X$ there is a positive real number $\varepsilon > 0$, such that $U_\varepsilon(x) = \{y \in X | \; d(x, y) < \varepsilon\} \subseteq U$.

Definition 2.1 *1. A topological space is a pair* $(X, \mathcal{U})$ *with* $\mathcal{U} \subseteq 2^X$ *a system of (open) subsets such that*

(a) $X, \emptyset \in \mathcal{U}$*;*

(b) Any union of open sets is open.

(c) Any finite *intersection of open sets is open.*

2. A subset $A \subseteq X$ *is* closed *if and only if its complement* $X \setminus A$ *is open.*

3. *Two points $x, y \in X$ can be* separated *if there are open sets $U_x, U_y \in \mathcal{U}$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.*

4. *A topological space such that any pair of points $x \neq y \in X$ can be separated is called* Hausdorff.

Example 2.2 *1. A metric space is Hausdorff.*

2. *A strange topology on $X = \mathbf{R}^2$ is given by: $U \subseteq X$ is open if and only if for every $(x, y) \in U$ there is an $\varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\times \mathbf{R} \subseteq U$. This is a topological space in which two points on the same vertical line* cannot *be separated.*

3. *Many computer scientists are familiar with the* Scott topology*.*

Maps between topological spaces that preserve nearness are called continuous. They are generalizations of the continuous maps between metric spaces mapping points "sufficiently close" to each other into points close to each other. A neat formulation is:

Definition 2.3 *A map $f : X \to Y$ between two topological spaces X and Y is* continuous *if and only if $f^{-1}(U) \subseteq X$ is open for every open set $U \subseteq Y$.*

Example 2.4 *Let $X = \mathbf{R}^2$ be endowed with the topology from Ex. 2.2 and $Y = \mathbf{R}^2$ endowed with the (standard) topology inherited from the standard metric. The identity map $id : X \to Y$ is* not *continuous, whereas the identity map $id : Y \to X$ is continuous.*

Definition 2.5 *1. A map $f : X \to Y$ between two topological spaces X and Y is called a* homeomorphism *if it is a bijection and if both f and its inverse $f^{-1} : Y \to X$ are continuous.*

2. *Two topological spaces are called* homeomorphic *if and only if there exists a homeomorphism $f : X \to Y$.*

Example 2.6 *1. The open interval $]0, 1[$ is homeomorphic to the real half-line $]0, \infty[$ (both with standard topology inherited from the standard metric). A homeomorphism is given by the map $f :]0, 1[\to]0, \infty[$, $f(x) = \frac{x}{1-x}$ with $f^{-1}(y) = \frac{y}{1+y}$. In particular, a bounded and a non-bounded space can be homeomorphic.*

2. *A 2-dimensional sphere (boundary of a 3-dimensional ball) is homeomorphic to an ellipsoid, but not to a torus (doughnut).*

3. *The two topologies on $\mathbf{R}^2$ from Ex. 2.2.2 give rise to non-homeomorphic spaces. It is easy to see that a space homeomorphic to a Hausdorff space has to be Hausdorff again.*

Homeomorphy is an equivalence relation. From the topological point of view, one should not discriminate between two homeomorphic topological spaces from each other.

2.2 Paths

Let $I = [0,1]$ denote the unit interval with standard metric and topology, and let X denote a topological space. Any continuous map $\alpha : I \to X$ is called a *path* in X.

How can one compose paths? In general this is not possible. But if the endpoint $\alpha_1(1)$ of α_1 agrees with the start point $\alpha_2(0)$ of α_2, their *concatenation* $\alpha_1 * \alpha_2 : I \to X$ is defined by $(\alpha_1 * \alpha_2)(s) = \begin{cases} \alpha_1(2s), & t \leq \frac{1}{2} \\ \alpha_2(2s-1), & t \geq \frac{1}{2}. \end{cases}$ (Both paths are pursued with "double speed"). Concatenation defines a (non-commutative, non-associative) monoidal structure on the path space $\mathcal{P}(X)$ of all paths on X. (OBS: Not all elements of $\mathcal{P}(X)$ can be composed with each other).

Definition 2.7 *A topological space X is called* path-connected *if and only if, for every pair of elements $x_0, x_1 \in X$, there exists a path $\alpha : I \to X$ with $\alpha(0) = x_0$ and $\alpha(1) = x_1$.*

2.3 Homotopy

What is a path in the space of maps between two topological spaces X and Y? Let again I denote the unit interval.

Definition 2.8

1. *A* homotopy *is a family $H_t : X \to Y$, $t \in I$ of maps, such that the associated map $H : X \times I \to Y$ is continuous.*
2. *Two continuous maps $f, g : X \to Y$ are homotopic if and only if there is a homotopy $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.*

Example 2.9

1. *Let $S^1 = \{(x,y) | x^2 + y^2 = 1\} \subset \mathbf{R}^2$ denote the unit circle. The map $H : S^1 \times I \to \mathbf{R}^2$, $H((x,y),t) = (tx, ty)$ is a homotopy between the constant map and the inclusion of the unit circle into $\mathbf{R}^2$.*
2. *There is no homotopy between the inclusion $i : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$ in the pointed plane and any constant map $c : S^1 \to \mathbf{R}^2 \setminus \{(0,0)\}$.*

Homeomorphy is still a quite fine relation between topological spaces. It is in general quite difficult to find algebraic counterparts to help with a classification of certain spaces up to homeomorphism. The following relation is coarser and often easier to handle algebraically:

Definition 2.10

1. *A continuous map $f : X \to Y$ is called a* homotopy equivalence *if there are a continuous map $g : Y \to X$ and two homotopies between $g \circ f : X \to X$ and id_X, resp. $f \circ g : Y \to Y$ and id_Y.*
2. *Two spaces X and Y are called* homotopy equivalent *if and only if there is a homotopy equivalence $f : X \to Y$.*

Example 2.11 *1. The spaces $X = S^1$ and $Y = \mathbf{R}^2 \setminus \{(0,0)\}$ are homotopy equivalent (though of different dimension) via the inclusion map $i : X \to Y$ and the "contraction" $c : Y \to X$ with $c(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. In fact, $c \circ i = id_X$; the map $H : Y \times I \to Y$, $H((x,y),t) = (1-t)(x,y) + t(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ defines a homotopy between $id_Y(t=0)$ and $i \circ c(t=1)$.*

2. The spaces $Z = \mathbf{R}^2$ and Y (from above) are not *homotopy equivalent, as will be shown in Sect. 2.5*

2.4 The fundamental group

2.4.1 Definitions

We shall now introduce the first algebraic construction associating to a topological space X a *group*. We shall make use of (some of the) paths considered in Sect. 2.2 "up to" a specific type of homotopy. More specifically: Let X denote a topological space, and let $x_0 \in X$ denote an (arbitrarily chosen) *basepoint.*

Definition 2.12 *1. A path $\alpha : I \to X$ is called a loop with basepoint x_0 if $\alpha(0) = \alpha(1) = x_0$. The set of loops with basepoint x_0 is denoted $\mathcal{P}_1(X; x_0)$.*

2. Concatenation defines a binary operation $C : \mathcal{P}_1(X; x_0) \times \mathcal{P}_1(X; x_0) \to \mathcal{P}_1(X; x_0)$.

3. A homotopy *of loops at x_0 is a family of loops $H_t : I \to X$ at x_0 such that the associated map $H : I \times I \to X$, $H(x,t) = H_t(x)$ is continuous.*

4. Two loops α and β at x_0 are homotopic *if there exists a homotopy H_t of loops with $H_0 = \alpha$ and $H_1 = \beta$. In that case, we write: $\alpha \simeq \beta$.*

It is essential that every path in the homotopy is a loop, i.e., that $H_t(0) = H_t(1)$ for all $t \in I$. Moreover, loops with the same basepoint can always be concatenated.

Example 2.13 *1. Let $X = \mathbf{R}^n$ and $x_0 \in \mathbf{R}^n$ any base point. Any two loops α, β at x_0 are homotopic via the linear homotopy $H_t = (1-t)\alpha + t\beta$. The same result holds for a* convex *subset of $\mathbf{R}^n$, and even for a subset X that is* star-shaped *with respect to $x_0 \in X$, i.e., containing the line segment between x_0 and every $y \in X$.*

2. The same argument does not work for $Y_n = \mathbf{R}^n \setminus \{\mathbf{0}\}$. It turns out that two loops in Y_n are always homotopic for $n > 2$, but not always for $n = 2$.

3. A reparametrization *of a path (loop) α in X is a composition $\beta = \alpha \circ \varphi$ where φ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$. Essentially, a reparametrization of α is a loop with the same base point running along the same trace as α, but possibly at another "speed".*

A loop α in X and every reparametrization $\beta = \alpha \circ \varphi$ are homotopic; a homotopy is given by $H_t(s) = \alpha((1-t)\varphi(s) + ts)$.

Proposition 2.14 *1. The homotopy relation on paths with fixed basepoint defines an equivalence relation. The set of equivalence classes is denoted* $\pi_1(X;x_0)$*.*

2. Concatenation factors over the homotopy relation and thus defines a binary operation. $C : \pi_1(X;x_0) \times \pi_1(X;x_0) \to \pi_1(X;x_0)$*. We write* $[\alpha] * [\beta]$ *for* $C([\alpha],[\beta])$*.*

3. $\pi_1(X;x_0)$ *with the operation* $*$ *is a group.*

Proof. (Sketch)

1. *Reflexivity*: Homotopy constant in t. *Symmetry*: $\tilde{H}(t) = H(1-t)$. *Transitivity*: Concatenation of two homotopies H^1 and H^2 "in the parameter t": $H_t = \begin{cases} H^1(2t) & t \leq \frac{1}{2} \\ H^2(2t-1) & t \geq \frac{1}{2}. \end{cases}$
2. Concatenation of two homotopies H^1 and H^2 "in the parameter s": $H_t = H_t^1 * H_t^2$.
3. *Associativity:* $\alpha_1 * (\alpha_2 * \alpha_3)$ is a reparametrization of $(\alpha_1 * \alpha_2) * \alpha_3$. Use Ex. 2.13.3. Concatenation of any loop α at x_0 with the constant loop c (with $c(s) = x_0$ for all $s \in I$) yields a reparametrization of α; hence $[c]$ is a *two-sided identity* in $\pi_1(X;x_0)$. The inverse path to a path in X is defined by $\bar{\alpha}(s) = \alpha(1-s)$. The path $\alpha^t(s) = \alpha(ts)$ runs from $\alpha(0)$ to $\alpha(t)$. For every $t \in I$, the concatenation $\alpha^t * \overline{\alpha^t}$ is a loop at x_0. Altogether, these maps define a homotopy of loops between $\alpha * \bar{\alpha}$ and $c = \alpha^0 * \overline{\alpha^0}$. Replacing α with $\bar{\alpha}$ yields a homotopy between $\bar{\alpha} * \alpha$ and c, i.e., $[\bar{\alpha}]$ is inverse to $[\alpha]$ in $\pi_1(X;x_0)$.

□

Example 2.15 *1.* $\pi_1(\mathbf{R}^n, x_0)$ *is the (one-element) trivial group.*

2. The fundamental group of a circle is isomorphic to the integers. (To a loop on the circle, you may associate its winding number counting the total number of – directed – turns around the circle.)
The fundamental group of a higher-idmensional sphere is trivial.

3. The fundamental group of a space is in general not *commutative. The simplest example of a space with non-commutative fundamental group consisting of two circles with one common point. It turns out that the fundamental group of this space (with the common point as base point) is a* free group on two generators, *cf. Ex. 2.23.2.*

The definition of the fundamental group depends on the base point. But it is easy to see, that fundamental groups corresponding to two points x_0, x_1 in the space X are isomorphic, if there exists a path β from x_0 to x_1. A concrete isomorphism is given by $[\alpha] \to [\beta * \alpha * \beta^{-1}]$.

Remark 2.16 *The geometric shapes under consideration are usually uncountable, and so is the set of loops through a given point. The homotopy relation has two important effects: it reduces the cardinality to something typically discrete (finite or at most countable) and it imposes an algebraic (group) structure.*

2.4.2 Induced homomorphisms

A continuous map $f : X \to Y$ induces a map $f_\# : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ between the associated fundamental groups. The definition is easy: Associate to a loop α in X the loop $f \circ \alpha$ in Y; this map factors over the homotopy relation. Moreover, $f_\#$ is a *group homomorphism.*

Example 2.17 *Let $f : S^1 \to S^1$ denote the circle self-map, that "doubles angles", i.e., $f(\exp(it)) = \exp(2it)$. The winding number of the loop $f \circ \alpha$ is twice the winding number of the loop α on S^1. Hence, $f_\# : \mathbf{Z} \cong \pi_1(S^1, 1) \to \pi_1(S^1, 1) \cong \mathbf{Z}$ corresponds to multiplication with 2.*

The following two properties of induced homomorphism are easy to derive, but essential:

1. Let $f_1, f_2 : X \to Y$ denote *homotopic*[3] maps from X to Y. Then, the induced maps $f_{j\#} : \pi_1(X; x_0) \to \pi_1(Y; f(x_0))$ coincide.

   Corollary 2.18 *Homotopy equivalent spaces have isomorphic fundamental groups.*

2. Let $g : Y \to Z$ denote another continuous map inducing the homomorphism $g_\# : \pi_1(Y; f(x_0)) \to \pi_1(Z; g(f(x_0)))$. The composite map $g \circ f : X \to Z$ induces the homomorphism $(g \circ f)_\# : \pi_1(X; x_0) \to \pi_1(Z; g(f(x_0)))$.

   Lemma 2.19 *The homomorphisms $(g \circ f)_\# = g_\# \circ f_\# : \pi_1(X; x_0) \to \pi_1(Z; g(f(x_0)))$ coincide.*

Generally speaking, we have the first example of a functor ("translator") that allows to associate to continuous geometric objects and their relations (topological spaces and continuous maps) discrete algebraic counterparts. The aim is to allow geometric conclusions based on properties of these algebaic images.

2.5 Functoriality: an example

The following is to serve as an example how the translation mechanims from topology to algebra can serve to yield non-trivial topological reuslts. Let $B^n := \{\mathbf{x} \in \mathbf{R}^n | ||x|| \leq 1\}$ denote an n-dimensional ball, and $S^{n-1} = \partial B^n = \{\mathbf{x} \in \mathbf{R}^n | \; ||x|| = 1\}$ denote an $(n-1)$-dimensional sphere.

Theorem 2.20 ***(Brouwer's fixed point theorem)*** *Every continuous self-map $f : B^n \to B^n$ has a fixed point $x_0 \in B^n$ $(f(x_0) = x_0)$.*

[3] The homotopy has to preserve base points.

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A proof for this theorem is elementary for $n = 1$. In that case, the continuous map $g : [0,1] \to \mathbf{R}$, $g(x) = f(x) - x$ has the number 0 amongst its values since $g(-1) \geq 0$ and $g(1) \leq 0$. For $n > 1$, it is a consequence of the following

Lemma 2.21 *There is no continuous map $F : B^n \to S^{n-1}$ extending the identity on S^{n-1}.*

Proof. The proof given here applies only to $n = 2$. For a proof in higher dimensions, one needs higher homotopy or homology groups cf. e.g. [9]):

Let $i : S^{n-1} \to B^n$ denote the continuous inclusion map. A map F as in the lemma would satisfy: $F \circ i = id$, the identity map on S^{n-1}. On the fundamental groups level (choose $x_0 \in S^{n-1}$), this amounts to

$$id_\# : \pi_1(S^{n-1}; x_0) = F_\# \circ i_\# : \pi_1(S^{n-1}; x_0) \to \pi_1(B^n; x_0) \to \pi_1(S^{n-1}; x_0).$$

Since B^n is convex (homotopy equivalent to a one-point space), we have $\pi_1(B^n; x_0) = 0$, and thus $id_\#$ has to be the zero-map. On the other hand, $id_\#$ is the identity map on $\pi_1(S^{n-1}; x_0)$. This yieldss a contradiction for $n = 2$, where $\pi_1(S^1; x_0) \cong \mathbf{Z}$: $id_{\mathbf{Z}} \neq 0$. $\square$

Proof. of Brouwer's fixed point theorem. Assume there is a continuous map $f : B^n \to B^n$ *without fixed point*. Then, one can construct a continuous map $F : B^n \to S^{n-1}$ by associating to x the intersection of the half-line starting at $f(x)$ through x with S^{n-1} (can be described by a formula using the solution of a quadratic equation and is thus continuous). Obviously, F restricts to the identity map on O^{n-1}. The existence of F contradicts Lemma 2.21. $\square$

The general idea is, that the (highly structured) discrete structure corresponding to a continuous structue is often easier to overlook than the original. Most often, the methods gives rise to *impossibility* results. In some cases, *existence* of objects or maps can be unveiled algebraically; this requires a proof that the vanishing of an algebraic obstruction is not only necessary, but indeed *sufficient* for the construction.

2.6 Compositions: The van Kampen theorem

The calculation of fundamental groups and of induced homomorphisms is difficult in general. One of the methods is a calculation "by recurrence", i.e., determining the fundamental group of a space by considering fundamental groups of subspaces and of relations between those. We look at the simplest case only:

Let $A_1, A_2 \subset X$ denote subsets each containing the base point x_0. Let $i_j : A_j \to X$, $i_{12} : A_1 \cap A_2 \to A_1$ and $i_{21} : A_1 \cap A_2 \to A_2$ denote the inclusion maps. They satisfy: $i_1 \circ i_{12} = i_2 \circ i_{21} : A_1 \cap A_2 \to X$, and the obvious relations can be seen from the diagrams

$$\begin{tikzcd} A_1 \cap A_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(A_1 \cap A_2; x_0) \arrow[r] \arrow[d] & \pi_1(A_1; x_0) \arrow[d] \\ \pi_1(A_2; x_0) \arrow[r] & \pi_1(X; x_0) \end{tikzcd}$$

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From the fundamental groups of the pieces A_j, one can construct the free group $\pi_1(A_1; x_0) * \pi_1(A_2; x_0)$ generated by the two fundamental groups. It consists of all words in the two "alphabets". It contains the *normal subgroup* N generated by all words of type $i_{12}(\alpha)i_{21}(\alpha^{-1})$ with $\alpha \in \pi_1(A_1 \cap A_2; x_0)$.

Theorem 2.22 ***(van Kampen theorem)*** *Let $A_1, A_2 \subset X$ denote* path-connected *(cf. Def. 2.7)* open *subsets with* path-connected intersection $A_1 \cap A_2$. *The fundamental group $pi_1(X; x_0)$ is then isomorphic to the quotient group of $\pi_1(A_1; x_0) * \pi_1(A_2; x_0)$ by the normal group N described above.*

A more categorial way to phrase van Kampen's theorem is as follows: The push-out diagram of spaces on the left-hand side of the diagram above is translated into a push-out diagram of groups on the right-hand side of that diagram.

Remark 2.23

1. *It is essential that the intersection is path-connected, as well. The van Kampen theorem does thus* not *apply to the calculation of the fundamental group of the circle from the (trivial) fundamental groups of two half-circles (well, a bit more than a half to ensure openness of the pieces). The intersection consists of two "intervals" that cannot be connected by a path.*
   On the other hand, the theorem shows that the fundamental group of an n-sphere S^n is trivial for $n > 1$: An n-sphere can be described as the union of two half-spheres, that are homeomorphic to n-dimensional balls with trivial fundamental groups. Their intersection is homotopy equivalent to an $(n-1)$-dimensional sphere, which is path-connected for $n > 1$.

2. *The fundamental group of the "one point union" of two subspaces (in which the base point has a neighborhood that is contractable, i.e., homotopy equivalent to a 1-point space) is the free product of the fundamental group of the subspaces.*

2.7 Further topics

Higher homotopy groups Definition. Abelian groups. Difficult to determine. Results on spheres.

Particular topological spaces Simplicial complexes. CW-complexes. Approximation.

Simplicial homology Definition. Induced maps.

Singular homology Definition. Induced maps. Naturality. Homotopy invariance.

Mayer-Vietoris Homology of unions and intersections. Long exact sequence.

Functoriality Brouwer. Euclidean spaces up to homeomorphism.

3 A tutorial in ditopology

3.1 Introduction

Ditopology is not yet a well-established disciplin. It presents our attempt to apply *methodology* from classical topology to the study of concurrency. The main difference compared to classical topology is, that we have to work with spaces and maps with an extra structure given by a *(local) partial order*. In the applications, the partial order reflects the time flow for the processors involved in the concurrent system under consideration.

Hence, we have to rephrase parts of the classical curriculum in topology in a category of partially ordered spaces and maps between them. The term *ditopology* (directed topology) was coined for this situation. It turns out, that this rephrasing is not just a dull exercise, and that the partial orders force you to invent notions that seem necessary for progressing with the applications – sometimes with help from neighbouring disciplines like, e.g., relativity theory.

Algebraic topology has been highly successful in deriving results about geometric structures that are *robust under deformations*. The key ingredient is very often an algebraization of the geometric structures to be considered and the use of *functoriality*, cf. Sect. 2.5. The introduction to [12] is a very readable account of our dream how this methodology might be applied in concurrency theory; moreover, it gives an elementary example (non-existence of a simulation), in which this dream actually works out.

We have to admit from the very beginning, that ditopology is not at all as advanced as classical topology is. The foundational definitions are still under debate, only few general results or calculations are achieved so far. Nevertheless, the few tools and results have shown to be useful in several applications; to mention

- An algorithm detecting deadlocks and associated safe/unsafe regions for concurrent systems generalising the progress graphs studied in the introduction [3, 4, 6];
- A topological underpinning of the result "2-phase locking is safe" used as a data engineering approach to ensure serialisability of procotcols for distributed databases [8, 5].

3.2 (Local) po-spaces

We start with elementary definitions and properties of po-spaces, cf. e.g. [7]:

Definition 3.1 *1. A* partial order $\leq$ *on a set* U *is a reflexive, transitive and antisymmetric relation. We wite* $x < y$ *for* ($x \leq y$ *and* $x \neq y$)*.*

2. A partial order $\leq$ *on a topological space* X *is called* closed *if* $\leq$ *is a closed subset (cf. Def. 2.1.2) of* $X \times X$ *in the product topology. If* $\leq$ *is closed, we call* $(X, \leq)$ *a* po-space*.*

Remark 3.2 *Let* $(X, \leq)$ *denote a po-space.*

1. *For every $x \in X$, the sets $\downarrow x = \{y \in X | y \le x\}$ and $\uparrow x = \{y \in X | y \ge x\}$ are closed.*

2. *For every pair of points $y_1, y_2 \in X$, the set $[y_1, y_2] = \{x \in X |\, y_1 \le x \le y_2\} = \downarrow y_2 \cap \uparrow y_1$ is closed.*

3. *A po-space is Hausdorff[7].*

Example 3.3 *The progress graph Γ of a concurrent system modelling mutual exclusion from Sect. 1.2 can be considered as a po-space as follows: $\mathbf{R}^n$ is equipped with the partial order*

$$(x_1, \ldots, x_n) \le (y_1, \ldots, y_n) \Leftrightarrow \ \forall 1 \le i \le n : \ x_i \le y_i.$$

The progress graph $\Gamma \subset \mathbf{R}^n$ inherits the partial order as a subspace.

A loop cannot be given a consistent partial order: anti-symmetry will always be violated. But locally, "within the loop", there is still an order between the steps. We have thus to generalise our framework to include situations where a partial order only can be established *locally*:

Definition 3.4 *Let X be a topological space. A collection $\mathcal{U}(X)$ of pairs $(U, \le_U)$ with partially ordered open subsets U covering X is a* local partial order *on X if for every $x \in X$ there is a nonempty open neighbourhood $W(x) \subset X$ such that the restrictions of $\le_U$ to $W(x)$ coincide for all $U \in \mathcal{U}(X)$ with $x \in U$, i.e.,*

$$y \le_{U_1} z \iff y \le_{U_2} z \qquad \text{for all } U_1, U_2 \in \mathcal{U}(X) \text{ such that } x \in U_i \text{ and for all } y, z \in W(x) \cap U_1 \cap U_2$$

A neighbourhood $W(x)$ with a well-determined partial order as above is called a po-neighbourhood *of x.*

Example 3.5 *The circle $S^1 = \{e^{i\theta} \in \mathbf{C}\}$ has a local partial order: the open subsets*

$$U_1 = \{e^{i\theta} \in S^1 |\, 0 < \theta < \frac{3\pi}{2}\} \quad \text{and} \quad U_2 = \{e^{i\theta} \in S^1 |\, \pi < \theta < \frac{5\pi}{2}\}$$

are (partially) ordered by the order on the θ's. Notice that the relation on S^1 generated by these local partial orders by taking the transitive closure is of no use: it is the trivial relation: $x \le y$ for any pair of elements x, $y \in S^1$.

Remark 3.6

1. *In the applications, only processes without loops can be moddelled by a partially ordered space. Processes allowing loops have to be modelled by locally partially ordered spaces.*

2. *It is necessary to define when two coverings by partially ordered subspaces define the* same *local partial order, cf. [5].*

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3.3 Dimaps and Dipaths

Looking back at our example on progress graphs, we observe that executions correspond to paths (defined on a closed interval I with the usual order) in the partially ordered space *preserving that partial order*. A generalisation of this concept is as follows:

Definition 3.7 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be locally partially ordered spaces. A continuous map $f : X \to Y$ is called a* dimap *(directed map) if for any $x \in X$ there are po-neighborhoods $W(x)$ and $W(f(x))$ such that*

$$x_1 \leq_{W(x)} x_2 \Rightarrow f(x_1) \leq_{W(f(x))} f(x_2) \text{ whenever } x_1, x_2 \in f^{-1}(W(f(x))) \cap W(x)$$

It is not hard to see, that this definition does not depend on the choice of representative $\mathcal{U}$ of the equivalence class of local po-structures (cf. Rem. 3.6.2). In the case of po-spaces (not just local ones), dimaps are the same as monotone continuous maps. It is straightforward to see that local po-spaces and dimaps form a category.

A *dipath* is a dimap defined on either the unit interval I (relevant for paths in compact po-spaces; this is the approach used e.g. in [5, 10]), or, the half-line $\mathbf{R}_{\geq 0} := \{t \in \mathbf{R} | t \geq 0\}$ (relevant for paths in local po- spaces) – both with the usual order as the partial order relation $\leq$ on the domain. An execution where one process loops infinitely often corresponds to the exponential map $\varphi : \mathbf{R}_{\geq 0} \to S^1,\ \varphi(t) = \exp(2\pi i t)$ considered as a dipath; two processors looping infinitely many times can be modelled by a dipath into the 2-torus of type $\psi : \mathbf{R}_{\geq 0} \to T = S^1 \times S^1,\ \psi(t) = (\varphi(mt), \varphi(nt)),\ m, n > 0$.

Definition 3.8 *Let X denote a local po-space.*

1. *A* dipath *in X is a dimap $\alpha : \mathbf{R}_{\geq 0} \to X$.*

2. *We call α* finite *it there is a real number $T > 0$ such that αs restriction to $[T, \infty[$ is constant.*

3. *We call a dipath β in X an* extension *of a dipath α in X if there is a real number $T > 0$ and a surjective dimap $\varphi : [0, T[\to \mathbf{R}_{\geq 0}$ such that the diagram*

$$\begin{tikzcd}
{[0,T[} \arrow[r, "\varphi"] \arrow[d, "\subset"'] & \mathbf{R}_{\geq 0} \arrow[d, "\alpha"] \\
\mathbf{R}_{\geq 0} \arrow[r, "\beta"] & X
\end{tikzcd}$$

   commutes and such that βs restriction to $[T, \infty[$ is non-constant.

4. *A dipath $\alpha : \mathbf{R}_{\geq 0} \to X$ is called* inextendible *if it does not admit any extension $\beta : \mathbf{R}_{\geq 0} \to X$.*

5. *A new* local *partial order $\prec$ on X is defined as follows: $x \prec y \Leftrightarrow$ there is a finite dipath from x to y.*

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6. For $X_0, X_1 \subset X$, we define the dipath spaces

$$\vec{P}_1(X; X_0, X_1) = \{\alpha : \mathbf{R}_{\geq 0} \to X \text{ finite} \mid \alpha(0) \in X_0,\ \alpha(T) \in X_1 \text{ for large } T\} \text{ and}$$

$$\vec{P}_1(X; X_0, \infty) = \{\alpha : \mathbf{R}_{\geq 0} \to X \text{ inextendible} \mid \alpha(0) \in X_0\}.$$

Example 3.9 *1. In a finite mutual exclusion model (with PV semantics) the state space is $X = I^n \setminus int(F)$, the complement of the interior of the forbidden region F in a cube. $X_0 = \{\mathbf{0}\}$ consists of the initial point; X_1 will typically either contain only the final point $\mathbf{1}$ or be a (finite) set of (deadlock – cf. Def. 3.11) points. For such a compact po-space, it is a bit artificial to consider dipaths defined on $\mathbf{R}_{\geq 0}$; dipaths defined on a closed interval I give rise to an equivalent notion.*

2. Let $X = S^1$ denote a circle, or more general, $X = (S^1)^n$ denote an n-torus (with the product local partial order) modelling concurrent loops. In that case, the interesting dipaths are the non-finite ones. If a forbidden region is removed from X, finite dipaths ending in a deadlock arise naturally, as well.

We would like to have a clean definition for dimaps that send inextendible dipaths to inextendible dipaths in order to imitate the set-up of homotopy alluded to in Sect. 2. This is work in progress.

3.4 Dihomotopy

State spaces for concurrent systems tend to have an enormous (but finite) size. The main idea with the ditopology approach is to replace the finite state space by a continuous higher-dimensional (infinite) one, and then to impose relevant equivalent relations on the associated space of dipaths (and, as a result, on the state space itself), yielding classification patterns that apply to the original state space. As an effect, the number of *essentially different* states can usually be reduced drastically.

The relevant equivalence relation is given by a special type of homotopy:

Definition 3.10 *Let X denote a local po-space with subspaces $X_0, X_1 \subset X$. A continuous family $H_t : \mathbf{R}_{\geq 0} \to X$ of dipaths (giving rise to a homotopy $H : \mathbf{R}_{\geq 0} \times I \to X$) is called*

1. *a* dihomotopy *from X_0 to X_1, if every map $H_t \in \vec{P}_1(X; X_0, X_1)$ is a finite dipath from X_0 to X_1.*

2. *an* inextendible dihomotopy *from X_0 if every map $H_t \in \vec{P}_1(X; X_0, \infty)$ is an inextendible dipath from X_0. .*

These notions give rise to equivalence relations on the path spaces. Their quotient sets are denoted by $\vec{\pi}_1(X; X_0, X_1)$, resp. $\vec{\pi}_1(X; X_0, \infty)$.

Why is there any relation between dihomotopy and concurrency? In fact, dihomotopy of dipaths corresponds to the *commutativity* of local actions. Consider the following basic example: There are (essentially, i.e., up to reparametrization) two dipaths in the *boundary of a rectangle* from the "bottom" edge to the "top" edge. As dipaths in the

boundary they are not dihomotopic (even not homotopic with end points fixed);

filled-in rectangle they are dihomotopic (connect them linearly).

At least in dimension two, it is quite convincing, that execution paths in the mutual exclusion models discussed in the introduction yield equivalent results if they are dihomotopic (and that you can invent situations where they yield inequivalent results, if not). A theoretical classification of dipaths up to dihomotopy in 2-dimensionial mutual exclusion models and an algorithm determining the (finite) set $\vec{P}_1(X;\mathbf{0},\mathbf{1})$ is described in [10].

The dihomotopy notion is certainly even more interesting and more promising – but also more involved in higher dimensions. Let us again consider the basic example, dipaths from the bottom to the top on the *boundary of a 3-dimensional cube.* This boundary models a piece of shared memory, that two, but not three processes can access and manipulate in a commutative way. In this simple case, it is elementary to see that any execution is equivalent to a serial one, and that all serial ones are equivalent – corresponding to the fact, that all dipaths in the model are dihomotopic to each other.

The following example of a space consisting of a cube from which 3 forbidden "bars" are removed (cf. Fig. 4), is a bit more involved and probably already quite difficult to analyse combinatorially: It describes 3 concurrent processes that access 3 shared objects,

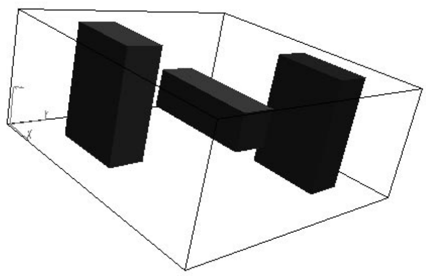

Figure 4: Room with 3 barriers

two of which can only handle one of them at any given time while the "middle" one can handle access of two of them in parallel. In this case, there exist five essentially different schedules corresponding to dihomotopy classes of dipaths. Two of those dipaths are in fact homotopic (with end points fixed), but *not di*homotopic. An example in which the schedules corresponding to those may lead to different results of a concurrent calculation is given in [5].

Which algebraic structures should one consider on top of the dihomotopy set $\vec{\pi}_1$? This question is not quite settled yet. The most promising so far is that of a partial order in non-published work of S. Sokołowski.

3.5 Deadlocks, unsafe and unreachable regions

We survey a fully-developped fast algorithm detecting *deadlocks, unsafe* and *unreachable regions* for mutual exclusion models. Though it does not use the general framework for local po-spaces nor the notion of dihomotopy, it was conceived in the same geometrical spirit. Details can be found in [3, 4].

3.5.1 Definitions

In the applications, a *deadlock* is a state in which the system under consideration is blocked, i.e., there is no execution leaving that particular state. The associated *unsafe region* is the set of states that are bound to be blocked in that deadlock somewhere in the future. If executions are modelled by dipaths in (local) po-spaces, both notions (and their relatives) have counterparts with nice and clear definitions:

Definition 3.11 *1. An element $x \in X$ with $\uparrow x = \{x\}$ is called a deadlock. The set of all deadlocks in X is denoted by $\mathcal{D}(X)$. (Sometimes, a particular final state is exempted from $\mathcal{D}(X)$).*

2. The unsafe region *$Uns(X; X_1) = U(X; X, X_1)$ consists of all $x \in X$ that* cannot be connected *to any point in X_1 by a dipath, i.e.,*

$$Uns(X; X_1) = \{x \in X | \vec{P}_1(X; x, X_1) = \emptyset\} = X \setminus (\downarrow X_1) = \{x \in X | (\uparrow x) \cap X_1 = \emptyset\}.$$

3. The unreachable region *$Unr(X; X_0) = U(X; X_0, X)$ consists of all $x \in X$ that* cannot be reached *from any point in X_0 by a dipath, i.e.,*

$$Unr(X; X_0) = \{x \in X | \vec{P}_1(X; X_0, x) = \emptyset\} = X \setminus (\uparrow X_0) = \{x \in X | (\downarrow x) \cap X_0 = \emptyset\}.$$

Remark 3.12 *1. The symbols $\uparrow$ and $\downarrow$ above have to be interpreted with respect to the partial order $\prec$ from Def. 3.8.*

2. Consider the po-space associated to a PV-program discussed in Sect. 1.2 with final state $\mathbf{1}$. Then $Uns(X; \mathbf{1})$ corresponds exactly to the unsafe region of those states that can only reach a deadlock (different from $\mathbf{1}$).

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3.5.2 Detection of deadlocks and unsafe areas for mutual exclusion models

Unsafe and unreachable regions can be algorithmically determined in the PV model [3, 4] and we recap here the basic idea of the algorithm.

Suppose the semantics of a PV program is given in terms of a forbidden region $F \subset I^n$ in a hypercube containing forbidden *hyperrectangles* $R^i = \prod_{j=1}^n [a_j^i, b_j^i] \subset I^n$ (with $n \geq 2$). Each of those hyperrectangles models a region that only a limited number of processes can enter simultaneously. We assume moreover that the coordinates a_j^i are pairwise different for every $1 \leq j \leq n$ (geometrically, this is a genericity assumption). The relevant state space is $X = I^n \setminus int(F)$.

For any nonempty index set $J = \{i_1, \ldots, i_k\}$ define

$$R^J = R^{i_1} \cap \cdots \cap R^{i_k} = [a_1^J, b_1^J] \times \cdots \times [a_n^J, b_n^J]$$

with $a_j^J = \max\{a_j^i | i \in J\}$ and $b_j^J = \min\{b_j^i | i \in J\}$. This set is again an $n-$rectangle unless it is empty (if $a_j^k > b_j^l$ for some $1 \leq j \leq n$ and $k, l \in J$). Let $\mathbf{a}^J = [a_1^J, \ldots, a_n^J] = \min R^J$ denote the *minimal* point in that hyperrectangle.

For every $1 \leq j \leq n$, we choose $\widetilde{a_j^J}$ as the "second largest" of the $a_j^{i_l}$, i.e., $\widetilde{a_j^J} = a_j^{i_s}$ with $a_j^{i_l} \leq a_j^{i_s} < a_j^J$ for all $a_j^{i_l} \neq a_j^J$, and consider the associated hyperrectangle $U^J = [\widetilde{a_1^J}, a_1^J] \times \cdots \times [\widetilde{a_n^J}, a_n^J]$ "below" R^J, the interior of which is unsafe with respect to $\mathbf{a}^J$. Usually, it models a large number of "states"; this is where we exploit higher-dimensionality.

Deadlock points in the interior of I^n are then exactly the minimal points $\min R^J$ of intersections with index set J *of cardinality* n (the number of processes, i.e. the dimension of the geometric shape we are studying) such that $R^J \neq \emptyset$ and with $\min R^J$ not contained in any R^i with $i \notin J$. Deadlock points on the boundary ∂I^n can be found using the same recipe after modification of the hyperrectangles used in the description (cf. [3, 4]).

This description allows to find the set $\mathcal{D}$ of deadlocks in X and, for every deadlock $\mathbf{a} \in \mathcal{D}$ corresponding to a set of indices $J_{\mathbf{a}}$, the unsafe hyperrectangle $U^{J_{\mathbf{a}}}$ "just below". To detect the *entire* unsafe region, let $F_1 = F \cup \bigcup_{\mathbf{a} \in \mathcal{D}} U^{J_{\mathbf{a}}}$. Find the set $\mathcal{D}_1$ of deadlocks in $X_1 = X \setminus int(F_1) \subset X$, and, for every deadlock $\mathbf{a} \in \mathcal{D}_1$, the unsafe corresponding hyperrectangle $U^{J_{\mathbf{a}}}$. Let $F_2 = F_1 \cup \bigcup_{\mathbf{a} \in \mathcal{D}_1} U^{J_{\mathbf{a}}}$ etc. (see Fig. 5 – 8 for an example).

The algorithm stops after a finite number l of loops ending with a set $U = F_l$ and such that $X_l = X \setminus int(U)$ does no longer contain any deadlocks. The set U consists precisely of the forbidden and of the unsafe points.

Literally the same algorithm will find the *unreachable* regions after a time reversal (reflection in the barycenter of I^n).

3.6 Further topics

Application 2-phase locked protocols

Related concepts Dicoverings. Po-structure. Homotopy history. Dicomponents.

Models: Cubical complexes with local partial order

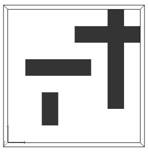

Figure 5: The forbidden region

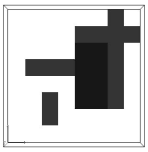

Figure 6: First step of the algorithm

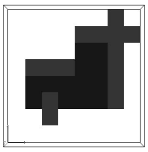

Figure 7: Second step of the algorithm

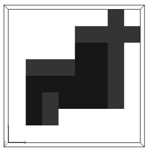

Figure 8: Last step of the algorithm

More structure higher dihomotopy, structure(s), dihomology, functoriality and applications, relations to classical paradigma in concurrency

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Investigating The Algebraic Structure of Dihomotopy Types

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July 2001

Abstract

This presentation is the sequel of a paper published in GETCO'00 proceedings where a research program to construct an appropriate algebraic setting for the study of deformations of higher dimensional automata was sketched. This paper focuses precisely on detailing some of its aspects. The main idea is that the category of homotopy types can be embedded in a new category of dihomotopy types, the embedding being realized by the Globe functor. In this latter category, isomorphism classes of objects are exactly higher dimensional automata up to deformations leaving invariant their computer scientific properties as presence or not of deadlocks (or everything similar or related). Some hints to study the algebraic structure of dihomotopy types are given, in particular a rule to decide whether a statement/notion concerning dihomotopy types is or not the lifting of another statement/notion concerning homotopy types. This rule does not enable to guess what is the lifting of a given notion/statement, it only enables to make the verification, once the lifting has been found.

Contents

1 Introduction

This paper is an expository paper which is the sequel of [Gau01c]. We will come back only very succinctly on the explanations given in this latter. A technical appendix explains some of the notions used in the core of the paper and fixes some notations. A reader who would need more information about algebraic topology or homological algebra could refer to [May67, Wei94, Rot88, Hat]. A reader who would need more information about the geometric point of view of concurrency theory could refer to [Gou95, FGR99b].

The purpose is indeed to explain with much more details [1] the speculations of the last paragraph of [Gau01c]. More precisely, we are going to describe a research program whose goal is to construct an appropriate algebraic theory of the deformations of higher dimensional automata (HDA) leaving invariant their computer-scientific properties. Most of the paper is as informal as the preceding one. The term *dihomotopy* (contraction of *directed homotopy*) will be used as an analogue in our context of the usual notion of *homotopy*.

There are two known ways of modeling higher dimensional automata for us to be able to study their deformations. 1) The ω-categorical approach, where strict globular ω-categories

[1] Even if the limited required number of pages for this paper too entails to make some shortcuts.

are supposed to encode the algebraic structure of the possible compositions of execution paths and homotopies between them, initiated by [Pra91] and continued in [Gau00] where connections with homological ideas of [Gou95] were made. 2) The topological approach which consists, loosely speaking, to locally endow a topological space with a closed partial ordering which is supposed to represent the time : this is the notion of local po-space developed for example in [FGR99b]. The description of these models is sketched in Section 2.

Section 3 is an exposition of the homological constructions which will play a role in the future algebraic investigations. Once again, the ω-categorical case and the topological case are described in parallel.

In Section 4, the notion of deformation of higher dimensional automata is succinctly recalled. For further details, see [Gau01c].

Afterwards Section 5 exposes the main ideas about the relation between homotopy types and dihomotopy types. And some hints to explore the algebraic structure of *dihomotopy types* are explained (this question is widely open).

Everything is presented in parallel because, as in usual algebraic topology, the ω-categorical approach and the topological approach present a lot of similarities. In a first version, the paper was organized with respect to the main result of [KV91], that is the category equivalence between CW-complexes up to weak homotopy equivalence and weak ω-groupoids up to weak homotopy equivalence. By [Sim98], it seems that this latter result cannot be true, at least with the functors used in Kapranov-Voevodsky's paper. Therefore in this new version, the presentation of some ideas is slightly changed. I thank Sjoerd Crans for letting me know this fact.

2 The formalization

2.1 The ω-categorical approach

Several authors have noticed that a higher dimensional automaton can be encoded in a structure of precubical set (Definition C.1). This idea is implemented in [Cri96] where a CaML program translating programs written in Concurrent Pascal into (huge !) text files is presented.

But such object does not contain any information about the way of composing n-transitions, hence the idea of adding composition laws. In an ω-category (Definition C.2), the 1-morphisms represent the execution paths, the 2-morphisms the concurrent execution of the 1-source and the 1-target of the 2-morphisms we are considering, etc... The link between this way of modeling higher dimensional automata and the formalization by precubical sets is the realization functor $K \mapsto \Pi(K)$ described in Appendix D.

There exist two equivalent notions of (strict) ω-categories, the globular one and the cubical one [AABS00] : the globular version will be used, although all notions could be adapted to the cubical version. In fact, for some technical reasons (Proposition 3.1), even a more restrictive notion will be necessary :

Definition 2.1. *[Gau01a] An ω-category $\mathcal{C}$ is non-contracting if s_1x and t_1x are 1-dimensional as soon as x is not 0-dimensional. Let f be an ω-functor from $\mathcal{C}$ to $\mathcal{D}$. The morphism f is non-contracting if for any 1-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a 1-dimensional morphism of $\mathcal{D}$. The category of non-contracting ω-categories with the non-contracting ω-functors is denoted by ωCat_1.*

The following proposition ensures that this technical restriction is not too small and that it does contain all precubical sets.

Proposition 2.1. *[Gau01a] For any precubical set K, $\Pi(K)$ is a non-contracting ω-category. The functor Π from the category of cubical sets $Sets^{\square^{preop}}$ to that of ω-categories ωCat yields a functor from $Sets^{\square^{preop}}$ to the category of non-contracting ω-categories ωCat_1.*

2.2 The topological approach

Another way of modeling higher dimensional automata is to use the notion of local po-space. A local po-space is a gluing of the following local situation : 1) a topological space, 2) a partial ordering, 3) as compatibility axiom between both structures, the graph of the partial ordering is supposed to be closed [FGR99b] (cf. Appendix A).

However the category of local po-spaces is too wide, and as in usual algebraic topology, a more restrictive notion is necessary to avoid too pathological situations (for instance think of the Cantor set). A new notion which would play in this context the role played by the CW-complexes in usual algebraic topology is necessary. This is precisely the subject of [GG01] (joined work with Eric Goubault).

Let $n \geq 1$. Let D^n be the closed n-dimensional disk defined by the set of points $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ such that $x_1^2 + \cdots + x_n^2 \leq 1$ endowed with the topology induced by that of $\mathbb{R}^n$. Let $S^{n-1} = \partial D^n$ be the boundary of D^n for $n \geq 1$, that is the set of $(x_1, \ldots, x_n) \in D^n$ such that $x_1^2 + \cdots + x_n^2 = 1$. Notice that S^0 is the discrete two-point topological space $\{-1, +1\}$. Let $I = [0,1]$. Let D^0 be the one-point topological space. And let $e^n := D^n - S^n$. Loosely speaking, globular CW-complexes are gluing of po-spaces $\overrightarrow{D}^{n+1} := Glob(D^n)$ along $\overrightarrow{S}^n := Glob(\partial D^n) = Glob(S^{n-1})$ where $Glob$ is the Globe functor (cf. Appendix A).

Notice that there is a canonical inclusion of po-spaces $\overrightarrow{S}^n \subset \overrightarrow{D}^{n+1}$ for $n \geq 1$. By convention, let $\overrightarrow{S}^0 := \{0, 1\}$ with the trivial ordering (0 and 1 are not comparable). There is a canonical inclusion $\overrightarrow{S}^0 \subset \overrightarrow{D}^1$ which is a morphism of po-spaces.

Proposition and Definition 2.2. *[GG01] For any $n \geq 1$, $\overrightarrow{D}^n - \overrightarrow{S}^{n-1}$ with the induced partial ordering is a po-space. It is called the n-dimensional globular cell. More generally,* *every local po-space isomorphic to $\overrightarrow{D}^n - \overrightarrow{S}^{n-1}$ for some n will be called a n-dimensional globular cell.*

Now we are going to describe the process of attaching globular cells.

1. Start with a discrete set of points X^0.
2. Inductively, form the n-skeleton X^n from X^{n-1} by attaching globular n-cells $\overrightarrow{e}^n_\alpha$ via maps $\phi_\alpha : \overrightarrow{S}^{n-1} \longrightarrow X^{n-1}$ with $\phi_\alpha(\underline{\iota}), \phi_\alpha(\underline{\sigma}) \in X^0$ such that[2] : for every non-decreasing map ϕ from $\overrightarrow{I}$ to $\overrightarrow{S}^{n-1}$ such that $\phi(0) = \underline{\iota}$ and $\phi(1) = \underline{\sigma}$, there exists $0 = t_0 < \cdots < t_k = 1$ such that $\phi_\alpha \circ \phi(t_i) \in X^0$ for any $0 \leq i \leq k$ which must satisfy
   (a) for any $0 \leq i \leq k-1$, there exists a globular cell of dimension d_i with $d_i \leq n-1$ $\psi_i : \overrightarrow{D}^{d_i} \to X^{n-1}$ such that for any $t \in [t_i, t_{i+1}]$, $\phi_\alpha \circ \phi(t) \in \psi_i(\overrightarrow{D}^{d_i})$;
   (b) for $0 \leq i \leq k-1$, the restriction of $\phi_\alpha \circ \phi$ to $[t_i, t_{i+1}]$ is non-decreasing;
   (c) the map $\phi_\alpha \circ \phi$ is non-constant ;

   Then X^n is the quotient space of the disjoint union $X^{n-1} \bigsqcup_\alpha \overrightarrow{D}^n_\alpha$ of X^{n-1} with a collection of $\overrightarrow{D}^n_\alpha$ under the identification $x \sim \phi_\alpha(x)$ for $x \in \overrightarrow{S}^{n-1}_\alpha \subset \partial \overrightarrow{D}^n_\alpha$. Thus as set, $X^n = X^{n-1} \bigsqcup_\alpha \overrightarrow{e}^n_\alpha$ where each $\overrightarrow{e}^n_\alpha$ is a n-dimensional globular cell.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case, X is given the weak topology : A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for some n (this topology is nothing else but the direct limit of the topology of the X^n, $n \in \mathbb{N}$). Such a X is called a globular CW-complex and X_0 and the collection of $\overrightarrow{e}^n_\alpha$ and its attaching maps $\phi_\alpha : \overrightarrow{S}^{n-1} \longrightarrow X^{n-1}$ is called the cellular decomposition of X.

As trivial examples of globular CW-complexes, there are $\overrightarrow{D}^{n+1}$ and $\overrightarrow{S}^n$ themselves where the 0-skeleton is, by convention, $\{\underline{\iota}, \underline{\sigma}\}$.

We will consider without further mentioning that the segment $\overrightarrow{I}$ is a globular CW-complex, with $\{0, 1\}$ as its 0-skeleton.

Proposition and Definition 2.3. *[GG01] Let X be a globular CW-complex with characteristic maps (ϕ_α). Let γ be a continuous map from $\overrightarrow{I}$ to X. Then $\gamma([0,1]) \cap X^0$ is finite. Suppose that there exists $0 \leq t_0 < \cdots < t_n \leq 1$ with $n \geq 1$ such that $t_0 = 0$, $t_n = 1$, such that for any $0 \leq i \leq n$, $\gamma(t_i) \in X^0$, and at last such that for any $0 \leq i \leq n-1$, there exists an α_i (necessarily unique) such that for $t \in [t_i, t_{i+1}]$, $\gamma(t) \in \phi_{\alpha_i}(\overrightarrow{D}^{n_\alpha})$. Then such a γ is called an execution path if the restriction $\gamma \upharpoonright_{[t_i, t_{i+1}]}$ is non-decreasing.*

[2] This condition will appear to be necessary in the sequel.

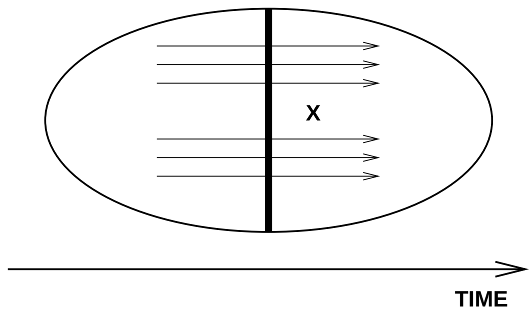

Figure 1: Symbolic representation of $Glob(X)$ for some topological space X

By constant execution paths, one means an execution paths γ such that $\gamma([0,1]) = \{\gamma(0)\}$. The points (i.e. elements of the 0-skeleton) of a given globular CW-complexes X are also called *states*. Some of them are fairly special:

Definition 2.2. *Let X be a globular CW-complex. A point α of X^0 is initial (resp. final) if for any execution path ϕ such that $\phi(1) = \alpha$ (resp. $\phi(0) = \alpha$), then ϕ is the constant path α.*

Let us now describe the category of *globular CW-complexes.*

Definition 2.3. *[GG01] The category* **glCW** *of globular CW-complexes is the category having as objects the globular CW-complexes and as morphisms the continuous maps $f : X \longrightarrow Y$ satisfying the two following properties :*

- $f(X^0) \subset Y^0$
- *for every non-constant execution path ϕ of X, $f \circ \phi$ must not only be an execution path (f must preserve partial order), but also $f \circ \phi$ must be non-constant as well : we say that f must be non-contracting.*

The condition of non-contractibility is very analogous to the notion of non-contracting ω-functors appearing in [Gau00], and is necessary for similar reasons. In particular, if the constant paths are not removed from $\mathbb{P}^{\pm}X$ (see Section 3.2 for the definition), then this latter spaces are homotopy equivalent to the discrete set X^0 (the 0-skeleton of X !). And the removing of the constant paths from $\mathbb{P}^{\pm}X$ entails to remove also the constant paths from $\mathbb{P}X$ in order to keep the existence of both natural transformations $\mathbb{P} \to \mathbb{P}^{\pm}$. Then the mappings $\mathbb{P}$ and $\mathbb{P}^{\pm}$ can be made functorial only if we work with non-contracting maps as above [GG01].

One can also notice that by construction, the attaching maps are morphisms of globular CW-complexes. Of course one has

$$
\begin{tikzcd}
 & \mathbb{P}M \arrow[dl, "h^-"'] \arrow[dr, "h^+"] & \\
\mathbb{P}^-M & & \mathbb{P}^+M
\end{tikzcd}
$$

Figure 2: The fundamental diagram

Theorem 2.4. *[GG01] Every globular CW-complex is a local po-space and this mapping induces a functor from the category of globular CW-complexes to the category of local po-spaces.*

3 The homological constructions

The three principal constructions are all based upon the idea of capturing the algebraic structure of the set of *achronal cuts* (cf. [Gau01c] for some explanations of this idea) included in the higher dimensional automaton M we are considering in three simplicial sets which seem to be the basement of an algebraic theory which remains to build. For that, one has to construct in both cases (the categorical and the topological approaches), three *spaces* :

1. the *space of non-constant execution paths* (this idea will become more precise below) : let us call it the *path space* $\mathbb{P}M$
2. the *space of equivalence classes of non-constant execution paths beginning in the same way* : let us call it the *negative semi-path space* $\mathbb{P}^-M$
3. the *space of equivalence classes of non-constant execution paths ending in the same way* : let us call it the *positive semi-path space* $\mathbb{P}^+M$.

and one will consider the simplicial nerve of each one.

[Gau01c] Figure 11 will become Figure 2 in both topological and ω-categorical situations. The construction of h^- and h^+ is straightforward in both situations.

3.1 The ω-categorical approach

Proposition 3.1. *[Gau01a] Let $\mathcal{C}$ be an ω-category. Consider the set $\mathbb{P}\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$. Then the operators s_n, t_n and $*_n$ for $n \geq 1$ are internal to $\mathbb{P}\mathcal{C}$ if and only if $\mathcal{C}$ is non-contracting. In that case, $\mathbb{P}\mathcal{C}$ can be endowed with a structure of ω-category whose n-source (resp. n-target, n-dimensional composition law) is the $(n+1)$-source (resp. $(n+1)$-target, $(n+1)$-dimensional composition law) of $\mathcal{C}$. The ω-category $\mathbb{P}\mathcal{C}$ is called the path ω-category of $\mathcal{C}$, and the mapping $\mathcal{C}$ induces a well-defined functor from ωCat_1 to ωCat.*

Definition 3.1. *Let $\mathcal{C}$ be a non-contracting ω-category. Denote by $\mathcal{R}^-$ (resp. $\mathcal{R}^+$) the reflexive symmetric and transitive closure of* $\{(x, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$ *(resp.* $\{(y, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$*) in* $\mathbb{P}\mathcal{C} \times \mathbb{P}\mathcal{C}$.

Proposition 3.2. *[Gau01d] Let $\alpha \in \{-, +\}$ and let $\mathcal{C}$ be a non-contracting ω-category. Then the universal problem*

> *"There exists a pair $(\mathcal{D}, \mu)$ such that $\mathcal{D}$ is an ω-category and μ an ω-functor from $\mathbb{P}\mathcal{C}$ to $\mathcal{D}$ such that for any $x, y \in \mathbb{P}\mathcal{C}$, $x\mathcal{R}^\alpha y$ implies $\mu(x) = \mu(y)$."*

has a solution $(\mathbb{P}^\alpha\mathcal{C}, (-)^\alpha)$. Moreover $\mathbb{P}^\alpha\mathcal{C}$ is generated by the elements of the form $(x)^\alpha$ for x running over $\mathbb{P}\mathcal{C}$. The mappings $\mathbb{P}^-$ and $\mathbb{P}^+$ induce two well-defined functors from ωCat_1 to ωCat.

Definition 3.2. *The ω-category $\mathbb{P}^-\mathcal{C}$ (resp. $\mathbb{P}^+\mathcal{C}$) is called the negative (resp. positive) semi-path ω-category of $\mathcal{C}$.*

In the sequel, $\mathbb{P}\mathcal{C}$ will be supposed to be a strict globular ω-groupoid in the sense of Brown-Higgins, which implies that $\mathbb{P}^-\mathcal{C}$ and $\mathbb{P}^+\mathcal{C}$ satisfy also the same property : this means concretely that if there exists an homotopy from a given execution path γ to another one γ', then there exists also an homotopy in the opposite direction [Gau01d].

Definition 3.3. *[Gau01a] The globular simplicial nerve $\mathcal{N}^{gl}$ is the functor from ωCat_1 to $Sets_+^{\Delta^{op}}$ defined by*

$$\mathcal{N}^{gl}_n(\mathcal{C}) := \omega Cat(\Delta^n, \mathbb{P}\mathcal{C})$$

for $n \geq 0$ and with $\mathcal{N}^{gl}_{-1}(\mathcal{C}) := \mathcal{C}_0 \times \mathcal{C}_0$, and endowed with the augmentation map ∂_{-1} from $\mathcal{N}^{gl}_0(\mathcal{C}) = \mathcal{C}_1$ to $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$ defined by $\partial_{-1}x := (s_0x, t_0x)$.

Geometrically, a simplex of this simplicial nerve looks as in Figure 3.

Definition 3.4. *Let $\mathcal{C}$ be a non-contracting ω-category. Then set*

$$\mathcal{N}^{gl^-}_n(\mathcal{C}) := \omega Cat(\Delta^n, \mathbb{P}^-\mathcal{C})$$

and $\mathcal{N}^{gl^-}_{-1}(\mathcal{C}) := \mathcal{C}_0$ with $\partial_{-1}(x) := s_0x$. Then $\mathcal{N}^{gl^-}$ induces a functor from ωCat_1 to $Sets_+^{\Delta^{op}}$ which is called the negative semi-globular nerve or (branching semi-globular homology) of $\mathcal{C}$.

The *positive semi-globular nerve* is defined in a similar way by replacing $-$ by $+$ everywhere in the above definition and by setting $\partial_{-1}(x) = t_0x$. Intuitively, the simplexes in the semi-globular nerves look as in Figure 4 : they correspond to the left or right half part of Figure 3.

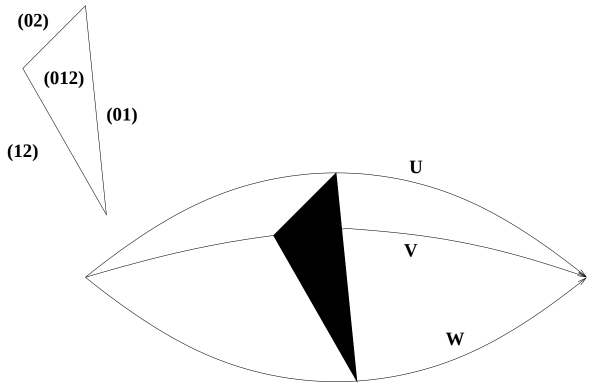

Figure 3: Globular 2-simplex

Figure 4: Negative and positive semi-globular 2-simplexes

3.2 The topological approach

Let X be a globular CW-complex. Let $\alpha, \beta \in X^0$. Denote by $(X, \alpha, \beta)^\perp$ the topological space of non-decreasing non-constant continuous maps γ from $[0,1]$ endowed with the usual order to X such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$ and endowed with the compact-open topology. Then

Definition 3.5. *Let X be a globular CW-complex. Then the path space of X is the disjoint union*

$$\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0 \times X^0} (X, \alpha, \beta)^\perp$$

endowed with the disjoint union topology.

Now denote by $(X, \alpha)^{\perp^-}$ (resp. $(X, \beta)^{\perp^+}$) the topological space of non-decreasing non-constant continuous maps from $[0,1]$ with the usual order to X such that $\gamma(0) = \alpha$ (resp. $\gamma(1) = \beta$), endowed with the compact-open topology. Then

Definition 3.6. *Let X be a globular CW-complex. Then the negative semi-path space $\mathbb{P}^- X$ (resp. positive semi-path space $\mathbb{P}^+ X$) of X are defined by*

$$\mathbb{P}^- X = \bigsqcup_{\alpha \in X^0} (X, \alpha)^{\perp^-}$$
$$\mathbb{P}^+ X = \bigsqcup_{\beta \in X^0} (X, \beta)^{\perp^+}$$

endowed with the disjoint union topology.

The reader can notice that in the topological context, we do not need anymore to consider something like the equivalence relations $\mathcal{R}^-$ and $\mathcal{R}^+$. The reason is that, ideologically ("moralement" in french !), a 1-morphism is of length 1. On contrary, a non-constant execution path is homotopic to any shorter execution path [3]

Definition 3.7. *[Gau01a] The globular simplicial nerve $\mathcal{N}^{gl}$ is the functor from* **glCW** *to $Sets_+^{\Delta^{op}}$ defined by*

$$\mathcal{N}^{gl}_n(X) := S_n(\mathbb{P}X)$$

for $n \geq 0$ where S_* is the singular simplicial nerve (cf. Appendix B) with $\mathcal{N}^{gl}_{-1}(X) := X^0 \times X^0$, and endowed with the augmentation map ∂_{-1} from $\mathcal{N}^{gl}_0(X) = \mathbb{P}X$ to $\mathcal{N}^{gl}_{-1}(X) = X^0 \times X^0$ defined by $\partial_{-1}\gamma := (\gamma(0), \gamma(1))$.

[3] in a "natural way" by considering $H(\gamma(t), u) = \gamma(tu)$. It is the reason why $\mathbb{P}^- X$ and $\mathbb{P}^+ X$ are homotopy equivalent to X^0 if one does not remove the constant paths from their definition.

$$\begin{tikzcd} \alpha \arrow[r, "u"] & \beta \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] & \gamma \end{tikzcd}$$

(a) $\mathcal{C}$

$$\begin{tikzcd} \alpha_1 \arrow[r, "u_1"] & \alpha_2 \arrow[r, "u_2"] & \beta \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] & \gamma \end{tikzcd}$$

(b) Subdivision of u in $\mathcal{C}$

Figure 5: Example of T-deformation

Definition 3.8. *Let X be a globular CW-complex. Then set*

$$\mathcal{N}^{gl^-}_n(X) := S_n(\mathbb{P}^- X)$$

for $n \geq 0$ and $\mathcal{N}^{gl^-}_{-1}(X) := X^0$ with $\partial_{-1}(\gamma) := \gamma(0)$. Then $\mathcal{N}^{gl^-}$ induces a functor from **glCW** *to $Sets_+^{\Delta^{op}}$ which is called the branching semi-globular nerve of X .*

The *merging semi-globular nerve* is defined in a similar way by replacing $-$ by $+$ everywhere in the above definition and by setting $\partial_{-1}(\gamma) := \gamma(1)$.

4 Deforming higher dimensional automata

As already seen in [Gau01c] in the ω-categorical context, there are two types of deformation leaving invariant the computer scientific properties of higher dimensional automata : the *temporal deformations* (or *T-deformations*) and the *spatial deformations* (*S-deformations*). The first type (temporal) is closely related to the notion of homeomorphism because a non-trivial execution path cannot be contracted in the same dihomotopy class [4], and the second one (spatial) to the classical notion of homotopy equivalence.

The ω-categorical case will be only briefly recalled. A temporal deformation corresponds informally to the reflexive symmetric and transitive closure of subdividing in an ω-category a 1-morphism in two parts as in Figure 5. A spatial deformation consists of deforming in the considered ω-category p-morphisms with $p \geq 2$, which is equivalent to deforming faces in one the three nerves in the usual sense of homotopy equivalence.

The topological approach is completely similar. A temporal deformation of a globular CW-complex X consists of dividing in two parts a globular 1-dimensional cell of the cellular decomposition of X, as in Figure 5. A spatial deformation consists of crushing globular cells of higher dimension.

[4] In fact, the T-dihomotopy equivalences in [GG01] are precisely the morphisms of globular CW-complexes inducing an homeomorphim between both underlying topological spaces.

Now what can we do with the previous homological constructions ? First of all consider the corresponding simplicial homology theories of all these augmented simplicial sets, with the following convention on indices : for $n \geq -1$ and $u \in \{gl, gl^-, gl^+\}$, set $H^u_{n+1}(M) = H_n(\mathcal{N}^u(M))$ for M either an ω-category or a globular CW-complex. We obtain this way three homology theories called as the corresponding nerve. One knows that the globular homology sees the globes included in the HDA [Gau00, Gau01a] and that the branching (resp. merging) semi-globular homology sees the branching areas (resp. merging areas) in the HDA [Gau00, Gau01b, Gau01d]. Since the three nerves are Kan [5], one can also consider the homotopy groups of these nerves, with the same convention for indices : for $n \geq 1$ and $u \in \{gl, gl^-, gl^+\}$, set $\pi^u_{n+1}(X) = \pi_n(\mathcal{N}^u(X), \phi)$ for X either an ω-category or a globular CW-complex. In this latter case, the base-point ϕ is in fact a 0-morphism of $\mathbb{P}\mathcal{C}$, that is a 1-morphism of $\mathcal{C}$ if $u = gl$, and an equivalence class of 1-morphism of $\mathcal{C}$ with respect to $\mathcal{R}^-$ (resp. $\mathcal{R}^+$) if $u = gl^-$ (resp. $u = gl^+$). Intuitively, elements of π^{gl}_{n+1} are $(n+1)$-dimensional cylinders with achronal basis.

The four first lines of Table 6 are explained in [Gau01c]. The branching and merging (semi-cubical) nerves $\mathcal{N}^\pm$ defined in [Gau00, Gau01b] are almost never Kan : in fact as soon as there exists in the ω-category $\mathcal{C}$ we are considering two 1-morphisms x and y such that $x *_0 y$ exists (see Proposition 5.6), both semi-cubical nerves are not Kan.

If the branching and merging (semi-cubical) nerves are replaced by the branching and merging semi-globular nerve, then the "almost" (in fact a "no") becomes a "yes" here because we are not disturbed anymore by the non-simplicial part of the elements of the branching and merging nerves (which is removed by construction).

The lines concerning the (globular, negative and positive semi-globular) homotopy groups need to be explained. The S-invariance of a given nerve implies of course the S-invariance of the corresponding homotopy groups. As for the T-invariance, it is due to the fact that in these homotopy groups, the "base-point" is an execution path (or eventually an equivalence class of). So these homotopy groups contain information only related to achronal cuts crossing the "base-point". Dividing this base-point or any other 1-morphism or 1-dimensional globular cell changes nothing.

The last lines are concerned with the bisimplicial set what we call *biglobular nerve* (for the contraction of bisimplicial globular nerve) described in [Gau01c, Gau01a] (cf. Appendix F) and constructed by considering the structure of augmented simplicial object of the category of small categories of the globular nerve. The biglobular nerve inherits the S-invariance of the globular nerve. And its T-invariance is due to the T-invariance of the simplicial nerve functor of small categories. The answer "yes ?" means that it is expected to find "yes" in some sense... It is worth noticing that in a true higher dimensional automaton, 1-morphisms are never invertible because the time is not reversible. So one cannot

[5] The ω-categorical versions are Kan as soon as $\mathbb{P}\mathcal{C}$ is an ω-groupoid [Gau01d] and the singular simplicial nerve is known to be Kan [May67].

Functors	S-invariant	T-invariant	Kan	$-\infty \to +\infty$
$\mathcal{N}^{gl}$	yes	no	yes	yes
$\mathcal{N}^{\pm}$	almost	no	almost never	yes
H^{gl}	yes	no	no meaning	yes
$H^{\pm}$	yes	yes	no meaning	yes
$\mathcal{N}^{gl^{\pm}}$	yes	no	yes	yes
$H^{gl^{\pm}}$	yes	yes	no meaning	yes
π^{gl}	yes	yes	no meaning	no
$\pi^{gl^{\pm}}$	yes	yes	no meaning	no
$\mathcal{N}^{bigl}$	yes	yes	yes ?	yes
H^{bigl}	yes	yes	no meaning	yes
(π^{bigl})	(yes)	(yes)	(no meaning)	(yes)

Figure 6: Behavior w.r.t the two types of deformations

expect to find a Kan bisimplicial set in the usual sense of the notion.

The last column is not directly concerned with the different types of deformations of HDA, but rather by the question of knowing if the functors contain information from $t = -\infty$ to $t = \infty$. The answer is yes everywhere except for the three homotopy groups functors : the latter contain indeed information only related to achronal cuts crossing the "base-point". One can by the way notice that, in the ω-categorical case :

Proposition 4.1. *Let ϕ and ψ be two 1-morphisms of a non-contracting ω-category $\mathcal{C}$. Suppose that $\phi *_0 \psi$ exists. Then*

1. *If $\phi *_0 \psi$ is 1-dimensional, then the mapping $(x, y) \mapsto x *_0 y$ partially defined on $\mathcal{C}_n \times \mathcal{C}_n$ induces a morphism of groups $\pi^{gl}_{n+1}(\mathcal{C}, \phi) \times \pi^{gl}_{n+1}(\mathcal{C}, \psi) \to \pi^{gl}_{n+1}(\mathcal{C}, \phi *_0 \psi)$.*
2. *If $\phi *_0 \psi$ is 0-dimensional, then the mapping $(x, y) \mapsto x *_0 y$ partially defined on $\mathcal{C}_n \times \mathcal{C}_n$ induces the constant map $\phi *_0 \psi$.*

Proof. It is due to the fact that

$$\pi^{gl}_{n+1}(\mathcal{C}, \phi) = \{x \in \mathcal{C}_{n+1}, s_1 x = s_2 x = \cdots = s_n x = t_1 x = t_2 x = \cdots = t_n x = \phi\}$$

with $*_n$ for group law. $\square$

The above proposition is a hint to correct the drawbacks of the globular and semi-globular homotopy groups.

The last line π^{bigl} is explained with Philosophy 5.8.

5 The category of dihomotopy types

5.1 Towards a construction

Here both approaches slightly diverge because of a lack of knowledge about the ω-categorical ways of constructing homotopy types. However one can certainly define in both contexts a notion of *weak dihomotopy equivalence* : see [GG01] for the topological context. Then let

- ωGrp be the category of strict globular ω-groupoids with the ω-functors as morphisms, and $\mathbf{Ho}(\omega Grp)$ its localization by the weak homotopy equivalences
- ωCat_1^{Kan} be the category of non-contracting ω-categories $\mathcal{C}$ such that $\mathbb{P}\mathcal{C}$ is an ω-groupoid with the non-contracting ω-functors as morphisms, and $\mathbf{Ho}(\omega Cat_1^{Kan})$ its localization by the weak dihomotopy equivalences
- **CW** the category of CW-complexes with the continuous maps as morphisms, and $\mathbf{Ho}(\mathbf{CW})$ its localization by the weak homotopy equivalences
- **glCW** the category of globular CW-complexes with the morphisms of globular CW-complexes as morphisms, and $\mathbf{Ho}(\mathbf{glCW})$ its localization by the weak dihomotopy equivalences.

Philosophy 5.1. *Both localizations* $\mathbf{Ho}(\omega Cat_1^{Kan})$ *and* $\mathbf{Ho}(\mathbf{glCW})$ *contain the precubical sets modulo spatial and temporal deformations. However, due to the fact that strict globular ω-groupoids do not represent all homotopy types [BH81a], but only those having a trivial Whitehead product,* $\mathbf{Ho}(\omega Cat_1^{Kan})$ *could be not big enough to construct an appropriate algebraic setting.*

After [Sim98], it is clear that the ω-categorical realization functor described in Section D loses some homotopical information and that keeping the complete information requires to work with ω-categories where the associativity of $*_n$ is weakened for any $n \geq 1$. However, this lost homotopical information is only related to the geometric situation in achronal cuts. In particular, this realization functor does not contract 1-morphisms. Therefore the $\mathbf{Ho}(\omega Cat_1^{Kan})$ framework could be sufficient to study questions concerning deadlocks or other similar 1-dimensional phenomena.

Definition 5.1. *The category* $\mathbf{Ho}(\mathbf{glCW})$ *is called the category of dihomotopy types.*

To describe the relation between the usual situation and the directed situation, we need two last propositions and definitions :

Proposition 5.2. *[GG01] Let X be a CW-complex. Let $Glob(X)^0 = \{\underline{\iota}, \underline{\sigma}\}$ where $\underline{\iota}$ (resp. $\underline{\sigma}$) is the equivalence class of $(x, 0)$ (resp. $(x, 1)$). Then the cellular decomposition of X yields a cellular decomposition of $Glob(X)$ and this way, $Glob(-)$ induces a functor from* **CW** *to* **glCW**.

Proposition 5.3. *Let G be an object of ωGrp. Then there exists a unique object $Glob(G)$ of ωCat_1^{Kan} such that $\mathbb{P}Glob(G) = G$, $Glob(G)_0 = \{\alpha, \beta\}$ is a two-element set, and such that $s_0(Glob(G)\backslash\{\beta\}) = \{\alpha\}$ and $t_0(Glob(G)\backslash\{\alpha\}) = \{\beta\}$. Moreover the mapping Glob induces a functor from ωGrp to ωCat_1^{Kan}.*

Both *Glob* functors (called *Globe functors*) yield two functors

$$\mathbf{Ho}(\mathbf{CW}) \to \mathbf{Ho}(\mathbf{glCW})$$

and

$$\mathbf{Ho}(\omega Grp) \to \mathbf{Ho}(\omega Cat_1^{Kan})$$

In the topological context, one has :

Proposition 5.4. *[GG01] Let X and Y be two CW-complexes. Then X and Y are homotopy equivalent if and only if $Glob(X)$ and $Glob(Y)$ are dihomotopy equivalent. Therefore the functor $\mathbf{Ho}(\mathbf{CW}) \to \mathbf{Ho}(\mathbf{glCW})$ is an embedding.*

Question 5.5. *Is it possible to find an ω-categorical construction of $\mathbf{Ho}(\mathbf{glCW})$?*

5.2 Investigating the algebraic structure of the category of dihomotopy types

One can check that in both topological and ω-categorical situations, the following fact holds

Proposition 5.6. *(partially in [Gau01b]) Let $\alpha \in \{-,+\}$. The morphism h^α induces an isomorphism of simplicial sets (not of augmented simplicial sets for trivial reason !) $\mathcal{N}^{gl}(Glob(M)) \simeq \mathcal{N}^{gl^\alpha}(Glob(M))$. Moreover in the ω-categorical case, $\mathcal{N}^{gl}(Glob(M)) \simeq \mathcal{N}^\alpha(Glob(M))$ where $\mathcal{N}^\alpha$ are the branching or the merging nerves (depending on the value of α) of an ω-category as defined and studied in [Gau00, Gau01b]. Moreover, this common simplicial set is homotopy equivalent to the simplicial nerve of M.*

This important proposition together with Proposition 5.4 suggests us a way of investigating the algebraic structure of the category of dihomotopy types.

Philosophy 5.7. *Let* $\mathbf{Th}$ *be a theorem (or a notion) in usual algebraic topology, i.e. concerning the category of homotopy types. Let* $\mathbf{Th}^{di}$ *be its lifting (i.e. its analogue) on the category of dihomotopy types. Then the statement* $\mathbf{Th}^{di}$ *must specialize into* $\mathbf{Th}$ *on the image of the Globe functor.*

Following Baues's philosophy [Bau99], a first goal would be then to lift from the usual situation to the directed situation the Whitehead theorem and the Hurewicz theorems.

Concerning the last one, it would be first necessary to understand what is the analogue [6] of the Hurewicz morphism for the category of dihomotopy types.

Philosophy 5.8. *The target of the Hurewicz morphism in the directed situation is likely to be the biglobular homology H^{bigl}. This new Hurewicz morphism must contain is some way all usual Hurewicz morphisms of all achronal cuts. At last, the source (let us denote it by π^{bigl}) of the Hurewicz morphism must be S-invariant, T-invariant and must contain information concerning the geometry of the HDA from $t = -\infty$ to $t = +\infty$.*

Suppose $n \geq 2$. After Proposition 4.1, a possible idea in the ω-categorical case would be then to build a chain complex of abelian groups by considering elements

$$(x_1, \dots, x_p) \in \pi^{gl}_{n+1}(\mathcal{C}, \phi_1) \times \dots \times \pi^{gl}_{n+1}(\mathcal{C}, \phi_p)$$

for all p and all p-uples $(\phi_1, \dots, \phi_p)$ such that $\phi_1 *_0 \dots *_0 \phi_p$ exists and by considering the simplicial differential map induced by $*_0$. Let us call the corresponding homology theory the *toroidal homology* $H^{tor}_*(\mathcal{C})$. Of course this construction makes sense only for $n \geq 2$ because the π^{gl}_2 are not necessarily abelian. Then the classical Hurewicz morphism induces a natural transformation from H^{tor}_* to the $E^2_{*,n+1}$-term of one of the canonical spectral sequences converging to H^{bigl}.

As explained in the introduction, the goal would be to reach an homological understanding of the geometry of flows modulo deformations. In particular, we would like to find exact sequences. It is then reasonable to think that

Philosophy 5.9. *An exact sequence $F_1(M) \to F_2(M) \to F_3(M)$ telling us something about flows M of execution paths modulo spatial and temporal deformations must use functors F_1, F_2 and F_3 invariant by spatial and temporal deformations.*

The weakness of the internal structure of the globular nerve (it is a disjoint union of simplicial sets), its non-invariance with respect to temporal deformations, and its natural correction by considering the *biglobular nerve* suggests that the *biglobular homology* (the total homology of this bisimplicial set) has more interesting homological properties than the globular homology.

Concerning the biglobular nerve, it is worth noticing that this object contains the whole information about the position of achronal simplexes and about the temporal structure of

[6] The solution given in [Gau00] is naturally wrong : the morphisms h^- and h^+ are not the analogues of the Hurewicz morphism. When [Gau00] was being written, It was not known that the correct definition of the globular homology would come from the simplicial homology of a simplicial nerve. Moreover the role of achronal cuts was also not yet understood. The globular homology was introduced as an answer of Goubault's suggestion of finding the analogue of the Hurewicz morphism in "directed homotopy" theory. Then starting from the principle that the branching and merging homology theories could be an analogue of the singular homology, I wondered whether it was possible to construct a morphism abutting to both corner homologies. The globular homology was then designed to be the source of this morphism.

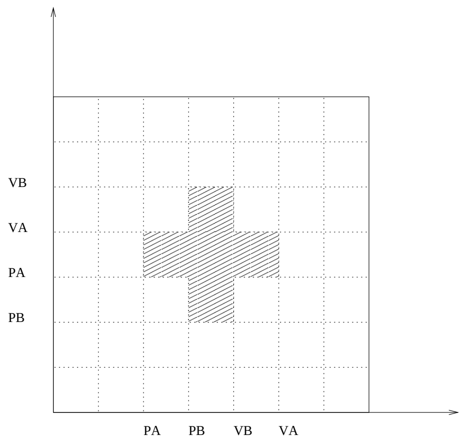

Figure 7: PV diagram

the underlying higher dimensional automaton. So in some sense, the biglobular nerve contains everything related to the geometry of HDA. Since the biglobular nerve is expected to be S-invariant and T-invariant, then it is natural to ask the following question :

Question 5.10. *Is it possible to recover all other S-invariant and T-invariant functors from the biglobular nerve ? For example, is it possible to recover the semi-globular homology theories ?*

Another natural question would be to relate a given dihomotopy type to the underlying homotopy type (when the flow of execution paths is removed). If the biglobular nerve really contains the complete information, then it should be possible to recover from it the underlying homotopy type.

As last remark, let us have a look at PV diagrams as in Figure 7. They are always constructed by considering a n-cube and by digging cubical holes inside. Such examples produce examples of ω-categories or globular CW-complexes whose all types of globular homologies do not have any torsion. To classify this kind of examples, the study of *rational dihomotopy types* could be sufficient.

6 Conclusion

We have described in this paper a way of constructing the category of dihomotopy types and we have given some hints to investigate its internal algebraic structure. Intuitively, the isomorphism classes of objects in this category represent exactly the higher dimensional automata modulo the deformations which leave invariant their computer-scientific properties. So a good knowledge of the algebraic structure of this category will enable us to classify higher dimensional automata up to dihomotopy and therefore, hopefully, to write new algorithms manipulating directly the equivalence classes of HDA.

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Technical Appendix

A Local po-space : definition and examples

If X is a topological space, a binary relation R on X is closed if the graph of R is a closed subset of the cartesian product $X \times X$. If R is a closed partial order $\leq$, then $(X, \leq)$ is called a *po-space* (see for instance [Nac65], [Joh82] and [FGR99a]). Notice that a po-space is necessarily Hausdorff. We say that $(U, \leq)$ is a sub-po-space of (X, R) if and only if it is a po-space such that U is a sub topological space of X and such that $\leq$ is the restriction of R to U.

A collection $\mathcal{U}(X)$ of po-spaces $(U, \leq_U)$ covering X is called a *local partial order* if for every $x \in X$, there exists a po-space $(W(x), \leq_{W(x)})$ such that:

- $W(x)$ is an open neighborhood containing x,
- the restrictions of $\leq_U$ and $\leq_{W(x)}$ to $W(x) \cap U$ coincide for all $U \in \mathcal{U}(X)$ such that $x \in U$. This can be stated as: $y \leq_U z$ iff $y \leq_{W(x)} z$ for all $U \in \mathcal{U}(X)$ such that $x \in U$ and for all $y, z \in W(x) \cap U$. Sometimes, $W(x)$ will be denoted by $W_X(x)$ to avoid ambiguities. Such a $W_X(x)$ is called a po-neighborhood.

Two local partial orders are equivalent if their union is a local partial order. This defines an equivalence relation on the set of local partial orders of X. A topological space together with an equivalence class of local partial order is called a *local po-space*.

A morphism f of local po-spaces (or *dimap*) from $(X,\mathcal{U})$ to $(Y, \mathcal{V})$ is a continuous map from X to Y such that for every $x \in X$,

- there is a po-neighborhood $W(f(x))$ of $f(x)$ in Y,
- there exists a po-neighborhood $W(x)$ of x in X with $W(x) \subset f^{-1}(W(f(x)),$
- for $y, z \in W(x)$, $y \leq z$ implies $f(y) \leq f(z)$.

In particular, a dimap f from a po-space X to a po-space Y is a continuous map from X to Y such that for any $y, z \in X$, $y \leq z$ implies $f(y) \leq f(z)$. A morphism f of local po-spaces from $[0, 1]$ endowed with the usual ordering (denoted by $\overrightarrow{I}$) to a local po-space X is called *dipath* or sometime *execution path*.

The category of Hausdorff topological spaces with the continuous maps as morphisms will be denoted by **Haus**. The category of local po-spaces with the dimaps as morphisms will be denoted by **LPoHaus**. The category of general topological spaces without further assumption will be denoted by **Top** and the category of general topological spaces endowed with a partial ordering not necessary closed will be denoted by **PoTop**.

We end this section by an example of po-spaces which matters for this paper. Let us construct the *Globe* $Glob(X)$ associated to a topological space X. It is defined as follows. As topological space, $Glob(X)$ is the quotient of the product space $X \times I$ by the relations $(x, 0) = (x', 0)$ and $(x, 1) = (x', 1)$ for any $x, x' \in X$. It is equipped with the closed partial order $(x, t) \leq (x', t')$ if and only if $x = x'$ and $t \leq t'$. The equivalence class of $(x, 0)$ (resp. $(x, 1)$) in $Glob(X)$ is denoted by $\underline{\iota}$ (resp. $\underline{\sigma}$).

B Simplicial set

For further details, cf. [May67, Wei94].

Definition B.1. *A simplicial set A_* is a family $(A_n)_{n\geq 0}$ together with face maps $\partial_i : A_n \to A_{n-1}$ and $\epsilon_i : A_n \to A_{n+1}$ for $i = 0, \ldots, n$ which satisfy the following identities :*

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \qquad \text{if } i < j \\ \epsilon_i \epsilon_j &= \epsilon_{j+1} \epsilon_i \qquad \text{if } i \leq j \\ \partial_i \epsilon_j &= \begin{cases} \epsilon_{j-1} \partial_i & \text{if } i < j \\ Identity & \text{if } i = j \text{ or } i = j+1 \\ \epsilon_j \partial_{i-1} & \text{if } i > j+1 \end{cases} \end{aligned}$$

A morphism of simplicial sets from A_* to B_* consists of a set map from A_n to B_n for each $n \geq 0$ commuting with all operators defined on both sides. The category of simplicial sets is denoted by $Sets^{\Delta^{op}}$.

Consider the **topological n-simplex** Δ^n defined by

$$\Delta^n = \{(t_0, \ldots, t_n), t_0 \geq 0, \ldots, t_n \geq 0 \text{ and } t_0 + \ldots t_n = 1\}$$

Here is now the most classical example of simplicial sets :

Definition B.2. *Let Y be a topological space. The singular simplicial nerve of Y is the simplicial set $S_*(XY)$ defined as follows : $S_n(Y) := \mathbf{Top}(\Delta^n, Y)$ with $\partial_i(f)(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, \ldots, t_{n-1})$ and $\epsilon_i(f)(t_0, \ldots, t_{n+1}) = f(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1})$.*

Definition B.3. *[Dus75] An augmented simplicial set is a simplicial set*

$$((X_n)_{n\geq 0}, (\partial_i : X_{n+1} \longrightarrow X_n)_{0\leq i\leq n+1}, (\epsilon_i : X_n \longrightarrow X_{n+1})_{0\leq i\leq n})$$

together with an additional set X_{-1} and an additional map ∂_{-1} from X_0 to X_{-1} such that $\partial_{-1}\partial_0 = \partial_{-1}\partial_1$. A morphism of augmented simplicial set is a map of $\mathbb{N}$-graded sets which commutes with all face and degeneracy maps. We denote by $Sets_+^{\Delta^{op}}$ the category of augmented simplicial sets.

If X_* is an augmented simplicial set, one obtains a chain complex of abelian groups ($\mathbb{Z}S$ being the free abelian group generated by the set S)

$$\cdots \longrightarrow \mathbb{Z}X_2 \xrightarrow{\partial_0-\partial_1+\partial_2} \mathbb{Z}X_1 \xrightarrow{\partial_0-\partial_1} \mathbb{Z}X_0 \xrightarrow{\partial_{-1}} \mathbb{Z}X_{-1} \longrightarrow 0$$

We will denote $H_{n+1}(X)$ for $n \geq -1$ the n-th simplicial homology group of X_*. This means for example that $H_1(X)$ will be the quotient of $\partial_{-1} : \mathcal{N}_0^{gl}(\mathcal{C}) \to \mathcal{N}_{-1}^{gl}(\mathcal{C})$ by the image of $\partial_0 - \partial_1 : \mathcal{N}_1^{gl}(\mathcal{C}) \to \mathcal{N}_0^{gl}(\mathcal{C})$.

C Precubical set, globular ω-category and globular set

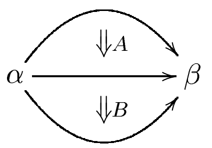
Definition C.1. *[BH81b] [KP97] A precubical set consists of a family of sets $(K_n)_{n\geqslant 0}$ and of a family of face maps $K_n \xrightarrow{\partial_i^\alpha} K_{n-1}$ for $\alpha \in \{-,+\}$ which satisfies the following axiom (called sometime the cube axiom) :*

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \text{ for all } i < j \leqslant n \text{ and } \alpha, \beta \in \{-,+\}.$$

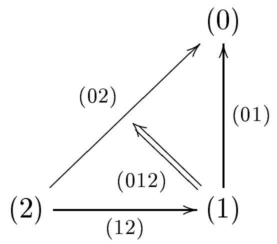
If K is a precubical set, the elements of K_n are called the n-cubes. An element of K_n is of dimension n. The elements of K_0 (resp. K_1) can be called the *vertices* (resp. the *arrows*) of K.

Definition C.2. *[BH81a, Str87, Ste91] An ω-category is a set A endowed with two families of maps $(s_n = d_n^-)_{n\geqslant 0}$ and $(t_n = d_n^+)_{n\geqslant 0}$ from A to A and with a family of partially defined 2-ary operations $(*_n)_{n\geqslant 0}$ where for any $n \geqslant 0$, $*_n$ is a map from $\{(a,b) \in A\times A, t_n(a) = s_n(b)\}$ to A ((a,b) being carried over $a *_n b$) which satisfies the following axioms for all α and β in $\{-,+\}$:*

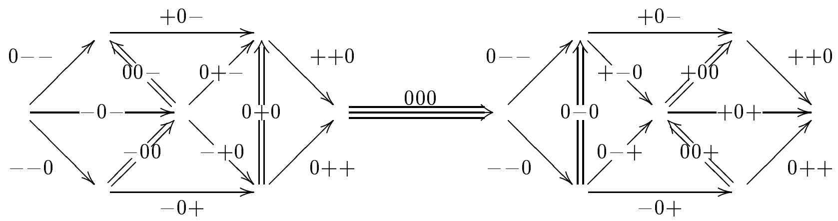
1. $d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x \text{ if } m < n \\ d_n^\alpha x \text{ if } m \geqslant n \end{cases}$
2. $s_n x *_n x = x *_n t_n x = x$
3. *if $x *_n y$ is well-defined, then $s_n(x *_n y) = s_n x$, $t_n(x *_n y) = t_n y$ and for $m \neq n$, $d_m^\alpha(x *_n y) = d_m^\alpha x *_n d_m^\alpha y$*
4. *as soon as the two members of the following equality exist, then $(x *_n y) *_n z = x *_n (y *_n z)$*
5. *if $m \neq n$ and if the two members of the equality make sense, then $(x *_n y) *_m (z *_n w) = (x *_m z) *_n (y *_m w)$*



(a) Composition of two 2-morphims



(b) The ω-category Δ^2



(c) The ω-category I^3

Figure 8: Some ω-categories (a k-fold arrow symbolizes a k-morphism)

6. *for any x in A, there exists a natural number n such that $s_n x = t_n x = x$ (the smallest of these numbers is called the dimension of x and is denoted by $dim(x)$).*

A n-dimensional element of $\mathcal{C}$ is called a n-morphism. A 0-morphism is also called a state of $\mathcal{C}$, and a 1-morphism an arrow. If x is a morphism of an ω-category $\mathcal{C}$, we call $s_n(x)$ the n-source of x and $t_n(x)$ the n-target of x. The category of all ω-categories (with the obvious morphisms) is denoted by ωCat. The corresponding morphisms are called ω-functors. The set of n-dimensional morphisms of $\mathcal{C}$ is denoted by $\mathcal{C}_n$.

As fundamental examples of ω-categories, there is the ω-category Δ^n freely generated by the faces of the n-simplex [Str87]. To characterize this ω-category, the first step consists of labeling all faces of the n-simplex. Its faces are indeed in bijection with strictly increasing sequences of elements of $\{0, 1, \ldots, n\}$. A sequence of length $p+1$ will be of dimension p. If x is a face, let $R(x)$ be the set of faces of x seen as a sub-simplex. If X is a set of faces, then let $R(X) = \bigcup_{x \in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$ and that $R(\{x\}) = R(x)$. Then Δ^n is the free ω-categories generated by the $R(x)$ with the rules

1. For x p-dimensional with $p \geq 1$, $s_{p-1}(R(x)) = R(s_x)$ and $t_{p-1}(R(x)) = R(t_x)$ where s_x and t_x are the sets of faces defined below.

2. If X and Y are two elements of Δ^n such that $t_p(X) = s_p(Y)$ for some p, then $X \cup Y$ belongs to Δ^n and $X \cup Y = X *_p Y$.

Let us give the definition of s_x and t_x on some example :

$$s_{(04589)} = \{(4589), (0489), (0458)\}$$

The elements in odd position are removed ;

$$t_{(04589)} = \{(0589), (0459)\}$$

The elements in even position are removed.

Let $\underline{\Delta}$ be the unique small category such that a pre-sheaf over $\underline{\Delta}$ is exactly a simplicial set [May67, Wei94]. The category $\underline{\Delta}$ has for objects the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$ and has for morphisms the non-decreasing monotone functions. One is used to distinguishing in this category the morphisms $\epsilon_i : [n-1] \to [n]$ and $\eta_i : [n+1] \to [n]$ defined as follows for each n and $i = 0, \ldots, n$:

$$\epsilon_i(j) = \left\{ \begin{array}{ll} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{array} \right\}, \quad \eta_i(j) = \left\{ \begin{array}{ll} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{array} \right\}$$

The mapping $n \mapsto \Delta^n$ yields a functor from $\underline{\Delta}$ to ωCat by setting $\epsilon_i \mapsto \Delta^{\epsilon_i}$ and $\eta_i \mapsto \Delta^{\eta_i}$ where

- for any face $(\sigma_0 < \cdots < \sigma_s)$ of Δ^{n-1}, $\Delta^{\epsilon_i}(\sigma_0 < \cdots < \sigma_s)$ is the only face of Δ^n having $\epsilon_i\{\sigma_0, \ldots, \sigma_s\}$ as set of vertices ;

- for any face $(\sigma_0 < \cdots < \sigma_r)$ of Δ^{n+1}, $\Delta^{\eta_i}(\sigma_0 < \cdots < \sigma_r)$ is the only face of Δ^n having $\eta_i\{\sigma_0, \ldots, \sigma_r\}$ as set of vertices.

Therefore

Definition C.3. *Let $\mathcal{C}$ be an ω-category. Then the graded set $\omega Cat(\Delta^*, \mathcal{C})$ is naturally endowed with a structure of simplicial sets. It is called the simplicial nerve of $\mathcal{C}$.*

D ω-categorical realization of a precubical set

Intuitively the ω-categorical realization $\Pi(K)$ of a precubical set K (also called the free ω-category generated by K) as defined below contains as n-morphisms all composites (or all concatenations) of cubes of K which are n-dimensional (this means that somewhere in the composite, a n-dimensional cube appears). In particular the 1-morphisms of $\Pi(K)$ will be exactly all arrows of K and all possible compositions of these arrows.

The free ω-category $\Pi(K)$ is constructed as follows. The main ingredient is the free ω-category I^n generated by the faces of the n-cube. Its characterization is very similar to that of the ω-category Δ^n generated by the faces of the n-simplex. The faces of the n-cube are labeled by the word of length n in the alphabet $\{-,0,+\}$, the number of zero corresponding to the dimension of the face. Everything is similar, except the definition of s_x and t_x. The set s_x is the set of sub-faces of the faces obtained by replacing the i-th zero of x by $(-)^i$, and the set t_x is the set of sub-faces of the faces obtained by replacing the i-th zero of x by $(-)^{i+1}$. For example, $s_{0+00} = \{\text{-+00}, \text{0++0}, \text{0+0-}\}$ and $t_{0+00} = \{\text{++00}, \text{0+-0}, \text{0+0+}\}$. Figure 8(c) represents the free ω-category generated by the 3-cube (cf. [Gau00] for some examples of calculations). The first construction of I^n is due to Aitchison in [Ait86].

Then to each $x \in K_n$, we associate a copy of I^n denoted by $\{x\} \times I^n$ whose corresponding faces will be denoted by $(x, k_1 \dots k_n)$. We then take the quotient of the direct sum of these $\{x\} \times I^{dim(x)}$ in ωCat (which corresponds for the underlying sets to the disjoint union) by the relations

$$(\partial_i^\alpha(x), k_1 \dots k_{n-1}) \sim (x, k_1 \dots k_{i-1}[\alpha]_i k_i \dots k_{n-1})$$

for any $n \geq 1$ and any $x \in K_n$ where the notation $[\alpha]_i$ means that α is put in i-th position. This expression means that in the copy of I^{n-1} corresponding to $\partial_i^\alpha(x)$, the face $k_1 \dots k_{n-1}$ must be identified to the face $k_1 \dots k_{i-1}[\alpha]_i k_i \dots k_{n-1}$ in the copy of I^n corresponding to x. And one has

Proposition D.1. *One obtains a well-defined ω-category $\Pi(K)$ and Π induces a well-defined functor from the category of precubical sets to that of ω-categories.*

The proof uses the coend construction (cf. [Mac71]).

E Localization of a category with respect to a collection of morphisms

Definition E.1. *Let $\mathcal{C}$ be a category (not necessarily small). Let S be a collection of morphisms of $\mathcal{C}$. Consider the following universal problem :*

> *"There exists a pair $(\mathcal{D}, \mu)$ such that μ is a functor from $\mathcal{C}$ to $\mathcal{D}$ and such that for any $s \in S$, $\mu(s)$ is an invertible morphism of $\mathcal{D}$."*

Then the solution $(\mathcal{C}[S^{-1}], Q)$, if there exists, is called the localization of $\mathcal{C}$ with respect to S.

F The biglobular nerve

Theorem F.1. *[Gau01a] Let $\mathcal{C}$ be a non-contracting ω-category.*

1. *Let x be an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$ for some $n \geq 0$. Then the set maps $(\sigma_0 \dots \sigma_r) \mapsto s_0 x((\sigma_0 \dots \sigma_r))$ and $(\sigma_0 \dots \sigma_r) \mapsto t_0 x((\sigma_0 \dots \sigma_r))$ from the underlying set of faces of Δ^n to $\mathcal{C}_0$ are constant. The unique value of $s_0 \circ x$ is denoted by $S(x)$ and the unique value of $t_0 \circ x$ is denoted by $T(x)$.*

2. *For any pair (α, β) of 0-morphisms of $\mathcal{C}$, for any $n \geq 1$, and for any $0 \leq i \leq n$, then $\partial_i \left(\mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])\right) \subset \mathcal{N}^{gl}_{n-1}(\mathcal{C}[\alpha,\beta])$.*

3. *For any pair (α, β) of 0-morphisms of $\mathcal{C}$, for any $n \geq 0$, and for any $0 \leq i \leq n$, then $\epsilon_i \left(\mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])\right) \subset \mathcal{N}^{gl}_{n+1}(\mathcal{C}[\alpha,\beta])$.*

4. *By setting, $G^{\alpha,\beta}\mathcal{N}^{gl}_n(\mathcal{C}) := \mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])$ for $n \geq 0$ and $G^{\alpha,\beta}\mathcal{N}^{gl}_{-1}(\mathcal{C}) := \{(\alpha,\beta),(\beta,\alpha)\}$, one obtains a $(\mathcal{C}_0 \times \mathcal{C}_0)$-graduation on the globular nerve ; in particular, one has the direct sum of augmented simplicial sets*

$$\mathcal{N}^{gl}_*(\mathcal{C}) = \bigsqcup_{(\alpha,\beta)\in\mathcal{C}_0\times\mathcal{C}_0} G^{\alpha,\beta}\mathcal{N}^{gl}_*(\mathcal{C})$$

   and $G^{\alpha,\beta}\mathcal{N}^{gl}_*(\mathcal{C}) = \mathcal{N}^{gl}_*(\mathcal{C}[\alpha,\beta])$.

Let $\mathcal{C}$ be a non-contracting ω-category. Using Theorem F.1, recall that for some ω-functor x from Δ^n to $\mathbb{P}\mathcal{C}$, one calls $S(x)$ the unique element of the image of $s_0 \circ x$ and $T(x)$ the unique element of the image of $t_0 \circ x$. If (α, β) is a pair of $\mathcal{N}^{gl}_{-1}(\mathcal{C})$, set $S(\alpha,\beta) = \alpha$ and $T(\alpha,\beta) = \beta$.

Proposition F.2. *[Gau01a] Let $\mathcal{C}$ be a non-contracting ω-category. Let x and y be two ω-functors from Δ^n to $\mathbb{P}\mathcal{C}$ with $n \geq 0$. Suppose that $T(x) = S(y)$. Let $x * y$ be the map from the faces of Δ^n to $\mathcal{C}$ defined by*

$$(x * y)((\sigma_0 \dots \sigma_r)) := x((\sigma_0 \dots \sigma_r)) *_0 y((\sigma_0 \dots \sigma_r)).$$

Then the following conditions are equivalent :

1. *The image of $x * y$ is a subset of $\mathbb{P}\mathcal{C}$.*

2. *The set map $x * y$ yields an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$ and $\partial_i(x * y) = \partial_i(x) * \partial_i(y)$ for any $0 \leq i \leq n$.*

On contrary, if for some $(\sigma_0 \dots \sigma_r) \in \Delta^n$, $(x * y)((\sigma_0 \dots \sigma_r))$ is 0-dimensional, then $x * y$ is the constant map $S(x) = T(y)$.

In the sequel, we set $(\alpha,\beta)*(\beta,\gamma)=(\alpha,\gamma)$, $S(\alpha,\beta)=\alpha$ and $T(\alpha,\beta)=\beta$. If x is an ω-functor from Δ^n to $\mathbb{P}\mathcal{C}$, and if y is the constant map $T(x)$ (resp. $S(x)$) from Δ^n to $\mathcal{C}_0$, then set $x*y:=x$ (resp. $y*x:=x$).

Theorem F.3. *Suppose that $\mathcal{C}$ is an object of ωCat_1. Then for $n\geq 0$, the operations S, T and $*$ allow to define a small category $\underline{\mathcal{N}^{gl}_n(\mathcal{C})}$ whose morphisms are the elements of $\mathcal{N}^{gl}_n(\mathcal{C})\cup\{$constant maps $\Delta^n\to\mathcal{C}_0\}$ and whose objects are the 0-morphisms of $\mathcal{C}$. If $\underline{\mathcal{N}^{gl}_{-1}(\mathcal{C})}$ is the small category whose morphisms are the elements of $\mathcal{C}_0\times\mathcal{C}_0$ and whose objects are the elements of $\mathcal{C}_0$ with the operations S, T and $*$ above defined, then one obtains (by defining the face maps ∂_i and degeneracy maps ϵ_i in an obvious way on $\{$constant maps $\Delta^n\to\mathcal{C}_0\}$) this way an augmented simplicial object $\underline{\mathcal{N}^{gl}_*}$ in the category of small categories.*

By composing by the classifying space functor of small categories (cf. for example [Qui73] for further details), one obtains a bisimplicial set which is called the *biglobular nerve*.

Cyclic and Partial Order Models for Concurrency

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July 24, 2001

Abstract

Ternary cyclic order relations can serve as models for non-sequential periodic processes iff they are *orientable*. We use translational symmetries and *windings* to give a new, twofold, characterization of orientable cyclic orders, connecting orientability to partial orders and to separation properties. Applied to Petri nets, windings of causal nets yield cyclic orders on synchronization graphs that describe the concurrent behavior of corresponding live and safe markings. The article finally outlines connections between cyclic order theory and dihomotopy.

1 Introduction

Unlike acyclic (partial) orders, cyclic orders *CyO's* are not binary relations; rather, they are modeled as sets of *triples* (x, y, z) that satisfy the system $\mathcal{A}$ of axioms

1. *inversion asymmetry*: $\prec (x, y, z) \Rightarrow \neg \prec (y, x, z)$,
2. *rotational symmetry*: $\prec (x, y, z) \Rightarrow \prec (z, x, y)$,
3. *ternary transitivity*: $[\prec (a, b, c) \wedge \prec (a, c, d)] \Rightarrow \prec (a, b, d)$.

If $(\mathcal{X}, \prec)$ is a CyO, then $\prec$ is simple: if $\prec (x, y, z)$ then $x \neq y$, $y \neq z$, and $z \neq x$. Note that ternary transitivity *resembles* binary transitivity (project onto the last two components); however, binary transitive relations are not adequate to express cyclic ordering. One verifies easily that, in the examples given below, any transitive binary extension of the arc relation yields the *full* relation $\mathcal{X} \times \mathcal{X}$, which contains no information at all. – In the simplest case, a cyclic order consists of just one cycle of elements; we will call this a *total cyclic order*. In the general case, there exist pairs of incomparable elements, i.e. pairs not contained in a cyclically ordered triple. There, the question arises whether it is possible to find a *totalization*, that is, a total cyclic order consistent with all ternary cyclic arrangements; by Szpilrajn's Theorem [Szp30], the analogous property always holds for acyclic partial orders.

For cyclic orders according to the above system $\mathcal{A}$ 1-3 of axioms, a total extension need not exist; examples were given by Megiddo [Meg76] and others, see also below. We call these CyO's *non-orientable*; Megiddo [Meg76] showed NP-hardness for the problem of *orientability*.

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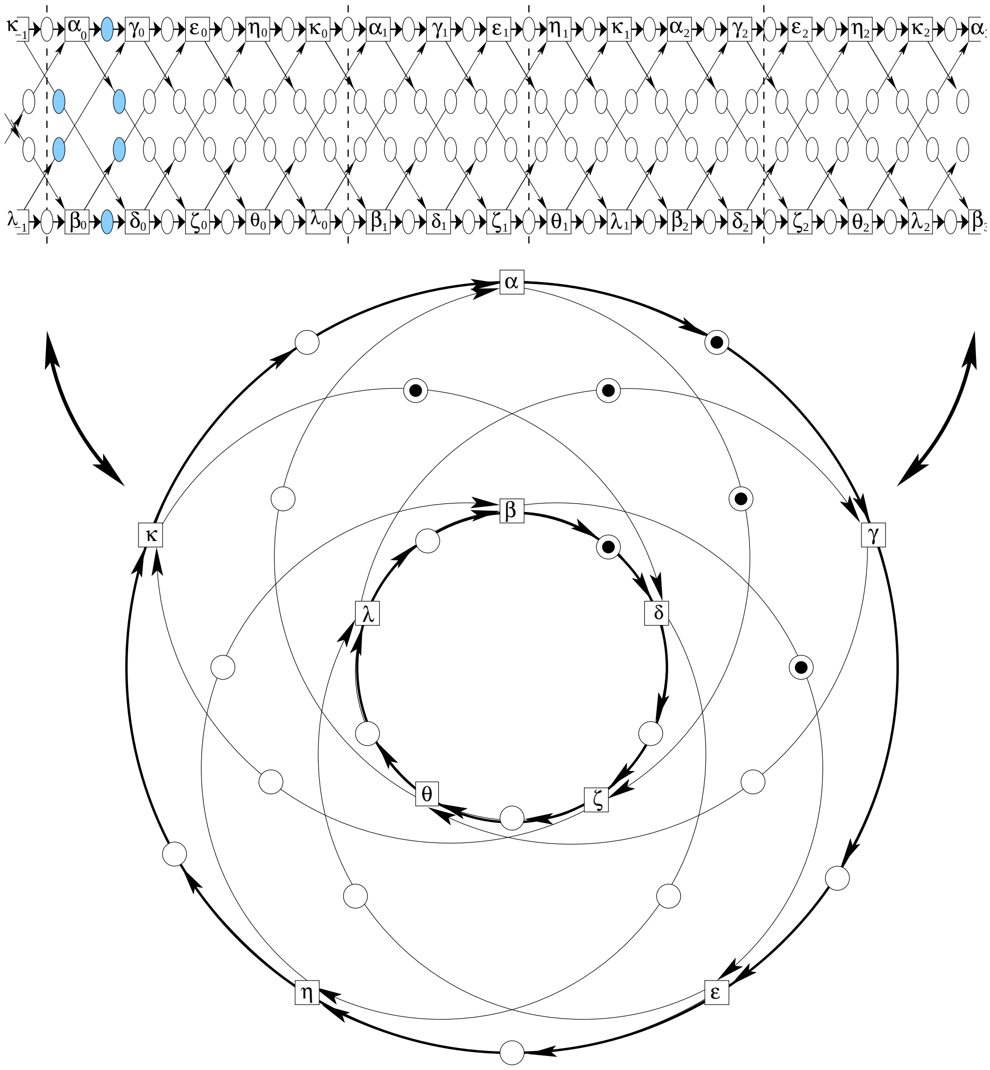

Figure 1: Winding (down) and unwinding (up)

Combinatorial characterizations of orientable cyclic orders have been given by several authors, [Gen71, ChN83, Qui89, Qui91, ANP91, Jak94, Ste98]. The approach by *windings* that we here differs from the above and connects cyclic orders to posets.

Consider Figure 1. It shows how cyclic orders can be associated to Petri net systems: below, one has a a cyclic *synchronization graph*, i.e. without branching of places, and above a *causal net* (sometimes also called *occurrence net*), i.e., in addition to non-branching places, the flow relation (indicated by arcs) is acyclic. Writing $\langle x_1, \dots, x_n \rangle$ to denote the cyclic arrangement – the *oriented simple cyclic path* or *opath* – from x_1 to x_2 to ... x_n to x_1, we have, for instance, $\langle \alpha, \zeta, \theta \rangle$ and all its rotations, $\langle \zeta, \theta, \alpha \rangle$ and $\langle \theta, \alpha, \zeta \rangle$. These cycles are reflected by a *line* or directed path in the acyclic net sitting on top of Figure 1, namely

$$\dots \to \alpha_0 \to \zeta_0 \to \theta_0 \to \alpha_1 \to \zeta_1 \to \theta_1 \to \alpha_2 \to \zeta_2 \to \theta_2 \to \dots$$

where we ignored the places without loss of information since every place in both nets is uniquely specified by its only pre- and only post-transition. We say (see below) that the *winding* associates every cycle a periodic directed path that passes through the same sequence of equivalence classes in every section of a partial order; the "periodicity" meaning invariance under some automorphism of the partial order.

Some further remarks to fix the intuitions. All periodic lines will be wound to cycles, more precisely, closed self-avoiding curves or, in graph theoretic terms, paths. Consider

$$\ldots \to \alpha_0 \to \zeta_0 \to \theta_0 \to \lambda_0 \to \beta_1 \to \theta_1 \to \varepsilon_1 \to \eta_1 \to \kappa_1 \to \alpha_2 \to \ldots .$$

This line is also periodic, yet is invariant under *fewer* automorphism than the above; the corresponding closed curve, call it $\gimel_2$, has an intuitive winding number of two, whereas the "winding number" of $\gimel_1$ is 1. This is reflected by the numbers of *tokens*, to which we direct our attention now: the marking of the net in the lower part of Figure 1 is live and 1-bounded. From classical results by Genrich and Lautenbach on synchronization graphs, we know that these markings are exactly those under which (i) every cycle contains at least one token, and (ii) every node lies on at least one cycle that contains *exactly* one token (see below). The latter type of cycles – called *elementary* cycles – are thus of particular interest; note that all their nodes are pairwise causally connected. In Figure 1, the two simplest elementary cycles are highlighted by thick arrows; there are 15 more. In contrast, cycles with two or more tokens – see the example – contain pairs of *concurrent* elements.

These observations show the connection between Petri net theory on the one hand and cyclic orders in the other. The next section gives the main characterization result from [Haa00]; Section 3 then takes a first look at the connection between cyclic orders and dipaths.

2 Orientable Cyclic Orders

Periodic concurrent processes generate CyO's; does, on the other hand, every Cyo also permit a physical interpretation ? Is there a *global sense of rotation* in the CyO as there is in periodic behavior ? In general, the answer is no; the question is in fact that of *orientation* as above. The existence of a totalization for Φ is equivalent to Φ having a graphical representation by *clock cycles*, i.e. as a collection of directed loops in the two-dimensional plane such that the origin is avoided and such that all loops run clockwise around the origin[1]; totalizable CyO's are therefore also called *globally oriented.* The central statement of this paper can be summed up as follows:

A CyO is globally oriented
if and only if
it can be represented by
concurrent periodic deterministic processes.

2.1 Posets and Cyclic Orders

Some terminology: if $, = (\mathcal{X}, <)$ is a partial order, let $\mathsf{li} := < \cup <^{-1}$ denote the relation of causal connection, $\mathsf{id}_{\mathcal{X}} := \{(x,x) \mid x \in \mathcal{X}\}$ that of identity, and $\mathsf{co} := \mathcal{X}^2 \Leftrightarrow (\mathsf{id}_{\mathcal{X}} \cup \mathsf{li})$ that of concurrency. A maximal clique of li – i.e. a maximal dipath – is called a *line*, and the set

[1] cf. the *arc orders* in Alles, Nešetřil, Poljak [ANP91].

of lines in , is denoted by $\mathsf{Lines}(,)$; the elements of $\mathsf{Cuts}(,)$ are the maximal cliques of co, called *cuts*, of , . Similarly, in a CyO, set $\mathsf{li} := \{(x,y) \mid \exists z : \ (\prec(x,y,z) \vee \prec(x,z,y))\}$ and $\mathsf{co} := \mathcal{X}^2 \Leftrightarrow (\mathsf{id}_{\mathcal{X}} \cup \mathsf{li})$, and denote the maximal cliques of li as *lines* and those of co as *cuts*. Note that all lines have at least three elements.

So, in the above example, $\beth_1$ – complete with the corresponding places – is a line, but $\beth_2$ is not.

Now, we ask for minimal additional properties that, together with the above axioms, ensure orientability. We will, in the following, consider only li-*oriented* CyO's, i.e. such that for any $\{a,b,c\} \in \mathcal{CLI}(\mathsf{li})$, either $\prec(a,b,c)$ or $\prec(b,a,c)$. CyO's that are not li-oriented may be completed to a CyO containing the corresponding $\prec(a,b,c)$ or $\prec(b,a,c)$; of course, not all CyOs will allow this. It is obvious, however, that only CyO's extensible to a li-oriented one can be globally oriented, and that they are globally oriented iff one of their li-oriented super-CyO's is; so we may restrict our attention to li-oriented CyO's or *LOCyO's*, and add li-orientedness to our axioms.

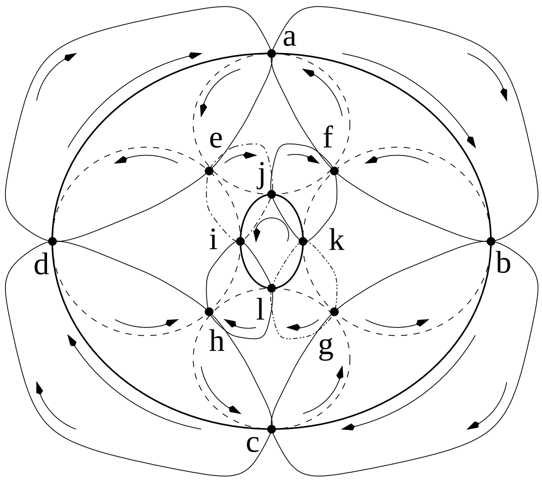

Figure 2: A non-totalizable CyO

Figure 2 shows a LOCyO that is not globally oriented (the example is due to Genrich [Gen71], with completion by Stehr). Thus li-orientation is insufficient for global orientation.

The following subsections introduce the concepts used in the Characterization Theorem, which will then be stated as Theorem 2.5 in Subsection 2.4.

2.2 COWs

We now come back to the idea of *winding* which we sketched above. Let τ be an automorphism of poset , $= (\mathcal{X}, <)$. τ is an *order translation* of , if either

1. $x <\leq \tau(x)$ for all $x \in \mathcal{X}$ *(forward),* or

2. $\tau(x) \leq x$ for all $x \in \mathcal{X}$ *(backward order translation).*

If $\leq$ can be replaced by $<$ in the above, we call τ a *proper* forward/backward order translation. The set of order translations plus the identity mapping of , form a group (with concatenation); f is a forward order translation iff its inverse is a backward order translation, and vice versa. Let τ be a proper forward order translation, and $\mathcal{G}$ be the group generated by τ acting on Ψ; $\mathcal{G}$ is isomorphic to $(\mathbb{Z}, +)$. Write $\overline{x} \equiv_\tau \overline{y}$ iff there exists $k \in \mathbb{Z}$ such that $\tau^k(\overline{x}) = \overline{y}$; then $\equiv_\tau$ is an equivalence relation on $\overline{\mathcal{X}}$, and the $\mathcal{G}$-orbit of x is $[x]_{\equiv_\tau}$ Let $\mathcal{X} := \overline{\mathcal{X}}_{/\equiv_\tau}$, and $\phi_\tau : \overline{\mathcal{X}} \to \mathcal{X}$, $\overline{x} \mapsto [x]_{\equiv_\tau}$ the associated quotient map. The *cyclic order winding* (COW) of Ψ is , $= (\mathcal{X}, \prec)$, where $\prec$ is given by:

$$\begin{array}{rcl} \prec(x,y,z) & :\Leftrightarrow & \exists\, \overline{x} \in \phi_\tau^{-1}(x),\ \overline{y} \in \phi_\tau^{-1}(y),\ \overline{z} \in \phi_\tau^{-1}(z) : \\ & & \overline{x} < \overline{y} < \overline{z} < \tau(\overline{x}). \end{array}$$

In the first figure, $\prec(\alpha, c, \gamma)$ holds since $\alpha_0 < c_0 < \gamma_0 < \alpha_1$, etc. It is easy (yet tedious) to verify that COWs are LOCyO's. Now, if Φ is a COW thus generated from a PO Ψ, we say that Ψ is *wound* to Φ and call ϕ_τ the associated *winding map.*

Windings abstract a cyclic order from a partial order that has translation symmetries, i.e. that is *periodic.* If a partial order admits one winding ϕ_τ, it admits infinitely many, namely at least ϕ_{τ^k} for all $k \in \mathbb{N}$. On the other hand, the representation of a cyclic order by a winding need not be unique (not even up to partial order isomorphisms).

Remark 2.1 *In Alles, Nešetřil, Poljak [ANP91], a CyO is generated from a poset , by simply taking the rotational (symmetric) closure; that is, set*

$$\prec^\circ \ := \ \{(a,b,c) \mid a \ < \ b \ < \ c\},$$

and let $\prec$ be the smallest superset of $\prec^\circ$ that is rotationally symmetric, i.e., $(x,y,z) \in \prec$ implies $(y,z,x) \in \prec$. This is not at all *equivalent to windings. There is an obvious difference and a more subtle one: Obviously, no abstraction takes place in the rotational closure process, so the cyclic order has as many elements as its generating poset, whereas a all pre-images under windings are infinite. But even the restriction to one section of the wound poset does not yield an isomorphic cyclic order: in Figure 1, consider only the elements with index* 0. *Then the cyclic order generated by rotational closure contains the triple $(\alpha_0, \gamma_0, \lambda_0)$, but $(\alpha, \gamma, \lambda)$ does not belong to the cyclic order winding since* $\mathsf{co}(\lambda_0, \alpha_1)$. *Concurrency is not respected by the rotational closure but is essential for windings.*

For a cyclic order, being representable as a COW is equivalent to orientability ([Haa01, Haa00]), see below.

2.3 Density Properties

A cut $\mathfrak{c}$ in a partial order can be viewed as a *global* state of the set of local processes that are represented by lines. The intersection of $\mathfrak{c}$ with a line $\mathfrak{l}$ then yields the local state of $\mathfrak{l}$ when the 'snapshot' $\mathfrak{c}$ is taken. From the semantic point of view, one is therefore interested to know whether $\mathfrak{l}$ and $\mathfrak{c}$ *do* intersect in the first place; more generally, density properties provide deeper insight into the order structure.

Definition 2.2 *Let* , $=$ $(\mathcal{X}, <)$ *be a partial order. Then we say that* $\mathfrak{c} \in \mathsf{Cuts}(,)$ *is* K-separating[2] *iff* $\mathfrak{c} \cap \mathfrak{l} \neq \emptyset$ *for all* $\mathfrak{l} \in \mathsf{Lines}(,)$. , *is* weakly K-dense *iff it contains a K-separating cut, and* (strongly) K-dense *iff every cut of* , *is K-separating.*

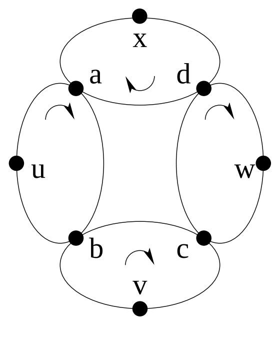

Figure 3: On K-/Q-density

[2]called a *true cut* in[Ste98]

The *density* terminology is due to Petri; *separating* has been introduced for the detailed study of intersection properties in [Haa01, Haa00]. Density properties of causal nets have been extensively studied, see for instance [BF88]. We note for our purposes that a causal net all of whose cuts are finite is K-dense; see, for instance, the one sitting on top in Figure 1.

Similarly, in a *cyclic* order, we say that $\mathfrak{c} \in \mathsf{Cuts}(,)$ is *K-separating* iff $\mathfrak{c} \cap \mathfrak{l} \neq \emptyset$ for all $\mathfrak{l} \in \mathsf{Lines}(,)$. , is *weakly K-dense* iff it contains a K-separating cut, and *(strongly) K-dense* iff every cut of , is K-separating.

Note that, in the context of cyclic orders, the notions of *cuts* and *lines* carry over, yet we need also to distinguish *lines* from *circuits*.

Intervals and *edges* can be defined in a similar fashion for posets and cyclic orders: Let , $= (\mathcal{X}, <)$ be a poset and $\Phi = (\mathcal{Y}, \prec)$ a LOCyO.

1. For $a, b \in \mathcal{X}$ such that $a < b$, define the following intervals:

$$
\begin{aligned}
&]a, b[:= \{x \in \mathcal{X} : \ a < x < b)\}, \\
\text{and} \quad &[a, b[\ := \ ]a, b[\ \cup \{a\}, \ ]a, b] \ := \ ]a, b[\ \cup \{b\}, \ [a, b] \ := \ ]a, b[\ \cup \{a, b\}.
\end{aligned}
$$

2. For $a, b \in \mathcal{Y}$ such that $\mathsf{li}(a, b)$, define:

$$
\begin{aligned}
&]a, b[:= \{x \in \mathcal{X} : \prec (a, x, b)\}, \\
\text{and} \quad &[a, b[\ := \ ]a, b[\ \cup \{a\}, \ ]a, b] \ := \ ]a, b[\ \cup \{b\}, \ [a, b] \ := \ ]a, b[\ \cup \{a, b\}.
\end{aligned}
$$

In either case, an *edge* of Φ is a li-clique $\mathcal{E}$ such that there exist $a, b \in \mathcal{X}$ satisfying $\mathsf{li}(a, b)$ and $\mathcal{E} \subseteq [a, b]$, and for any $u \in [a, b]$ such that $\forall\, v \in \mathcal{E} : \mathsf{li}(v, u)$ one has $u \in \mathcal{E}$. Set $\mathfrak{start}(\mathcal{E}) := a$ and $\mathfrak{end}(\mathcal{E}) := b$.

In both cases, if $\mathcal{E}$ is an edge, $\mathfrak{start}(\mathcal{E}) \in \mathcal{E}$ and $\mathfrak{end}(\mathcal{E}) \in \mathcal{E}$, and every edge $\mathcal{E}$ can be represented as the intersection of an appropriate line $\mathfrak{l}_{\mathcal{E}}$ with $[\mathfrak{start}(\mathcal{E}), \mathfrak{end}(\mathcal{E})]$.

Using edges, we can state the following definition that generalizes the graph theoretic notion of a circuit:

Definition 2.3 (Circuits of a LOCyO) *Let $\Phi = (\mathcal{X}, \prec)$ be a LOCyO . $\mathcal{C} \subseteq \mathcal{X}$ is a* circuit *of Φ iff there exists $n \geqslant 2$ and edges $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that, for $1 \leqslant i \leqslant n \Leftrightarrow 1$, $start(\mathcal{E}_1) = \mathfrak{end}(\mathcal{E}_n)$ and $\mathfrak{start}(\mathcal{E}_{i+1}) = \mathfrak{end}(\mathcal{E}_i)$, and $\mathcal{C} = \bigcup_{i=1}^{n-1} \mathcal{E}_i$. $\mathsf{Circuit}(\Phi)$ denotes the set of circuits of Φ.*

Every line in Φ is a circuit (note that no line contains fewer than three elements). In general, however, not every circuit is a line of Φ, as Figure 1 shows. For an extreme example, take

$$\daleth_3 \quad : \quad \langle \alpha, \zeta, \kappa, \delta, \eta, \beta, \varepsilon, \lambda, \gamma, \theta \rangle;$$

this circuit is obviously not a line since, e.g., $\mathsf{co}(\alpha, \beta)$; the places along $\daleth_3$ contain as many as four tokens.

Definition 2.4 *Let Φ be a LOCyO . A cut $\mathfrak{c}$ is called* Q-separating *iff, for all $\mathfrak{q} \in \mathsf{Circuit}(\Phi)$, $\mathfrak{c} \cap \mathfrak{q} \neq \emptyset$; Φ is called* weakly Q-dense[3] *iff there exists a Q-separating cut $\mathfrak{c} \in \mathsf{Cuts}(\Phi)$, and* strongly Q-dense *iff all its cuts are Q-separating. If Φ has a weakly (strongly) Q-dense elementary LOCyO extension, it is called* (strongly) saturable.

[3]In the special case of nets, Q-density has been introduced as 'F-density' in [KS97], [Ste98]

Thus Q-separation implies K-separation etc., but the converse is not true: Figure 3 shows a cyclic order with a cut $\{u, v, w, x\}$ that is obviously K-separating but fails to intersect the circuit $\langle a, b, c, d\rangle$. Note that the structure is nonetheless weakly Q-dense since $\{a, c\}$ is a Q-separating cut.

Since a total Cyo contains only one circuit, every singleton is a Q-separating cut; hence every total Cyo is strongly Q-dense.

2.4 Characterization of Global Orientation

We are now ready to state the following characterization theorem:

Theorem 2.5 *For a LOCyO $\Phi = (\mathcal{X}, \prec)$, the following are equivalent:*

1. *Φ is saturable;*
2. *Φ has a representation as a* COW*;*
3. *Φ has a totalization.*

Proof: We only sketch the ideas for this lengthy proof; for the details, see ([Haa01, Haa00]). The implication 3)→1) is shown by giving a Q-separating cut, first using the axiom of choice to saturate every line of Φ and then showing Q-separation. From 1) to 2), one "cuts Φ open" at the Q-separating cut; one then glues together infinitely many copies of the resulting acyclic structure to obtain a winding representation. Finally, using 2), one applies Szpilrajn's Theorem to obtain linear extensions of the unwinding, from which to construct a totalization of the winding and shows 3). □

The non-totalizable LOCyO 's of Figure 2 is not saturable and, a fortiori, *not* weakly Q-dense itself; this can be verified by explicitly inspecting all circuits and cuts.

In the case of finite $\mathcal{X}$, we retrieve from Theorem 2.5 the classical result by Genrich and Lautenbach [GL73]: A synchronization graph $\mathcal{N}$ is l.s. iff

1. all its circuits[4] contain at least one token, and
2. the net is covered by circuits containing exactly one token each.

For this, one constructs an unwinding as in Figure 1; that has all reachable markings as P-cuts (in Figure 1, a P-cuts corresponding to the initial marking below is indicated by the shaded places in the causal net). For the dynamics, the LOCyO's structure ensures that every place is contained in at least one line $\mathfrak{l}$, and since $\mathfrak{l}$ can intersect any cut in at most one element, there is never more than one token on $\mathfrak{l}$; weak Q-density, on the other hand, corresponds to the absence of unmarked circuits (where we allow transitions to 'contain one token' during their firing). So we ask whether, for weakly discrete LOCyO's on *nets*, saturability implies weak Q-density of the original LOCyO. The affirmative answer is given by Theorem 2.6 below; thus Genrich and Lautenbach's result is in fact the Petri Net interpretation of the finite case of Theorem 2.5.

Theorem 2.6 *Let $\Phi = (\mathcal{X}, \prec)$ be the net version of a weakly discrete* LOCyO*. If Φ is saturable, then Φ is itself weakly Q-dense.*

Proof: See ([Haa01, Haa00]). □

[4] in the graph theoretic sense, this coincides here with the circuits of the CyO

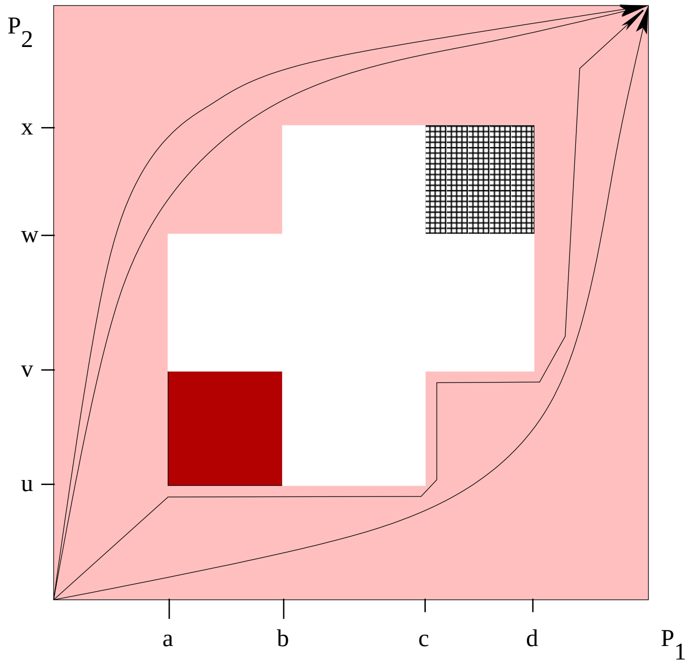

Figure 4: A concurrent system of two finite processes

3 Cyclic Orders without Petri nets

We saw that the above characterization of oriented cyclic orders is a natural extension of classical results from Petri Net theory; here, we outline how they can also be applied to concurrent systems and their dipaths.

3.1 Products of Cyclic Orders

For $\Phi_i = (\mathcal{X}_i, \prec_i)$, where $i \in \mathcal{I}$, define the product

$$\Phi \quad := \quad \bigotimes_{i \in \mathcal{I}} \Phi_i$$

as follows: $\Phi = (\mathcal{X}, \prec)$, where $\mathcal{X}$ is the Cartesian product of the $\mathcal{X}_i$, and $\prec$ given by: for all $x, y, z \in \mathcal{X}$, with $x = (x_i)_{i \in \mathcal{I}}$, $y = (y_i)_{i \in \mathcal{I}}$, $z = (z_i)_{i \in \mathcal{I}}$

$$\prec (x, y, z) \quad \Longleftrightarrow \quad \forall\, i \;\in\; \mathcal{I} : \prec_i (x_i, y_i, z_i)\,;$$

Now, it is straightforward to check that $\prec$ is a cyclic order; the product also respects and preserves global orientation:

Theorem 3.1 *In the above construction, Φ is globally oriented iff Φ_i is globally oriented for all $i \in \mathcal{I}$.*

Proof: *If*: For every $i \in \mathcal{I}$, let $\tilde{\Phi}_i = (\mathcal{X}_i, \tilde{\prec})$ be a totalization of Φ_i. One verifies that

$$\tilde{\Phi} \quad := \quad \bigotimes_{i \in \mathcal{I}} \tilde{\Phi}_i$$

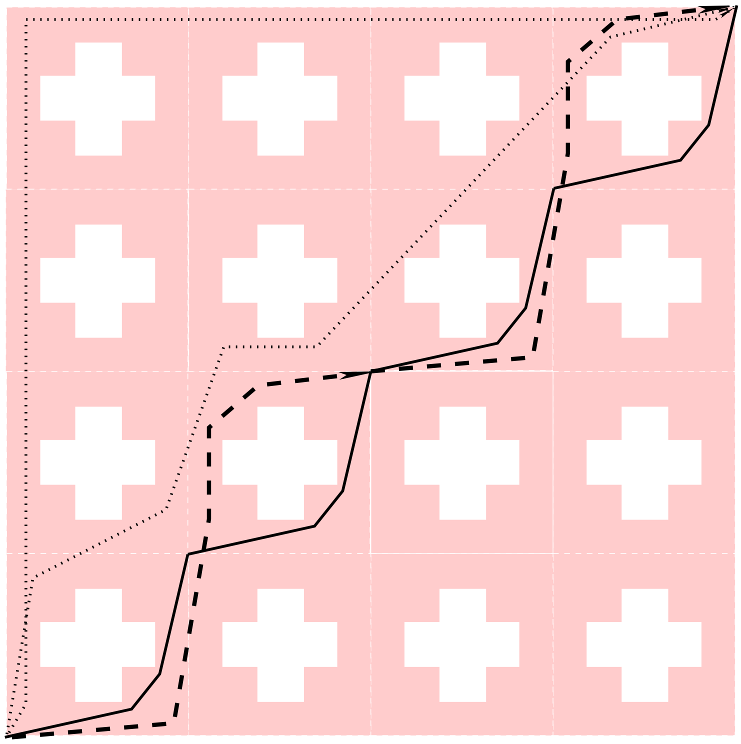

Figure 5: On winding and dipaths

is a totalization of Φ. Similarly, for the *Only if* part, take a totalization $\tilde{\Phi}$ of Φ and its projections $\tilde{\Phi}_i$ to the components $i \in \mathcal{I}$. Then $\tilde{\Phi}_i$ is a totalization of Φ. □

In the rest of the paper, we will assume for simplicity that $\mathcal{I}$ is finite. Then, using Theorem 2.5, unwindings can be produced for Φ from those for the $i \in \mathcal{I}$, by subsequently unwinding in each component dimension: let Ψ_i the partial order from which Φ_i is wound by ϕ_{τ_i} for some automorphism τ_i of Ψ_i. Then for the product poset

$$\Psi \quad := \quad \bigotimes_{i \in \mathcal{I}} \Psi_i$$

and all i, τ_i induces an automorphism $\overline{\tau}_i$ of Ψ. The concatenation of *all* $\overline{\tau}_i$ yields a winding of Ψ to Φ, where the order of applying the $\overline{\tau}_i$ is indifferent.

3.2 Windings for Dihomotopy

The connection between cyclic order theory and dihomotopy has not yet been fully elaborated; the following are some indications.

Recall that concurrent systems can be modeled as products of partial (in fact, typically *linear*) orders, each of which representing the phases of one of the processes. The possible evolutions of the system are then given by dipaths leading from the bottom to the top element. Further, for instance, forbidden, unsafe, and inaccessible regions are obtained from the order intervals: in Figure 4, we have two processes P_1 and P_2, with the (white) forbidden region given by

$$\{P_1 \ \in \ [a,d],\ P_2 \ \in [v,w]\} \ \ \cup \ \ \{P_1 \ \in \ [b,c],\ P_2 \ \in [u,x]\} \tag{1}$$

(similar descriptions can be given for the (dark) unsafe and the (checkered) inaccessible regions), as well as two pairs of dihomotopic finite dipaths.

Now, consider the axes of P_1 and P_2 extended both ways to infinity by repetition, see Figure 5. Some allowed dipaths are indicated in the figure; consider their extensions to infinite dipaths invariant under $(\tau_1 \circ \tau_2)^4$. The winding $\phi_{\tau_1 \circ \tau_2}$ creates a cyclic order $\Phi = (\mathcal{X}, \prec)$ on a torus, with connected forbidden region given by (1), where the intervals have to be interpreted in terms of $\prec$ rather than a partial order $<$.

Under $\phi_{\tau_1 \circ \tau_2}$, the only dipath yielding a line of Φ is drawn in solid; the others belong to two other homotopy classes. However, the thick dashed dipath is taken to a line by $\phi_{(\tau_1 \circ \tau_2)^2}$, and the same holds for all four dipaths drawn in Figure 5 under $\phi_{(\tau_1 \circ \tau_2)^4}$.

The approach to looped processes here is different to that taken by Fajstrup [Faj00]. Again, there are inessential and essential differences:

- The systems considered here are necessarily built from cyclic, non-terminating processes, whereas those of [Faj00] have distinguished start- and end- points and contain loops only in the interior. This means no restriction for our approach since each terminating process can be made into a looped one by adding a new least and maximal point, and then identifying both.
- The computation of forbidden, unsafe, inaccessible regions etc. is possible directly on the cyclic structure (once the cyclic order relation is known), without a need for unwinding; that is, the cyclic order is a natural abstraction of the looped concurrent system.

We appeal to the reader's imagination (not disposing of a convincing figure of the torus associated to Figure 5) to see that this is true in the case of the “swiss cross” example; the winding construction preserves the forbidden rectangles (some extra care must be taken when the image of such a rectangles under the winding map does not have the form of a rectangle; here, however, no such problem arises) and their unions, complements, etc.

4 Outlook

Several lines of research can only be mentioned here.

- We note first that the successful application of CyOs to both conflict-free Petri Nets and synchronized products would provide an important link. These two formalisms are not extensionally equal, as Figure 4 shows: there is no persistent (i.e. conflict free) Petri net model generating it.
- There exist cyclic equivalents for *lattices* (as for other order theoretic objects); the structure of the cuts is also transported from a partial order to a cyclic order. Their study involves a closer look at the "tightness" of a winding. If the translation defining the winding does not map its arguments sufficiently far away (with respect to the ordering, not a metric !) from their position, then the winding may cease to be *faithful* ([Haa01, Haa00]) and destroy lattice and other properties. The Cycloids and Orthoids of Petri [Pet96] provide abundant material to further investigation into symmetries and cyclic orders for *nets*.

- Finally, the above characterization of orientability uses a merely order-theoretic notion of circuit, extending directly the graph theoretic one. A characterization in an appropriate topological representation with closed curves should be tried.

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A geometric semantics for dialogue game protocols for autonomous agent interactions

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July 19, 2001

Abstract

Formal dialogue games studied by philosophers since the Middle Ages have recently found application in Artificial Intelligence as the basis for protocols for interactions between autonomous software agents. For instance, such protocols have been proposed for dialogues involving persuasion, negotiation and deliberation. We provide a geometric semantics for these protocols and define a notion of equivalence between protocols. We then demonstrate an algebraic property of equivalence, and use this to show non-equivalence of two similar generic protocols. Our result has implication of the design and evaluation of such dialogue-game protocols.

KEYWORDS: Agent Communication Languages, Argumentation, Autonomous Agents, Computational Dialectics, Dialogue Games.

1 Introduction

Over the last decade, philosophical theories of argument and argumentation have found increasingly-widespread application in Artificial Intelligence [2]. One important application has been for the design of protocols for interactions between autonomous software agents in multi-agent systems, where agents may need to convince others of some claim in order to achieve a desired objective [14]. Such ideas have been operationalized by the use of formal dialogue games adopted from philosophy; these are games between two or more players, where the "moves" made by the players are locutions, i.e. spoken utterances, which adhere to specified rules. Dialogue game protocols have now been proposed for agent interactions involving: persuasion [3, 10]; negotiation [1, 13, 15]; deliberation [5]; information-seeking [6]; and chance discovery [11].[1]

[1]Note that these different types of dialogue have precise definitions according to a standard argumentation theory classification of dialogues [16].

However, most of these protocols have been defined in a stand-alone manner, without comparison to other game protocols and with scant regard for their formal properties. In an effort to remedy this, we have recently initiated discussion over the criteria which may be appropriate for the design and evaluation of such protocols in multi-agent systems [12]. In undertaking this work, we have recognized the need for a formal language in which to explore the properties of dialogue game protocols, and, in particular, to compare one protocol with another. It is such a language we propose in this paper. Our approach is to define the elements of a generic dialogue game protocol in such a way that we can translate any protocol into a geometric representation, specifically a subset of a real, multi-dimensioned space. We are then able to use some simple ideas motivated by algebraic topology to consider the equivalence of two protocols. For reasons of space, some definitions are presented informally or semi-formally.

The paper is structured as follows: Section 2 reprises a formal model of dialogue games from our previous work, and Section 3 defines our geometric semantics for dialogue games protocols and for dialogues conducted according to such games. Section 4 defines a topological notion of equivalence of two protocols, and deduces an algebraic property of this notion. We then demonstrate that two generic protocols which are very similar are in fact not equivalent. Section 5 concludes the paper.

2 A Generic Dialogue-Game

Formal dialogue games have been studied by philosophers since at least the Middle Ages, and were revived in modern times to better understand fallacious modes of reasoning [4, 8] and as a game-theoretic semantics for intuitionistic logic [7]. In earlier work [9], we proposed a formal model for the elements of a dialogue-game, which we reprise here. In this model, it is assumed that the topics of discussion between the participants are represented in some logical language, whose well-formed formulae are denoted by the lower-case Greek letters, ϕ, θ, etc. The dialogue game then consists of several types of rules:

Commencement Rules: Rules which define the circumstances under which the dialogue begins.

Locutions: Rules which indicate what utterances are permitted. Typically, legal locutions permit participants to assert propositions, permit others to question or contest prior assertions, and permit those asserting propositions which are subsequently questioned or contested to justify their assertions. Justifications may involve the presentation of a proof of the proposition or an argument for it, and such presentations may also be legal utterances. In some multi-agent system applications of dialogue games, e.g. [1]), rationality conditions are imposed on utterances, for example allowing agents to assert statements only when they themselves have a prior argument or proof from their own knowledge base.

Combination Rules: Rules which define the dialogical contexts under which particular locutions are permitted or not, or obligatory or not. For instance, it may not be permitted for a participant to assert a proposition ϕ and subsequently the proposition $\neg\phi$ in the same dialogue, without in the interim having retracted the former assertion. Similarly, assertion of a proposition by a participant may oblige that participant to defend it in defined ways following contestation by other participants.

Commitments: Rules which define the circumstances under which participants express commitment to a proposition. Typically, assertion of a claim ϕ in the debate is defined as indicating to the other participants some degree of willingness to defend it in the dialogue if questioned or contested by other participants. Since Hamblin [4], it is common to track commitments in a set of publicly-readable blackboards called Commitment Stores.

Termination Rules: Rules which define the circumstances under which the dialogue ends. These rules may be expressible in terms of the contents of the Commitment Stores of one or more participants; for example, a persuasion dialogue may terminate when all participants utter a locution accepting the proposition at issue, a locution which inserts the proposition into the agent's Commitment Store.

In the dialogue game tradition in philosophy, *commitment* is a dialogical concept: a commitment expresses a willingness by an agent to defend in the dialogue a claim which that agent has previously asserted, and does not necessarily have any connection to any reality external to the dialogue. Thus, for example, acceptance in a dialogue by a participant of some proposition does not necessarily entail belief in that proposition by the participant. Dialogue locutions will only have such connections with external reality if the participants in the dialogue together agree to vest such meanings in the locutions. However, for dialogues of interest to agent designers, particularly negotiations and deliberations, connections between locutions and external reality is important. For example, parties to a negotiation dialogue may agree that its successful conclusion will lead to the execution of a purchase transaction following dialogue termination.

We therefore assume that participants to a dialogue may utter locutions which all participants understand to imply a willingness to execute an action or actions, external and subsequent to the dialogue. We further assume that these actions may be expressed in the same logical language as are the topics of discussion.[2] In the next section, we present a geometric semantics for dialogue-games and dialogues.

[2]The problem of semantic verification of agent communications languages – how may we verify that all participants have the same sincere understanding of an interaction protocol – is a thorny one, which we do not discuss here [17].

3 A Geometric Semantics

We now present a geometric semantics for dialogue games and dialogues. We use $\mathcal{R}^+$ and $\mathcal{Z}^+$ to denote spaces consisting of all non-negative real numbers and integers, respectively. We use the notation $\mathcal{R}^{n+}$ for the n-fold product of $\mathcal{R}^+$, i.e. the sub-space of $\mathcal{R}^n$ where all points have non-negative co-ordinates on each dimension.

3.1 Dialectical systems

We assume a finite set $\mathcal{A} = \{P_i | i = 1, \ldots, p\}$, of dialogue participants, or agents. We denote dialogue games by possibly-subscripted upper case script Roman letters, $\mathcal{D}, \mathcal{E}$, etc. Each dialogue game comprises a finite set of legal locution-types, denoted $\mathcal{L} = \{L_j | j = 1, \ldots, l\}$, and a number of rules which we will consider presently. Dialogues conducted according to the rules of a dialogue game are assumed to concern a fixed, finite set $\Phi = \{\phi_i | i = 1, \ldots, q\}$ of well-formed formulae in some propositional language, which we call the universe of discourse.[3] Some or all of these propositions may represent external actions which can be the subject of commitments incurred by speakers in the dialogue uttering specific locutions, and so we distinguish a specific subset $\Phi_a \subseteq \Phi$, which we call the action set.

We further assume that all locution types can be categorized into one or more of three types: (a) *Commitment locutions:* locutions which express some external or dialogical commitment, e.g. a commitment to execute a purchase action upon completion of the dialogue; these locutions are instantiated with one or more elements from Φ_a and possibly one or more elements of $\Phi \setminus \Phi_a$; (b) *Information locutions:* locutions which transmit some information from speaker to audience, e.g. a statement by a speaker of a preference-ordering over some set of objects; these locutions are instantiated with one or more elements of $\Phi \setminus \Phi_a$; and (c) *Procedural locutions:* locutions which neither transmit information, nor commit to actions; such locutions may, for instance, encode a theory of argumentation, e.g. questions to or challenges of other speakers, responses to such challenges, etc; these locutions are not directly instantiated with elements of Φ, although they may be instantiated with other locutions in which elements of Φ appear.

We assume that utterances of dialogue locutions occur at discrete time-points, called *rounds*, represented by the non-negative integers. A locution is executed by a speaker P_i instantiating the appropriate locution type with: a positive integer representing the time of utterance (a time-stamp); the identifier P_i of the speaker of the locution; and possibly a proposition ϕ_j from the set Φ.[4] The actions referred to by instantiated utterances of Commitment locutions and the information transmitted

[3]Of course, such a propositional language may have an infinite set of wffs. We leave discussion of the infinite case to another occasion.

[4]Some dialogue-game protocols, e.g. [13], permit speakers to target utterances at particular audiences, in which case the locution-type would also be instantiated with a subset of $\mathcal{A}$. For simplicity in this paper, we assume all utterances are intended for and heard by all participants.

by instantiated utterances of Information locutions are assumed to be expressible in formulae contained in the set Φ. Because some locutions may require instantiation with more than one proposition, e.g. a locution expressing a preference ordering over two purchase options in a negotiation, we assume Φ is closed under the combinations of propositions required by the syntax of the locution-types. We denote by $\mathcal{L}^*$ the finite set of locution-types $\mathcal{L}$ instantiated with possible speakers from $\mathcal{A}$ and discussion topics from Φ. In other words, $\mathcal{L}^*$ is isomorphic to a subset of the finite set of 3-tuples, $\{ (\mathcal{L}, \mathcal{A}, \Phi) \}$.

In the previous section we identified the different types of rules in our model of a dialogue game. Because initiation of a dialogue of specific type on a specific topic happens outside it, we do not assume commencement rules are part of the definition of the dialogue game protocol. For all other rules, however, it is possible to define each rule as a mapping from appropriate dialogue histories (the instantiated locutions actually uttered prior to the next round) to the set $\mathcal{L}^*$ in the case of Combination and Termination rules, or to the set Φ, in the case of Commitment rules. Details of these definitions can be found in [9]. For present purposes, it is sufficient to note that a dialogue game protocol includes a specification of such mappings, and that they induce a partition of the possible instantiated locutions for each agent at each round: *Obligatory moves:* instantiated locutions, one of which must be uttered by the agent at the next round; *Legal moves:* instantiated locutions which may be uttered by the agent at the next round; *Forbidden moves:* instantiated locutions which may not be uttered by the agent at the next round; and *Termination moves:* instantiated locutions which if uttered by the agent at the next round will result in termination of the dialogue. We next define a *dialectical system.*

Definition 1: *Suppose $\mathcal{A}$ is a finite set of agents, Φ a finite set of topics of discussion (including a possible subset of action commitments), and $\mathcal{D}$ a dialogue game protocol with a set of locutions $\mathcal{L}$. We say that the 4-tuple $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$ is a* dialectical system. *If the size of $\mathcal{A}$ is p, the size of Φ is q and the size of $\mathcal{L}$ is l, we say that $\tilde{\mathcal{D}}$ has dimension $n = pql$. A* dialogue *undertaken in accordance with such a dialectical system is a time-ordered sequence of locutions uttered by the agents in $\mathcal{A}$, each element of which consists of an instantiated locution from the set $\mathcal{L}^*$, uttered in accordance with the rules of dialogue game $\mathcal{D}$.*

We refer to dialogues undertaken in accordance with a dialectical system as being *associated with* or *under* the dialectical system. We also refer to such dialogues as *legal* dialogues, although we do not permit sequences of locutions under a dialectical system which do not conform to the rules of the corresponding dialogue game. In the next section, we present a geometric semantics for dialogue systems and their associated dialogues.

3.2 Dialogue paths

Let $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$ be a dialectical system, and set $n = pql$, where these constants are defined as in Section 3.1. We interpret the associated dialogues as paths in the real n-dimensional space $\mathcal{R}^{n+}$. We do this by labeling each axis of $\mathcal{R}^{n+}$ with a triple (P_i, ϕ_k, L_j), for all $P_i \in \mathcal{A}$, for all $\phi_k \in \Phi$ and for all $L_j \in \mathcal{L}$. The path corresponding to a dialogue, which we call a *dialogue path*, commences from the origin, and proceeds as follows: Whenever participant P_i utters locution L_j concerning topic ϕ_k, the path moves from whatever is its current position forward one unit in a direction parallel to the axis labeled (P_i, ϕ_k, L_j). We first define such paths formally as follows:

Definition 2: *A* dialogue path *is a function $d(.) : \mathcal{R}^+ \to \mathcal{R}^{n+}$, such that conditions (a), (b) and (c) are each satisfied:*

(a) $d(0) = \tilde{0}$;

(b) Either:

(i) For all integers $k \in \mathcal{R}^+$, $d(k) = (y_1, y_2, \ldots, y_n)$ where each $y_j \in \mathcal{Z}^+$ and $\sum_{j=1}^{n} y_j = k$; or

(ii) There is an integer $m \geq 0$ such that for all positive integers $k \leq m$, $d(k) = (y_1, y_2, \ldots, y_n)$ where each $y_j \in \mathcal{Z}^+$ and $\sum_{j=1}^{n} y_j = k$, and for all integers $k > m$, $d(k) = d(m)$;

(c) If $x \in \mathcal{R}^+$ not an integer, then $d(x) = d([x] + 1)$, where $[x]$ is the integer part of x.

Definition 3: *Let $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$ be a dialectical system, and let d_L be an associated dialogue, that is, a possibly-infinite time-ordered sequence of locutions from $\mathcal{L}$ instantiated by the topics in Φ, uttered by agents in $\mathcal{A}$ in accordance with the rules of game $\mathcal{D}$. Suppose the t-th element of d_L is the instantiated locution $\mathcal{L}_j(t, P_i, \phi_k)$, for $t = 1, 2, \ldots$. The dialogue path d associated with d_L is the function $d : \mathcal{R}^+ \to \mathcal{R}^{n+}$ obtained by setting $d(0) = \tilde{0}$, setting each $d(t)$ equal to: $d(t-1) + (0, \ldots, 0, 1, 0, \ldots, 0)$, where the non-zero co-ordinate corresponds to that axis labeled (P_i, ϕ_k, L_j); and where $d(x) = d([x] + 1)$, when $x \in [t-1, t)$. We say that d is the dialogue path which* corresponds to *or* matches *d_L.*

It is easy to see that such a d is a dialogue path. Thus, for each legal dialogue we have an associated continuous directed path through $\mathcal{R}^{n+}$, starting from the origin, made up of straight-line segments each parallel to an axis of the space, and each one unit in length. Such paths may represent infinite (condition b i of Definition 2) or terminating (condition b ii) dialogues. In the latter case, we say that the path d is *a terminating path with terminal time-point* m or that the path *terminates at* m, where m is the integer mentioned in condition (b) (ii) of Definition 2. In this case, we also say the dialogue path is of length m. If the former case, we say that the path d is *non-terminating* or *infinite*. It will be useful to distinguish generic dialogue paths (as defined above) from those which obey the rules of the dialogue game, $\mathcal{D}$.

Definition 4: *Let $d : \mathcal{R}^+ \rightarrow \mathcal{R}^{n+}$ be a dialogue path. Suppose $\mathcal{A}$ is a set of agents of size p, Φ a set of topics of discussion of size q, and $\mathcal{D}$ a dialogue game protocol with a set of locutions $\mathcal{L}$ of size l, with $n = pql$, such that there is a legal dialogue between the agents in $\mathcal{A}$ concerning the topics in Φ and conducted according to the rules of $\mathcal{D}$ using the set of locutions $\mathcal{L}$ whose dialogue path matches d. Then we say that d is* a dialogue path under *or* is legal under *the dialectical system* $(\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$.

In general, not all dialogue paths will correspond to legal dialogues. This is because the various combination, termination and commitment rules create subsets of $\mathcal{R}^{n+}$ which a legal dialogue path either cannot enter or must traverse. Because the combination rules typically specify which instantiated locutions may, may not or must be uttered depending on the previous utterances in the dialogue and the identity of the agent speaking, these forbidden and obligatory regions will differ at each round in the dialogue. Thus, because the histories of two dialogues at any one time may be different, one dialogue path may traverse a region which is forbidden to another dialoue path; a dialogue path may even traverse a region which is forbidden to itself later or earlier in the same dialogue. In the sections below we will refer to the Forbidden Region and the Termination Region for a dialogue path under particular dialectical system at a particular time point, with the obvious meanings.

3.3 Path and system equivalence

Because our focus is on specific types of agent interactions, such as negotiations or persuasions, we are concerned to see what external action commitments are made in the course of a dialogue, and what information is transferred between participants to achieve these commitments. We therefore require some measure of these, which we obtain by first examining the syntax of each locution-type.

Definition 5: *Suppose $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$ is a dialectical system, with $L \in \mathcal{L}$ a locution-type. We define the information possibly transferred by speakers uttering locution L, denoted* Poss_Info(L)*, by the set of all possible subsets of Φ which could be instantiated into L. Similarly, the actions possibly committed to by participants uttering locution L, denoted* Poss_Acts(L)*, is the set of all possible subsets of Φ_a which could be instantiated as actions in L. A procedural locution L has* Poss_Info(L) = Poss_Acts(L) $= \{\emptyset\}$*, i.e. the set containing only the empty set.*

For example, the locution with syntax *seek_price*(k, P_i, ϕ), with k is positive integer (a time stamp), $P_i \in \mathcal{A}$ an agent, and $\phi \in \Phi$ a proposition, will have *Poss_Info(seek_price)* = $\{\{\theta\} \mid \theta \in \Phi\}$, i.e. all singleton subsets of Φ. By contrast, the locution with syntax *prefer*(k, P_i, ϕ, ψ) with k and P_i as before, and with $\phi, \psi \in \Phi$ propositions, will have *Poss_Info(prefer)* = $\{\{\theta, \gamma\} \mid \theta, \gamma \in \Phi, \theta \neq \gamma\}$, i.e. all subsets of two distinct elements of Φ. Thus, *Poss_Info(seek_price)* $\neq$ *Poss_Info(prefer)*. Thus, each locution defines a subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$. Each utterance of an information-transferring locution L in a given dialogue is instantiated with the contents of one element of the subset *Poss_Info(L)* of $\mathcal{P}(\Phi)$. Considering instantiation leads us to define the information actually transferred and actions actually committed to by participants in a given dialogue.

Definition 6: *Let d be a dialogue path under a dialectical system $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$, and let d_L be the corresponding dialogue. We define the* information transferred *by d, denoted* Info(d)*, as the set of information transferred by the locutions uttered in d_L, i.e.*

Info(d) = $\{\phi \in \Phi \mid \exists \mathcal{L}_j(t, P_i, \phi) \in d_L\}$.

Likewise, we define the actions committed to *by d, denoted* Acts(d)*, as the set of action propositions committed to by the speakers of locutions uttered in d_L, i.e.*

Acts(d) = $\{\phi \in \Phi_a \mid \exists \mathcal{L}_j(t, P_i, \phi) \in d_L\}$.

Using these definitions, we next define a notion of "closeness" of two dialogue paths.

Definition 7: *Let $d, e : \mathcal{R}^+ \to \mathcal{R}^{n+}$, be two legal dialogue paths under the same dialectical system $(\mathcal{A}, \Phi, \mathcal{L}, \mathcal{D})$. We say that d* is close to e *precisely in the case when both paths are terminating and Info(d) = Info(e) and Acts(d) = Acts(e).*

Note that two close paths may terminate at different time-points, i.e. be of different length. We have the following result, whose straightforward proof is omitted.

Proposition 1: *The relation of closeness between two legal dialogue paths is an equivalence relation.* $\square$

We can therefore speak of two dialogue paths d and e under the same dialogue system being *equivalent*, which we denote by $d \sim e$. We also refer to the corresponding dialogues being equivalent. We denote the equivalence class of a dialogue path d by $[d]$, which we call a *path-equivalence*. We also have the following.

Proposition 2: *For any dialectical system $\tilde{\mathcal{D}}$, the set of path-equivalence classes of legal dialogue paths under $\tilde{\mathcal{D}}$ is finite.*

Proof. This result follows from the assumption that the set Φ is finite. $\square$

Using the notion of path-equivalence, we now define a relationship of similarity between dialectical systems.

Definition 8: *Suppose $\tilde{\mathcal{D}} = (\mathcal{A}_{\mathcal{D}}, \Phi_{\mathcal{D}}, \mathcal{L}_{\mathcal{D}}, \mathcal{D})$ and $\tilde{\mathcal{E}} = (\mathcal{A}_{\mathcal{E}}, \Phi_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}}, \mathcal{E})$ are two dialectical systems of dimension m and n respectively. We say that $\tilde{\mathcal{D}}$* is similar *to $\tilde{\mathcal{E}}$ if there exists a one-to-one and onto function $h : \mathcal{R}^{m+} \to \mathcal{R}^{n+}$ such that for* *every terminating dialogue path d under $\tilde{\mathcal{D}}$ there is a terminating dialogue path e under $\tilde{\mathcal{E}}$ with $h(d) = e$ and with* Info(d) = Info(e) *and with* Acts(d) = Acts(e)*, and such that for all dialogue paths $e' \sim e$ under $\tilde{\mathcal{E}}$, there exists a dialogue path $d' \sim d$ such that $h(d') = e'$.*

In other words, two dialogue systems are similar if the first can be mapped to the second so that terminating dialogues are mapped to terminating dialogues while preserving information-transfers and action-commitments, and so that equivalent dialogue paths are mapped to equivalent dialogue paths. Clearly, for such a map h to exist, the two universes of discourse Φ_D and Φ_E must intersect, as must their respective action subsets. The second condition in the definition of h, the existence of a d' for every e' with $h(d') = e'$, may be considered as analogous to a continuity requirement on h, since dialogue paths which are close to one another in the second dialectical system are required to be the images of dialogue paths close to one another in the first. As with path-equivalence, our notion of similarity of dialectical systems is an equivalence relationship, a statement whose straightforward proof we omit.

Proposition 3: *The relationship $\sim$ between dialectical systems is an equivalence relation.* $\Box$

We can therefore speak of two dialectical systems $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{E}}$ being *equivalent*, denoted $\tilde{\mathcal{D}} \sim \tilde{\mathcal{E}}$. This notion of equivalence of dialectical systems is a global property defined in terms of the existence of local properties, i.e. similarity of dialogue paths. Note that the equivalence mapping h maps legal dialogue paths to legal dialogue paths; since such paths avoid Forbidden Regions in their respective spaces, then h preserves this structure in mapping $\mathcal{R}^{m+}$ to $\mathcal{R}^{n+}$.

4 Comparing Dialectical Systems

In this section, we begin with a connection between the equivalence of dialectical systems and the sets of equivalence classes of the dialogue paths under them.

Proposition 4: *Suppose $\tilde{\mathcal{D}} \sim \tilde{\mathcal{E}}$ are two equivalent dialectical systems. Then the respective sets of path-equivalence classes generated by legal terminating dialogue paths under each system are isomorphic.*

Proof. We need to show that there is a one-to-one and onto map between the two sets of path-equivalence classes. Let $\Phi = \Phi_D \cup \Phi_E$ be the union of the two universes of discourse, and $\Phi_a = \Phi_D a \cup \Phi_E a$ the union of the two subsets of action propositions. By the definition of path-equivalence, each class in the set of path-equivalence classes for a specific dialectical system corresponds to a unique subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$, where $\mathcal{P}(X)$ is the power set of X. For each path-equivalence under $\tilde{\mathcal{D}}$, assign to it the path-equivalence under $\tilde{\mathcal{E}}$ corresponding to the same subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$. Such a path-equivalence class under $\tilde{\mathcal{E}}$ exists because the equivalence mapping h from $\tilde{\mathcal{D}}$ to $\tilde{\mathcal{E}}$ preserves the information transmitted and the actions committed to by each dialogue. This mapping is one-to-one because each path-equivalence class is associated with a unique subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$.

To prove it is onto, suppose, for purposes of contradiction, there is some path-equivalence class $[e]$ under $\tilde{\mathcal{E}}$ which corresponds to a subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$ to which no path-equivalence class under $\tilde{\mathcal{D}}$ corresponds. Consider a dialogue path e in the path-equivalence class $[e]$. Because $\tilde{\mathcal{D}} \sim \tilde{\mathcal{E}}$, then there must be a dialogue path d under $\tilde{\mathcal{D}}$ such that $h(d) = e$. Thus, *Info(d) = Info(e)* and *Acts(d) = Acts(e)*. But this just means that $[d]$ is associated with the same subset of $\mathcal{P}(\Phi) \times \mathcal{P}(\Phi_a)$ as is $[e]$, thus contradicting the assumption. □

Proposition 4 shows that dialogue equivalence, which is defined in terms of mappings between real spaces, preserves the structure of the sets of associated path-equivalence classes. This is really not surprising given the definition of dialogue equivalence. However, it allows us to deduce the following interesting corollary:

Proposition 5: *Suppose* $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}_{\mathcal{D}}, \mathcal{D})$ *and* $\tilde{\mathcal{E}} = (\mathcal{A}, \Phi, \mathcal{L}_{\mathcal{E}}, \mathcal{E})$ *are two dialectical systems of dimension* m *and* n *respectively, such that* $\mathcal{L}_{\mathcal{E}} = \mathcal{L}_{\mathcal{D}} \cup \{L'\}$. *Moreover, suppose that* $\{\emptyset\} \neq$ Poss_Info(L') $\neq \bigcup_{L_j \in J}$ Poss_Info(L_j), *for all* $J \subseteq \mathcal{L}_{\mathcal{D}}$. *Then* $\tilde{\mathcal{D}}$ *and* $\tilde{\mathcal{E}}$ *are not equivalent.*
Proof. If we had $\tilde{\mathcal{D}} \sim \tilde{\mathcal{E}}$ then, by Proposition 4, we would have an isomorphism of the two sets of path-equivalence classes. However, since $\{\emptyset\} \neq$ *Poss_Info(L')* $\neq \bigcup_{L_j \in J}$ *Poss_Info(L_j)*, for all $J \subseteq \mathcal{L}_{\mathcal{D}}$, there is at least one instantiation of locution L' which transfers a non-empty subset of $\mathcal{P}(\Phi)$ not transferred by any dialogue without this locution. Since the dialogues of system $\tilde{\mathcal{D}}$ consist only of the locutions in $\mathcal{L}_{\mathcal{D}}$, we thus have a contradiction. Hence $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{E}}$ are not equivalent.□

A similar result applies to dialogue systems which differ only by a locution which commits to actions not committed to by the other locutions. Moreover, both results apply if there is more than one locution $L' \in \mathcal{L}_{\mathcal{E}} \setminus \mathcal{L}_{\mathcal{D}}$ which transfers information or commits to actions not possible using the locutions in $\mathcal{L}_{\mathcal{D}}$. Similarly, we have the following corollary, whose similar proof we omit.

Proposition 6: *Suppose* $\tilde{\mathcal{D}} = (\mathcal{A}, \Phi, \mathcal{L}_{\mathcal{D}}, \mathcal{D})$ *and* $\tilde{\mathcal{E}} = (\mathcal{A}, \Phi, \mathcal{L}_{\mathcal{E}}, \mathcal{E})$ *are two dialectical systems of dimension* m *and* n *respectively, such that the protocols* $\mathcal{D}$ *and* $\mathcal{E}$ *differ only by a rule in* $\mathcal{E}$ *which terminates a dialogue under some conditions. Then* $\tilde{\mathcal{D}} \sim \tilde{\mathcal{E}}$ *only if for each terminating dialogue-path* e *under* $\mathcal{E}$ *in which the rule is invoked and leads to termination there is a terminating dialogue path* d *under* $\mathcal{D}$ *with* Info(d) = Info(e) *and* Acts(d) = Acts(e). □

These corollaries are important because they provide us with some guidance for the design of dialogue systems. With them, we know that adding a locution or a termination rule which transfers information not transferred in the current set of locutions will create a non-equivalent system, i.e. a dialogue system in which there will be dialogues transferring different information or committing to different actions. Dialogue-system equivalence, as we have defined it here, is not the only criterion one could use for design and assessment of dialogue game protocols; one may wish to add or not add such locutions or rules for other reasons, such as overall simplicity or to encode a particular theory of argumentation [12]. However, we believe dialogue equivalence is an important criterion and these results give us a purchase on understanding its implications for dialogue-game protocol design.

5 Conclusion

In this paper, we have defined a novel geometric semantics for dialogue game protocols for interactions between autonomous agents and begun to explore its formal properties. We have used these properties to show that two protocols which differ only in one locution are not equivalent if that additional locution transmits information or commits its speaker to actions which are not expressible by any combination of the other locutions. Although our methods are not sophisticated mathematically, we believe our results are important because of the guidance they provide for the design and evaluation of dialogue game protocols. We believe this application domain contains further potential for investigation. Firstly, we have not yet used the fact that dialogue paths are time-directed and are monotonically non-decreasing. Nor have we used the fact that our geometric semantics embeds dialogue paths in a real space with holes (the Forbidden Regions) and thus may make them liable to continuous approximation. Secondly, we believe the results here could be readily expressed in category-theoretic terms. Both these streams are the subject of future work.[5]

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[5]This work was partly funded by the EU IST Programme through the Sustainable Lifecycles in Information Ecosystems (SLIE) Project (IST-1999-10948) and by the British EPSRC through a PhD studentship, and this support is gratefully acknowledged. The first author is also grateful to Neville Smythe for introducing him to algebraic topology.

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Categories of dimaps and their dihomotopies in po-spaces and local po-spaces*

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Gdańsk, 6 August 2001

This report investigates a number of different notions of dimaps in [local] po-spaces, and of their dihomotopies. It discusses their respective advantages and drawbacks in modeling concurrency. This should be considered as a contribution towards putting some order into the foundations of the approach.

Contents

1 Introduction

As pointed out by many authors, geometric and topological methods have a potential for applications in the theory of concurrent processes (for a review and a rich bibilography,

*This research was supported by a 3-week visiting fellowship from the University of Aalborg, Denmark, by the State Committee for Scientific Research in Poland (grant 8 T11C 037 16) and by the Institute of Computer Science PAS.

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see [1] by Fajstrup, Goubault and Raussen). Possible executions of a family of concurrent processes are modeled in this framework as trajectories (dipaths) in the spaces of their possible configurations. The competition of processes for resources may be portrayed as "holes" in these spaces. Now, small deformations of a trajectory, not involving "jumping" over the holes, accounts for insignificant changes of the order of execution without any rescheduling of the use of resources. Such harmless deformations give rise to an analog of the topological notion of homotopy. In principle, the investigation of such holes and deformations should be similar to the investigation of topological spaces by algebraic topological tools.

However, the foundations of the geometric and algebraic-topological theory of concurrency are not yet firmly set. While the notion of *po-space* as a home of the finitely running processes (see, e.g., [1] by Fajstrup, Goubault and Raussen) seems appropriate and sufficiently well understood, the same is only partially true of the *local po-spaces* serving the infinitely running processes. In fact, as recently as a year ago we have felt it needed tuning up (cf. [2] by Fajstrup and Sokołowski).

There are more doubts on the side of the morphisms. The natural notion of a *dipath* in a po-space X (a continuous and monotone mapping of the ordered interval $\vec{\mathbb{I}}$ into X) does not readily generalize to a *dimap* of arbitrary po-spaces amenable to usual homotopical (or rather *dihomotopical*) investigations. The natural notion of dihomotopy equivalence of po-spaces seems a lot too permissive in that it identifies too many po-spaces which, from a computational point of view, had better be kept apart. And most importantly, the natural construction of the *dihomotopy posets* (cf. [7] by Sokołowski), corresponding to the homotopy groups in algebraic topology, fails to be functorial.

The situation is even more unstable in the local po-spaces. The notion of *long dipath* in X (cf. [5] by Raussen), which is a continuous and locally monotone mapping of the non-negative real numbers $\mathbb{R}_{\geq 0}$ into X, gives rise to a classification of such dipaths into *finite* ones that correspond to terminating computations (e.g., deadlocks) and *infinite* ones that correspond to infinite computations. This is nice. But there are examples of finite long dipaths which are dihomotopic to infinite ones. And this is most unwelcome, even unacceptable, if the notion of dihomotopy should have a computational meaning.

This report presents some of the problems and, in some cases, puts forward restrictions on the notions of dimap and dihomotopy designed to keep the problems away. But there seems to be no hope for a single notion of dimap and a single notion of dihomotopy, nice and simple. We had better get used to dealing with many categories of dimaps, each good for one purpose and useless for another.

Acknowledgements:

This report was born out of discussions with Lisbeth Fajstrup and Martin Raussen, and they are, in a "moral" sense, its coauthors.

1.1 Basic notions and notations

This report deals with *po-spaces*, i.e., topological spaces X with a partial order $\leq$ which is closed as a subset of $X \times X$. In most cases (not always), the po-spaces are assumed to be *pointed*, i.e., to have a distinguished least point 0 with respect to the order. The pointedness of the po-spaces does not affect the theory a lot, it just simplifies some formulations.

This report deals with local po-spaces too. Technically, this is a complicated concept, cf. [2] by Fajstrup and Sokołowski. But for the purposes of this report, a *local po-space* is a topological space with an open cover by po-spaces and such that, in a loose sense, the local partial orders on these open sets coincide[1].

Throughout this note, the following po-spaces are used for auxiliary purposes:

- $\mathbb{I}$ — the interval $[0\,..\,1]$ with the standard topology and the discrete order:

$$s \leq s' \overset{\text{def}}{\Longleftrightarrow} s = s'$$

- $\vec{\mathbb{I}}$ — the interval $[0\,..\,1]$ with the standard topology and the standard order.
- $\mathbb{R}_{\geq 0}$ — the set of non-negative reals $[0\,..\,+\infty)$ with the standard topology and the standard order.

By a *dimap* between [local] po-spaces will always be meant a continuous and [locally] monotone mapping between these spaces. Sometimes additional restrictions will be put on dimaps giving rise to subcategories of the full category of [local] po-spaces. A *dihomeomorphism* is a dimap which has an inverse dimap.

A *dihomotopy* from a [local] po-space X to a [local] po-space Y is a continuous and [locally] monotone mapping from $\mathbb{I} \times X$ to Y. Again, sometimes restrictions wil lbe put on this general notion of dihomotopy. Two dimaps $\varphi, \psi : X \to Y$ are *dihomotopic* (denoted by $\varphi \simeq \psi$) iff there exists a dihomotopy H from X to Y such that $\varphi = H\langle 0, \_\rangle$ and $\psi = H\langle 1, \_\rangle$.

A *dipath* in a po-space X is a dimap $\alpha : \vec{\mathbb{I}} \to X$. The set of dipaths in X will be denoted by $\mathcal{D}_1 X$. The set of *initial dipaths* in a pointed po-space X is

$$\mathcal{I}_1 X \overset{\text{def}}{=} \left\{\alpha \in \mathcal{D}_1 X \,\middle|\, \alpha 0 = 0\right\}$$

where 0 is the least point in X. A dipath models a finite computation of a system of concurrent processes. Its monotonicity accounts for the fact that time only flows forward. Its continuity rules out a "teleportation" that would, out of the blue, change the state of such a system into a completely different and remote state.

2 Global po-spaces

2.1 Fixed-ends dihomotopy of dipaths

In a pointed po-space X, which is our main concern, any two initial dipaths are dihomotopic to each other. To see this, take dihomotopies that contract each dipath to a constant dipath $\alpha t \overset{\text{def}}{=} 0$, and glue them together in the obvious way. This shows that the unrestricted dihomotopy is useless for a classification of dipaths.

[1]The complicated part of this definition is the exact formulation of this coincidence, and of the independence from a particular cover.

The movement of the end-point of a dipath throughout the dihomotopy has to be somehow restricted. A very general approach is adopted in [1] by Fajstrup, Goubault and Raussen, where the notion of dihomotopy of dipaths is parameterized by an arbitrary subset $A \subseteq X$ and a requirement that the dihomotopy keeps the end-point of the dipath within A [2]. I will assume A to be a singleton set, but I will allow every dihomotopy to have a different singleton set restriction:

1 Definition:
A *fixed-ends dihomotopy of dipaths* in X is a continuous and monotone mapping

$$H : \mathbb{I} \times \vec{\mathbb{I}} \to X \qquad \text{such that } H\langle s,0\rangle = H\langle 0,0\rangle \text{ and } H\langle s,1\rangle = H\langle 0,1\rangle \text{ for all } s \in \mathbb{I}$$

(i.e., the end-points of the dipaths are kept fixed). Two dipaths α and β are *fixed-ends dihomotopic* (denoted by $\alpha \simeq_d \beta$) if there exists a fixed-ends dihomotopy H of dipaths such that $\alpha = H\langle 0, \_\rangle$ and $\beta = H\langle 1, \_\rangle$.

Of course, a necessary condition for two dipaths to be fixed-ends dihomotopic is that they have the same end-point. There is, therefore, at least one fixed-ends dihomotopy class for each reachable point of X. If there are holes in X, the set of fixed-ends dihomotopy classes of dipaths may be bigger.

Computationally, fixed-ends dihomotopies identify the executions of the concurrent system in which the shared resources are allocated to particular processes in the same order. This corresponds to going around the holes in a po-space in the same way.

Note that the definition of fixed-ends dihomotopy of dipaths does not readily generalize to a dihomotopy of other dimaps of po-spaces, not even of pointed po-spaces. In particular, in other po-spaces, there is no obvious counterpart of the requirement that the end-points of the dipaths are fixed, because there may be no (or many) end-points. The property of pointedness is computationally natural, because it accounts for a system of concurrent processes beginning in a well-defined initial state, but the existence of a biggest point would rule out many useful examples. On the other hand, since two dimaps from a general po-space X to Y are not necessarily dihomotopic to each other, the dihomotopy classifications of general dimaps is not necessarily as meaningless as the dihomotopy classification (without fixed-ends) of dipaths.

For the time being, we need to put up with the fact that the dihomotopy classification of dipaths is different from the dihomotopy classification of other dimaps. In particular, that $\alpha \simeq_d \beta$ implies $\alpha \simeq \beta$ but not the other way round.

2.2 Problems with dihomotopy equivalence

It is natural to define the *dihomotopy equivalence* of two (local) po-spaces X and Y as a pair of dimaps $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ such that $\psi \circ \varphi \simeq id_X$ and $\varphi \circ \psi \simeq id_Y$. But it is unclear whether the dihomotopy equivalence of two po-spaces has any computational consequences. In view of the restriction on the notion of dipath dihomotopy discussed in

[2] Actually, there are two subsets of X restricting, respectively, the begin-point and the end-point of the dipath. For simplicity, I am assuming the former set to be $\{0\}$.

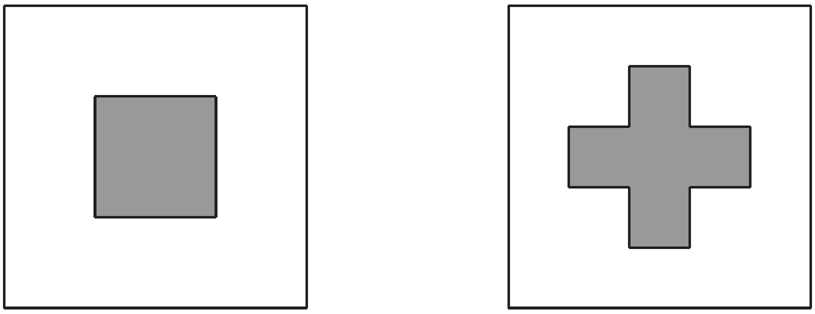


Figure 1: The single square hole and the Swiss flag.

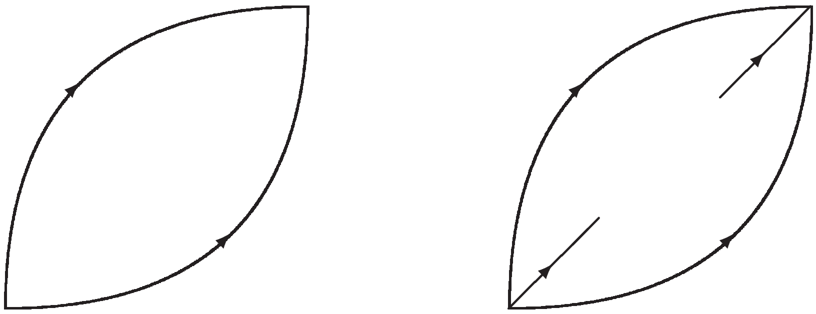


Figure 2: Two "thin" spaces corresponding to the ones in Fig. 1.

Sec. 2.1, two (local) po-spaces may be dihomotopy equivalent without any natural relation on the sets of their dipaths, even up to dipath dihomotopy.

2 Example

Consider the square with a single square hole and the Swiss flag in Fig. 1[3]. It is easy to realize that they are dihomotopy equivalent, respectively, to the "thin" po-spaces in Fig. 2 (cf. [3] by Gaucher). Now, a dihomotopy may contract the two straight-line segments in the right-hand side po-space, showing that the original po-spaces are dihomotopy equivalent[4], even though one of them allows for a deadlock while the other does not. This is an effect that we most certainly do not want since this is our flagship example of the applicability of dihomotopy considerations in concurrency.

On the other hand, the dipaths in the single square hole po-space behave quite differently from the dipaths in the Swiss flag. This demonstrates that the dihomotopy equivalence of po-spaces without further requirement may have no deeper computational meaning.

□

The particular anomaly referred to in Example 2 may be ruled out by requiring that the dimaps and the dihomotopies involved are strict. For po-spaces X and Y, $\varphi : X \to Y$

[3]The implied ordering of the po-spaces in the figures in this report is from the left to the right and from the bottom upwards.

[4]As pointed out by a referee, there is a more direct way of demonstrating that the po-spaces in Fig. 1 are dihomotopy equivalent.

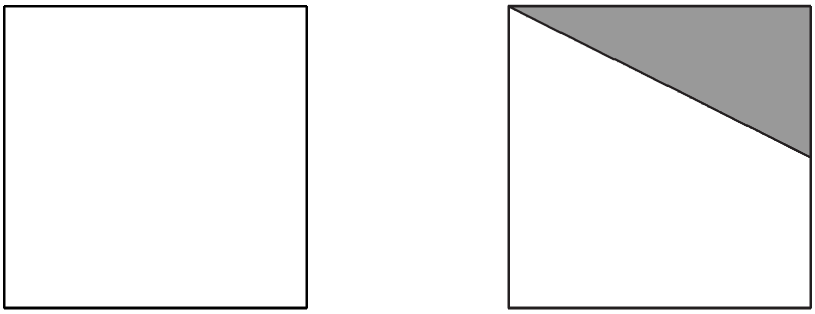


Figure 3: The square without holes and the square with the upper triangle clipped off.

is a *strict dimap* if and only if φ is continuous and

$$\forall_{x,x'}\ x < x' \Rightarrow \varphi x < \varphi x'$$

(for local po-spaces, use the obvious counterpart). A dihomotopy $H : \mathbb{I} \times X \to Y$ is *strict* iff $H\,\langle t, \_\rangle$ is a strict dimap for all $t \in \mathbb{I}$. The fact that strict dimaps φ and ψ are strictly dihomotopic will be denoted by $\varphi \simeq_s \psi$.

However, two strictly dihomotopic po-spaces may still dramatically differ in the behaviour of the processes they model.

3 Example

The po-spaces in Fig. 3 are dihomotopy equivalent, as shown in the last section of [7]. It is easy to see that the dihomotopy equivalence given there is strict. But there is only one maximal point in the first one and a continuum of maximal points in the second one.

□

The maximal points have an important computational meaning, corresponding to possible outcomes of the execution of a system of concurrent processes. For this reason, these two po-spaces had better not be identified. This casts more doubt on the usefulness of the strict dihomotopy equivalence of the po-spaces.

A strict dihomotopy may contract “space” but not “time” to single points — but please, take this informal statement with a big grain of salt, because there are hardly any space contractions that would leave time unaffected. The notion is not nice, because the set of strict dimaps is not closed in the set of all dimaps: the limit of a sequence of strict dimaps may fail to be strict (see Example 31 on page 20).

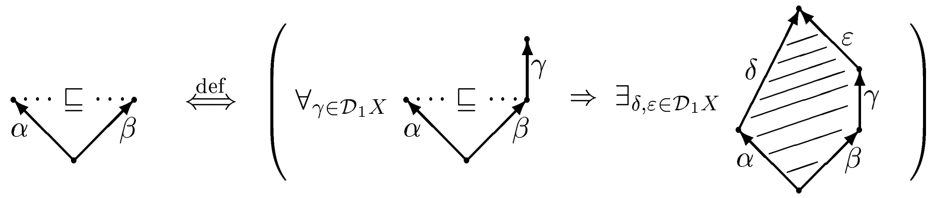
2.3 Fundamental posets

When the initial dipaths in a certain pointed po-space X are identified up to fixed-ends dihomotopy, the resulting set $\vec{\pi}_1 X \stackrel{\text{def}}{=} \mathcal{I}_1 X \big/ \simeq_d$ of equivalence classes has a natural partial order inherited from the prefix relation on initial dipaths: $\alpha \sqsubseteq \beta$ iff α is the restriction, up to a reparameterization, of β to the interval $[0\,..\,t_0]$ for a certain $t_0 \leq 1$. The *dihomotopy posets* $\vec{\pi}_1 X$ were studied in [8] and in [7] by Sokołowski. A related notion of *dihomotopy set* (without the partial order) appears in [1] by Fajstrup, Goubault and Raussen and is further studied in a number of unpublished notes by Raussen, where it gives rise to interesting considerations concerning dicomponents (cf. [4]).

The construction $\vec{\pi}_1$ of the dihomotopy poset has certain good properties for investigating po-spaces. It is functorial: dimaps are in a natural way translated to monotone functions on partial orders. In fact, this functoriality is used in [8] for proving a certain unimplementability result. And, as demonstrated in [7], it generalizes easily to dihomotopy posets $\vec{\pi}_n$ in higher dimensions. These posets may, in principle, detect higher-dimensional holes that arise from the investigations of critical regions that serve more than one process at the same time.

One problem with dihomotopy posets is that, in general, they are enormous. As noticed in Sec. 2.1, in the absence of any holes in a po-space, its dihomotopy poset is as big as the space itself, while one would want it to be trivial. In higher dimensions, the dihomotopy posets of uninteresting po-spaces without holes become huge function spaces, so the situation is even worse. For a useful tool, all this wealth of uninteresting complexity had better be factored out. This is done by collapsing the dihomotopy poset to a much smaller one. The definition of collapsing given below is much simpler than the one from [7], although the relations defined are the same[5].

4 Definition:
The *collapsing preorder* is the relation $\sqsubseteq\ \subseteq \mathcal{I}_1X \times \mathcal{I}_1X$ defined as follows[6]:



Informally, we may view dipaths as partial realizations of maximal (not further extendible) dipaths. While time flows, dipaths may grow and this growth is irreversible. At a given moment, a dipath has already made some "decisions" where to be heading, while some other decisions may still be open. The collapsing preorder $\sqsubseteq$ compares the numbers of decisions already made. Informally, $\alpha \sqsubseteq \beta$ means that α can still make all the decisions that β can make and, possibly, some more.

5 Proposition:

1. *The collapsing preorder $\sqsubseteq$ is a preorder in $\mathcal{I}_1X$ (reflexive and transitive).*
2. *$\simeq_d \subseteq \sqsubseteq$, i.e., the fixed-ends dihomotopy is finer than the collapsing preorder.*

6 Definition:
The *collapsing equivalence* or *collapse* is the relation

$$\alpha \mathrel{\overset{\sqsubset}{\sqsupset}} \beta \overset{\text{def}}{\Longleftrightarrow} \alpha \sqsubseteq \beta \ \&\ \beta \sqsubseteq \alpha$$

7 Corollary:
The collapsing equivalence $\overset{\sqsubset}{\sqsupset}$ is an equivalence relation in $\mathcal{I}_1X$. The collapsing preorder $\sqsubseteq$ induces a partial order (denoted by $\sqsubseteq$ too) on the set of equivalence classes.

[5]This is proven in [6].

[6]Such pictures will appear instead of mathematical formulae. I find them easier to read and not difficult to turn into a more standard notation. When a closed diagram is filled in with dashes, this means that there is a dipath dihomotopy between the edges of the filled in shape.

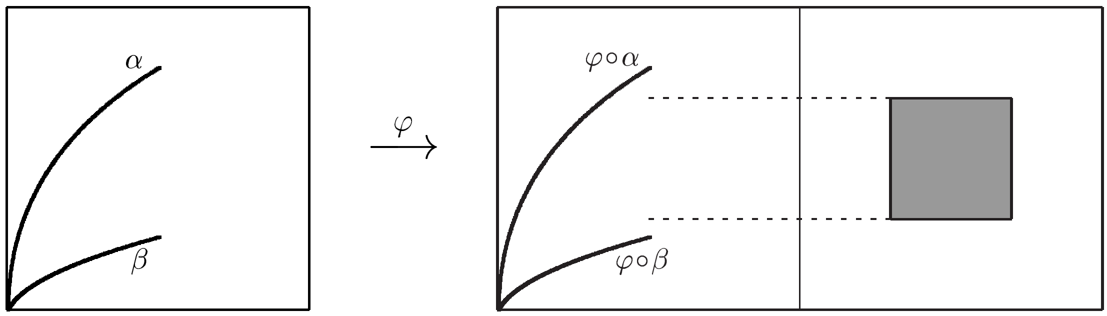


Figure 4: The $\sqsubseteq\!\!\!\sqsupseteq$-equivalence of dipaths may not be preserved by a dimap.

8 Definition:
The *fundamental po-set* of a po-space X is the set of equivalence classes

$$\Omega_1 X \stackrel{\text{def}}{=} \mathcal{I}_1 X \big/ \sqsubseteq\!\!\!\sqsupseteq$$

with the partial order $\sqsubseteq$.

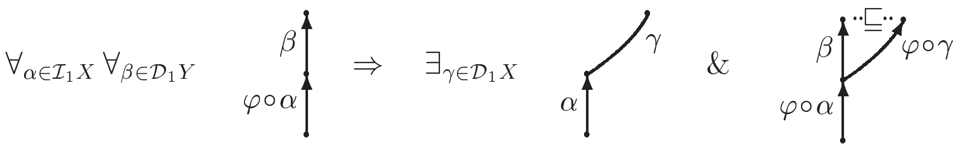
Since the equivalence classes wrt the collapse are larger than the fixed-ends dihomotopy classes (Prop. 5), the transition from a po-space X to $\Omega_1 X$ is a further reaching simplification than the one from X to $\vec{\pi}_1 X$. [7] gives many examples of po-spaces and their fundamental po-sets. By the usual category-theoretical nonsense, dihomeomorphic po-spaces have isomorphic fundamental po-sets (some clues about the way of proving this follow in Sec. 2.4 — Thm. 14). On the sad side: Examples 2 and 3 show that dihomotopy equivalent po-spaces may have non-isomorphic fundamental po-sets.

2.4 Functoriality of the fundamental poset

But [7] also gives the evidence that the construction Ω_1 of fundamental po-sets is not functorial with respect to dimaps[7]. A counterexample is provided by the embedding of the left-hand side square in Fig. 4 into the right-hand side rectangle with a hole. $\Omega_1\varphi$ is not well-defined, since $\alpha \sqsubseteq\!\!\!\sqsupseteq \beta$ but not $\varphi \circ \alpha \sqsubseteq \varphi \circ \beta$ — because $\varphi \circ \beta$ may still decide to go under the hole while $\varphi \circ \alpha$ is already committed to go over the hole.

This does not preclude, however, that the fundamental poset construction may be functorial with respect to a smaller category of dimaps. One proposal is discussed below.

9 Definition:
A dimap $\varphi : X \to Y$ will be called a *superior dimap* if it satisfies the following condition:



(1)

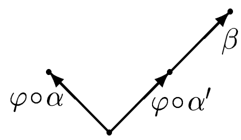
This means that in the target po-space Y, for every dipath there is a bigger (superior) dipath which is the image via φ of a dipath in X.

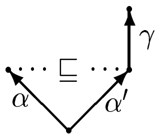
[7]Under the natural definition of $\Omega_1\varphi$ for a dimap $\varphi : X \to Y$: $\Omega_1\varphi[\alpha] \stackrel{\text{def}}{=} [\varphi \circ \alpha]$.

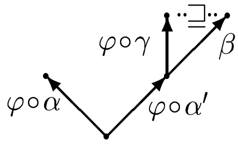
10 Theorem:
Assume X and Y are pointed po-spaces and $\varphi : X \to Y$ is a superior dimap that preserves the least points: $\varphi 0 = 0$. Then φ is monotone with respect to the preorder $\sqsubseteq$ in the following sense:

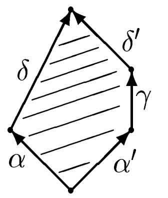
$$\forall_{\alpha,\alpha' \in \mathcal{I}_1 X} \;\; \alpha \sqsubseteq \alpha' \;\Rightarrow\; \varphi \circ \alpha \sqsubseteq \varphi \circ \alpha'$$

Proof of Thm. 10:

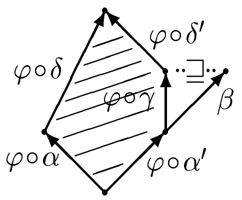
Take an arbitrary dipath $\beta \in \mathcal{D}_1 Y$ such that 

Condition (1) implies 

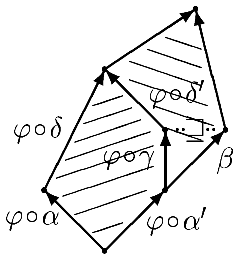
and 

By Def. 4 of the collapsing preorder $\sqsubseteq$, there exist dipaths $\delta, \delta' \in \mathcal{D}_1 X$ such that 

This translates via φ to



Again by the definition of the collapsing preorder:



This implies that $\varphi\alpha \sqsubseteq \varphi\alpha'$.

□

11 Proposition: *The composition of superior dimaps is a superior dimap.*

The po-spaces with superior dimaps form, therefore, a category.

12 Corollary:
Any superior dimap $\varphi : X \to Y$ induces a monotone function $\Omega_1\varphi : \Omega_1 X \to \Omega_1 Y$ by

$$\Omega_1\varphi[\alpha] \stackrel{\text{def}}{=} [\varphi \circ \alpha]$$

13 Corollary:
Ω_1 is a functor from the category of po-spaces with superior dimaps to po-sets with monotone functions.

14 Theorem:
Any two dihomeomorphic po-spaces have isomorphic fundamental posets.

Proof of Thm. 14:

Just note that dihomeomorphisms are superior dimaps[8].

□

So at last we have achieved the functoriality of the fundamental poset construction. The price to pay is a restriction on the class of allowed dimaps. An embedding of a square into a bigger square is a superior dimap, hence a superior dimap need not be surjective in the strict sense, but it has to be "surjective enough" to cover all inextendible dipaths in the target po-space:

15 Proposition:
For any superior dimap $\varphi : X \to Y$, all the maximal points of $\Omega_1 Y$ belong to the image $\Omega_1\varphi\,(\Omega_1 X)$.

The following examples should give more insight into the severity of the restriction.

16 Example

An embedding of the "letter C" shape into a square with a hole is or is not a superior dimap, depending on the positioning of that shape with respect to the hole. Fig. 5 shows an embedding which is a superior dimap (the letter C shape — light gray, the hole — dark gray). The right-hand side picture is the fundamental poset of both the letter C and the square with the hole; the induced monotone mapping is the identity on this poset.

The two embeddings in Fig. 6 are not superior dimaps. In each case, the offending dipaths α (fat curve within the letter C) and β (thin curve extending the fat one) have the same meaning as in Def. 9 of superior dimap[9].

Filling a hole in a square, as does φ in Fig. 7, is a superior dimap. Fig. 8 presents the induced monotone function on the fundamental posets.

□

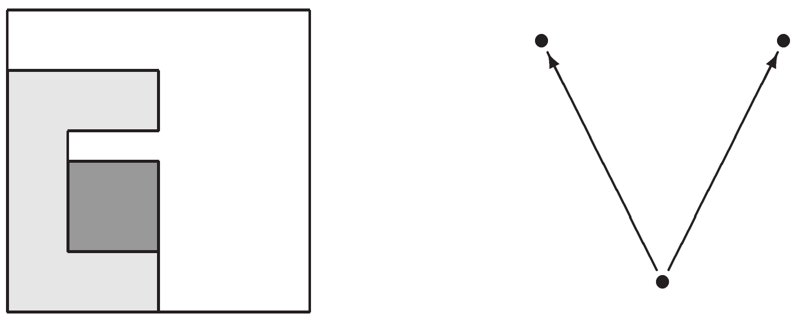


Figure 5: A superior dimap embedding and the corresponding Ω_1 po-set.

[8] In [7], the same fact has been proven by showing that dihomeomorphisms are so called *dipath surjections*. Incidentally, every dipath surjection is a superior dimap.

[9] As pointed out by Martin Raussen, the first version of this example given in [6] contained a faulty statement.

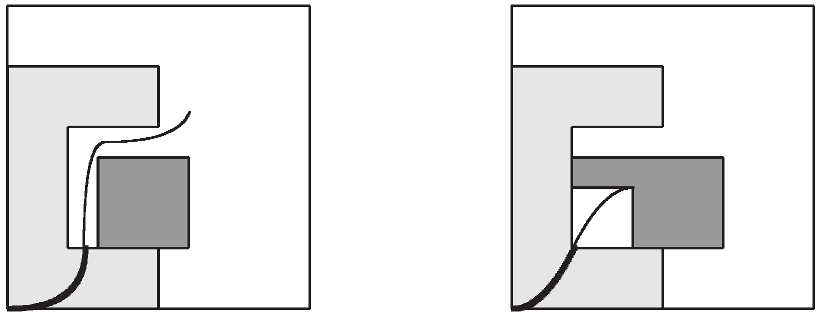


Figure 6: Non-superior dimap embeddings.

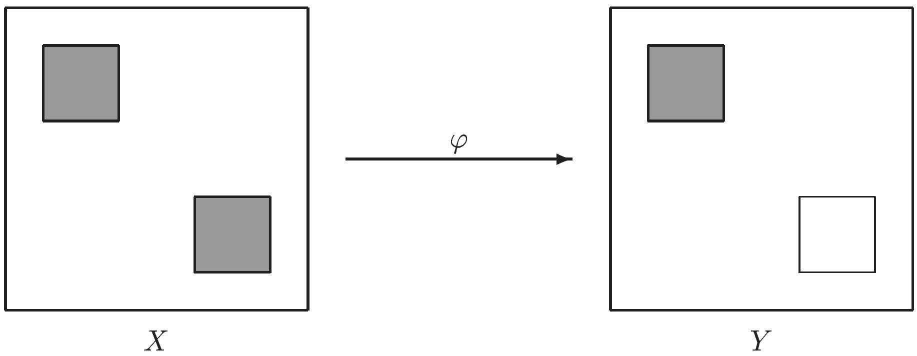
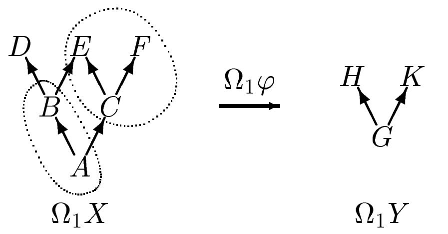


Figure 7: Filling a hole: a superior dimap.

And now a word of caution. Even though the restricted category of dimaps has some good properties, they are still not good enough. Namely, the fundamental posets of po-spaces, which are superior dimap dihomotopy equivalent, are not necessarily isomorphic. Consider the following example:



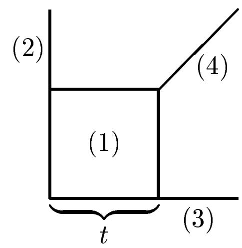
$$\begin{aligned}&\Omega_1\varphi\, A = \Omega_1\varphi\, B = G\\ &\Omega_1\varphi\, D = H\\ &\Omega_1\varphi\, C = \Omega_1\varphi\, E = \Omega_1\varphi\, F = K\end{aligned}$$

Figure 8: The monotone function induced by filling the hole (Fig. 7).

17 Example

For any $0 \le t < 1$, define the following po-space X_t:

$$X_t \stackrel{\text{def}}{=} \left\{ \langle x, y\rangle \in \vec{\mathbb{I}} \times \vec{\mathbb{I}} \;\middle|\; \begin{array}{ll} \scriptstyle(1) & (0 \le x \le t \;\&\; 0 \le x \le t) \vee \\ \scriptstyle(2) & (x = 0 \;\&\; y > t) \vee \\ \scriptstyle(3) & (x > t \;\&\; y = 0) \vee \\ \scriptstyle(4) & x = y > t \end{array} \right\}$$



Note that X_0 consists of three straight-line segments only. Now define a pair of dimaps $X_0 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X_{\frac{1}{2}}$ by

$$\varphi\,\langle x, y\rangle \stackrel{\text{def}}{=} \langle x, y\rangle \qquad \psi\,\langle x, y\rangle \stackrel{\text{def}}{=} \begin{cases} \langle 0, 0\rangle & \text{if } 0 \le x \le \frac{1}{2} \;\&\; 0 \le x \le \frac{1}{2} \\ \langle 0, 2y-1\rangle & \text{if } x = 0 \;\&\; y > \frac{1}{2} \\ \langle 2x-1, 0\rangle & \text{if } x > \frac{1}{2} \;\&\; y = 0 \\ \langle 2x-1, 2y-1\rangle & \text{if } x = y > \frac{1}{2} \end{cases}$$

It is easy to see that both φ and ψ are superior dimaps; and that there exists a superior dimap dihomotopy between $\psi \circ \varphi$ and id_{X_0} and another superior dimap dihomotopy between $\varphi \circ \psi$ and $id_{X_{\frac{1}{2}}}$. On the other hand, the respective fundamental posets are different, as shown in Fig. 9.

□

Figure 9: Different fundamental posets of superior dimap dihomotopy equivalent po-spaces.

This would suggest that our notion of dihomotopy equivalence is not yet refined enough.

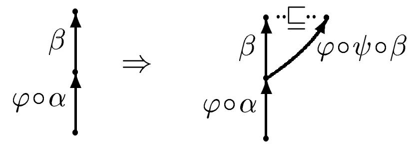
2.5 Weak dipath retractions

Condition (1) in Def. 9 of a superior dimap does not give any systematic assignment of γ's to various β's. If the existence of such a systematic assignment is assumed, the dimap φ becomes a weak dipath retraction:

18 Definition:
A dimap $\varphi : X \to Y$ is a *weak dipath retraction* iff

1. it is a coretraction in the usual sense, i.e., there exists a dimap $\psi : Y \to X$ such that $\psi \circ \varphi = id_X$, and
2. the dimaps φ and ψ satisfy the following condition:

$$\forall_{\alpha \in \mathcal{I}_1 X} \forall_{\beta \in \mathcal{D}_1 Y}$$



So a weak dipath retraction is a coretraction on points and almost a retraction on dipaths. In a related report [4], Martin Raussen assigned to po-spaces *fundamental categories*, which seemed to have a lot in common with the fundamental po-sets. In particular, Martin pointed out that the assignment was functorial for continuous and monotone retractions on dipaths. Later, he thought that being a retraction on dipaths was so much to ask from a mapping, as to make the notion useless. The weak dipath retractions presented here may be a way out.

On the other hand, retractions play a special rôle in modelling computer systems because they have the flavour of implementations; a coretraction corresponds to encoding and a retraction corresponds to decoding.

19 Proposition: *Every weak dipath retraction $\varphi : X \to Y$ is a superior dimap.*

Therefore, the function $\Omega_1\varphi$ is well-defined and monotone; and Ω_1 is a functor from the po-spaces with weak retractions to po-sets with monotone functions.

A weak dipath retraction is always injective and, in general, not surjective.

20 Example

An injection of a square into a bigger square is a weak dipath retraction. Since Ω_1 assigns a singleton to a square, the monotone function induced by this injection is the identity.

Filling the hole (Example 16) is a weak dipath retraction; indeed, its inverse ψ is defined by

$$\psi\,\langle x, y\rangle \stackrel{\text{def}}{=} \begin{cases} \langle x, y\rangle & \text{if } x \leq 1/2 \\ \langle 1/2, y\rangle & \text{if } x \geq 1/2 \end{cases}$$

Note that ψ itself is *not* a superior dimap, Def. 18 does not require it to be. Consequently, there may be no monotone function induced on fundamental posets by an inverse dimap of a weak dipath retraction.

Fig. 10 presents an embedding of the po-space composed of two faces and the diagonal of a square, into a square with two holes adjacent to its edges. This is a superior dimap but not a weak dipath retraction. Indeed, an inverse dimap would have to contract the whole E-area to the initial point A to be monotone; this contradicts the requirement that φ is a retraction on points. This example shows that the monotone function induced by a superior dimap is not necessarily surjective on the respective fundamental posets. This function is given in Fig. 11.

□

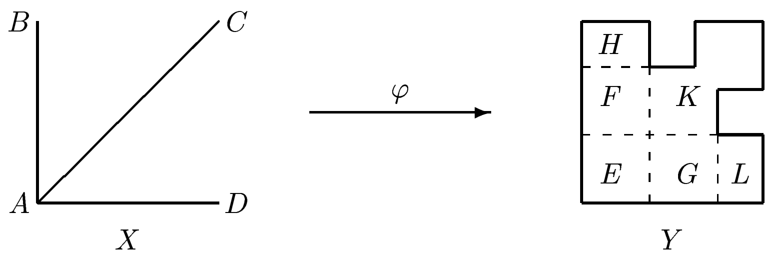
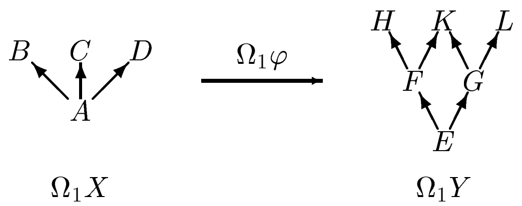


Figure 10: A superior dimap which is not a weak dipath retraction.



$$\Omega_1\varphi\,A = E$$
$$\Omega_1\varphi\,B = H$$
$$\Omega_1\varphi\,C = K$$
$$\Omega_1\varphi\,D = L$$

Figure 11: The non-surjective monotone function induced by the superior dimap from Fig. 10.

3 Local po-spaces

Martin Raussen [5] proposed a framework for handling long dipaths — i.e., dimaps of $\mathbb{R}_{\geq 0}$ into local po-spaces, cf. the definition in the beginning of Sec. 3.1 — and their dihomotopies. In particular, he made the distinction between the dipaths which have a limit (*extendible* and *finite inextendible*) and the ones that have no limit (*infinite inextendible*). The two kinds of long dipaths play very different rôles. The ones that have a limit correspond to the computations that either successfully or unsuccessfully terminate. The ones without a limit model infinitely running processes.

As noted in [5], a dipath with a limit may be dihomotopic to a dipath without a limit — see Example 21 below. This is unfortunate, because a computationally very crucial distinction escapes the formalism of algebraic topology of concurrent processes. Martin conjectures that this cannot happen in cubical complexes.

Rather than restricting the class of local po-spaces under consideration, I propose a critical look at the classical notion of dihomotopy: an arbitrary dimap from $\mathbb{I} \times X$ to Y. When the notion of dihomotopy is applied to long dipaths, as in [5], it is not clear how a dihomotopic deformation should behave "in the infinity". My answer, described below, is: it had better be continuous there as well. More precisely, dihomotopies of long dipaths had better be "uniform" mappings — as defined in Sec. 3.2 below. As demonstrated in the sequel, a uniform dihomotopy never identifies an infinitely running process with a terminating one.

Uniformity is normally studied in the context of metric rather than topological spaces, at least as much as I know about them. My area of interest is compact local po-spaces[10]. I have therefore generalized the uniformity requirement so that it does not require a metric[11].

3.1 Long dipaths

A *long dipath* in a local po-space X is a dimap $\alpha : \mathbb{R}_{\geq 0} \to X$. It may or may not have a limit in X. A long dipath is:

- *extendible* if an $x = \lim_{t\to+\infty} \alpha t$ exists and it is not a final point (i.e., there exists a non-constant dipath beginning in x);
- *finite inextendible* if an $x = \lim_{t\to+\infty} \alpha t$ exists and it is a final point;
- *infinite inextendible* if $\lim_{t\to+\infty} \alpha t$ does not exist.

The finite inextendible long dipaths model the executions that have terminated, either successfully or in a deadlock. The infinite inextendible long dipaths model the executions that go on for ever. The extendible long dipaths do not have a counterpart in computing since their "longness" may disappear after a reparameterization. A deadlocked execution (finite inextendible) may be dihomotopic to an infinitely running (infinite inextendible) even in a compact local po-space:

[10] Which is a larger class than finite cubical complexes.

[11] As pointed out by a referee, uniformity is studied in general topological spaces. I do not know whether or not these studies have something to do with the notion from Def. 22

21 Example

Take the infinite strip $\mathbb{R}_{\geq 0} \times \mathbb{I}$ and introduce a non-standard partial order:

$$\langle x_1, y_1 \rangle \leq \langle x_2, y_2 \rangle \overset{\text{def}}{\iff} x_1 \leq x_2 \ \&\ y_1 \leq y_2 \ \&\ x_1 \cdot y_2 \geq x_2 \cdot y_1$$

The last condition in the above conjunction means that the vector $\langle x_2, y_2 \rangle$ is "steeper" than $\langle x_1, y_1 \rangle$.

The verification, that this is a partial order and that it is closed in the standard topology, is straightforward. Note that two distinct points of the same height, $\langle x_1, y \rangle$ and $\langle x_2, y \rangle$ with $x_1 \neq x_2$, may be $\leq$-related only if $y = 0$. A dipath going through a point $\langle x, y \rangle$ with $y > 0$ must eventually converge to a point on the horizontal axis $y = 1$, whose points are incomparable. Fig. 12 gives some example long dipaths. It also suggests a dihomotopy between the many finite long dipaths converging to a point on the horizontal axis $t = 1$, and the single infinite long dipath going along the horizontal axis $t = 0$. Formally, this dihomotopy is given by:

$$\begin{aligned} &H : \mathbb{I} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{I} \\ &H \langle \tau, t \rangle \overset{\text{def}}{=} \left\langle \frac{\tau \cdot t}{1 + (1-\tau) \cdot t}, \frac{(1-\tau) \cdot t}{1 + (1-\tau) \cdot t} \right\rangle \end{aligned}$$

This is a non-compact global po-space. It can be wrapped on a cylinder $\mathbb{I} \times \mathbb{S}^1$, which is a compact local po-space, without losing the unwelcome dihomotopicity of a finite and an infinite long dipaths.

□

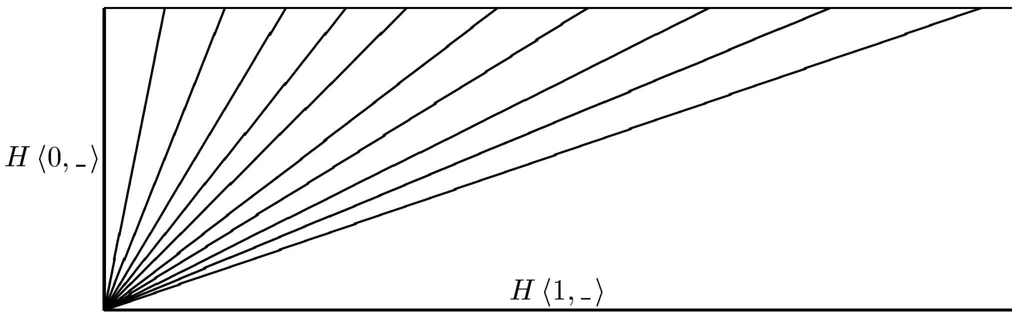


Figure 12: Dihomotopy between finite and infinite dipaths.

From now on, X will denote a compact local po-space, while α and β will denote long dipaths.

3.2 Uniform dihomotopies

22 Definition:

Assume X is a compact local po-space and Y is a local po-space (not necessarily compact). A dimap $H : \mathbb{I} \times Y \to X$ is a *uniform dihomotopy* iff H is *uniform*, i.e., for every finite cover $\mathcal{U}$ of X by open sets there exists a real $\delta > 0$ such that

$$\forall_{y \in Y} \forall_{s,s' \in \mathbb{I}} \mid s - s' \mid < \delta \Rightarrow H \langle [s, s'], y \rangle \subseteq U \text{ for a certain } U \in \mathcal{U}.$$

The above definition makes a technical sense for any local po-space X but it may be meaningless if all finite covers of X contain “big” sets, which is the case in non-compact spaces.

Two dimaps $\varphi, \psi : Y \to X$ are said to be *uniformly dihomotopic* (denoted: $\varphi \simeq_u \psi$) if there exists a uniform dihomotopy $H : \mathbb{I} \times Y \to X$ such that $H \langle 0, \_ \rangle = \varphi$ and $H \langle 1, \_ \rangle = \psi$.

23 Proposition:
The uniform dihomotopy relation $\simeq_u$ is an equivalence in the set of dimaps from Y to X.

24 Example
By $\mathbb{S}^1$ denote the circle $\left\{ e^{t \cdot 2\pi i} \,\middle|\, 0 \leq t \leq 1 \right\}$. The following two long dipaths

$$\begin{cases} \alpha : \mathbb{R}_{\geq 0} \to \mathbb{S}^1 \\ \alpha t \stackrel{\text{def}}{=} e^{0 \cdot 2\pi i} \end{cases} \quad \text{and} \quad \begin{cases} \beta : \mathbb{R}_{\geq 0} \to \mathbb{S}^1 \\ \beta t \stackrel{\text{def}}{=} e^{t \cdot 2\pi i} \end{cases}$$

(α is constant) are dihomotopic:

$$\begin{cases} H : \mathbb{I} \times \mathbb{R}_{\geq 0} \to \mathbb{S}^1 \\ H \langle s, t \rangle \stackrel{\text{def}}{=} e^{s \cdot t \cdot 2\pi i} \end{cases}$$

But they are not uniformly dihomotopic, which may be seen by investigating their liftings to the universal covering of $\mathbb{S}^1$ by a helix.

□

25 Proposition:
If both X and Y are compact then every dihomotopy $H : \mathbb{I} \times Y \to X$ is uniform.

Proof of Prop. 25 (actually, a draft of a proof):

Let $\mathcal{U}$ be a finite cover of X. Since both $\mathbb{I}$ and Y are compact, there exists a finite cover $\mathcal{V}$ of $\mathbb{I}$ and a finite cover $\mathcal{W}$ of Y such that

- each set $V \in \mathcal{V}$ is an interval; and
- for every $\langle s, y \rangle \in \mathbb{I} \times Y$ there exist $V \in \mathcal{V}$ and $W \in \mathcal{W}$ such that

  $$\langle s, y \rangle \in V \times W \subseteq H^{-1} U$$

  for a certain $U \in \mathcal{U}$.

This means the Cartesian product $\mathbb{I} \times Y$ has been covered by a finite “grid” of base open sets whose images by H are completely contained in the original cover $\mathcal{U}$ of X. Now, define

$$\delta \stackrel{\text{def}}{=} \min \left\{ \operatorname{diam} (V \cap V') \,\middle|\, V, V' \in \mathcal{V} \,\&\, V \cap V' \neq \emptyset \right\}$$

($\operatorname{diam} A \stackrel{\text{def}}{=} \inf \left\{ \mid s - s' \mid \,\middle|\, s, s' \in A \right\}$ is the *diameter* of a set $A \subseteq \mathbb{I}$).

□

3.3 Uniform invariance of the finiteness of long dipaths

The following proposition is the local po-spaces' counterpart of the elementary fact that a bounded increasing sequence has a limit. Note that its satisfaction is based on the compactness of X:

26 Proposition:
If there exists a po-open set $V \subseteq X$ such that $\alpha[t_0, +\infty) \subseteq V$ for a certain $t_0 \in \mathbb{R}_{\geq 0}$ then $\lim_{t\to+\infty} \alpha t$ *exists.*

I will also need the following topological (rather than ditopological) fact:

27 Proposition:
If $x \in V \subseteq X$ where V is open, then there exists a cover $\mathcal{U}$ of X so fine that

$$x \in U \in \mathcal{U} \ \&\ U' \cap U \neq \emptyset \Rightarrow U' \subseteq V$$

for all $U, U' \in \mathcal{U}$.

Informally, Prop. 27 says that a point sits so "deeply" inside its neighbourhood that it can be separated not only from the exterior of that neighbourhood but also from the open sets that intersect the exterior. All the proof needs are separation properties, which is OK, because X is compact and therefore normal.

Now assume $H : \mathbb{I} \times \mathbb{R}_{\geq 0} \to X$ is a uniform dihomotopy on long dipaths. To distinguish the finite from the infinite ones, consider the set

$$L_H \stackrel{\text{def}}{=} \left\{ s \in \mathbb{I} \,\middle|\, \lim_{t\to+\infty} H\langle s, t\rangle \text{ exists} \right\}$$

28 Lemma: *L_H is open in $\mathbb{I}$.*

Proof of Lemma 28:

For an $\bar{s} \in L_H$ we have to find an open neighbourhood totally contained in L_H. By the definition of L_H, the limit

$$\bar{x} \stackrel{\text{def}}{=} \lim_{t\to+\infty} H\langle \bar{s}, t\rangle \tag{2}$$

exists. Take a po-neighbourhood V of $\bar{x}$ and a finite cover $\mathcal{U}$ of X selected by Prop. 27. Select a δ for this cover from the uniformity condition:

$$\forall_{t\in\mathbb{R}_{\geq 0}} \forall_{s,s'\in\mathbb{I}} \mid s - s' \mid < \delta \Rightarrow H\langle [s, s'], t\rangle \subseteq U \text{ for a certain } U \in \mathcal{U}. \tag{3}$$

Let s be close to $\bar{s}$: $\mid s - \bar{s} \mid < \delta$; and let $\bar{x} \in U \in \mathcal{U}$. By (2), there is a $\bar{t}$ such that

$$H\langle \bar{s}, [\bar{t}, +\infty)\rangle \subseteq U \tag{4}$$

and by (3), there exist sets $U_t \in \mathcal{U}$ such that

$$H\langle [\bar{s}, s], t\rangle \in U_t \quad \text{for all } t \geq \bar{t}. \tag{5}$$

Since $H\langle \bar{s}, t\rangle \in U_t \cap U$ (cf. (4) and (5)), $U_t \subseteq V$ for all $t \geq \bar{t}$. Therefore, $H\langle s, t\rangle \in V$ for $t \geq \bar{t}$ and thus $H\langle s, [\bar{t}, +\infty)\rangle \subseteq V$. Now, Prop. 26 implies the existence of the limit $\lim_{t\to+\infty} H\langle s, t\rangle$.

This completes the proof of Lemma 28.

□

29 Lemma: *L_H is closed in $\mathbb{I}$.*

Proof of Lemma 29:
Let $s_0, s_1, s_2, \ldots$ be a convergent sequence in L_H:

$$\bar{s} \stackrel{\text{def}}{=} \lim_{n\to+\infty} s_n \quad \text{and} \tag{6}$$

$$x_n \stackrel{\text{def}}{=} \lim_{t\to+\infty} H\langle s_n, t\rangle \quad \text{for } n \in \mathbb{N}. \tag{7}$$

We need to prove that $\bar{s} \in L_H$.
Since X is compact, there exists a convergent subsequence

$$\bar{x} \stackrel{\text{def}}{=} \lim_{k\to+\infty} x_{n_k} \tag{8}$$

To prove that $\lim_{t\to+\infty} H\langle \bar{s}, t\rangle = \bar{x}$, take an arbitrary neighbourhood V of $\bar{x}$ and find a finite cover $\mathcal{U}$ of X as in Prop. 27. Select a δ for that cover from the uniformity condition. Let $\bar{x} \in U \in \mathcal{U}$; then $U \subseteq V$. Let ℓ be such that for any $k \geq \ell$:

$$x_{n_k} \in U \quad \text{(cf. (8))} \tag{9}$$

and

$$\mid s_{n_k} - \bar{s} \mid < \delta \quad \text{(cf. (6))}.$$

Let $\bar{t}$ be such that

$$H\langle s_{n_\ell}, [\bar{t}, +\infty)\rangle \subseteq U \quad \text{(cf. (7) and (9))}.$$

The uniformity condition implies the existence of sets $U_t \in \mathcal{U}$ such that

$$H\langle [s_{n_\ell}, \bar{s}], t\rangle \subseteq U_t \quad \text{for all } t \geq \bar{t}.$$

Since $H\langle s_{n_\ell}, t\rangle \in U_t \cap U$, $U_t \subseteq V$ for all $t \geq \bar{t}$. Therefore,

$$H\langle \bar{s}, t\rangle \in V \quad \text{for all } t \geq \bar{t}.$$

This completes the proof of Lemma 29.

□

30 Corollary:
If the long dipaths $\alpha, \beta : \mathbb{R}_{\geq 0} \to X$ are uniformly dihomotopic then

$$\lim_{t\to+\infty} \alpha t \text{ exists} \quad \text{if and only if} \quad \lim_{t\to+\infty} \beta t \text{ exists.}$$

Proof of Cor. 30:
By Lemma 28 and Lemma 29, L_H is either empty or the whole $\mathbb{I}$.

□

3.4 Discussion of related notions

Uniform dihomotopies are not only continuous, they are “continuous in the infinity” too. As demonstrated in Sec. 3.3, the uniformity of the dihomotopies implies some good properties of respective maps, thus answering the problem raised in [5]. But there are other problems worth mentioning.

In [5], Martin Raussen considers a different restriction on the set of allowed morphisms of (local) po-spaces: a dimap $f : Y \to Z$ is *busy* if

$$\alpha \text{ is infinite} \quad \Rightarrow \quad f \circ \alpha \text{ is infinite}$$

for an arbitrary long dipath $\alpha : \mathbb{R}_{\geq 0} \to Y$ [12].

Note that the set of busy dimaps is not closed in the set of all dimaps:

31 Example

Consider the sequence of dimaps

$$f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$
$$f_n t \stackrel{\text{def}}{=} \frac{t}{n}$$

for $n \in \mathbb{N}$. Each of them is strict (as defined on p. 6) and busy, but the limit $\bar{f}t \stackrel{\text{def}}{=} 0$ is neither strict nor busy.

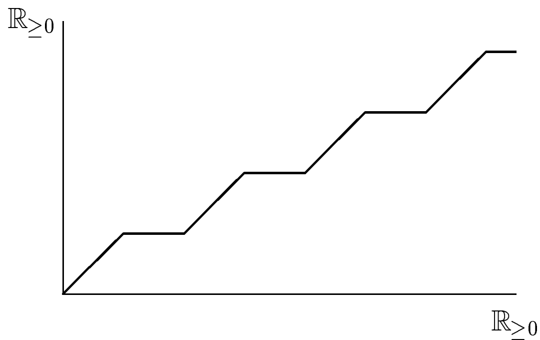
□

Unlike strictness, business does not contribute anything new in the compact case; in particular, busy dimaps cannot serve for distinguishing the dihomotopy types of the po-space with a square hole and the Swiss flag:

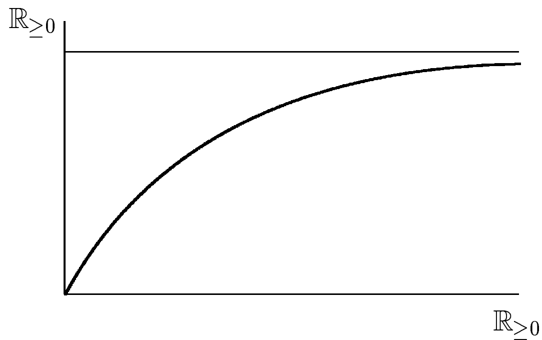
32 Proposition: *If Y is compact then every dimap $f : Y \to Z$ is busy.*

33 Example

In the case when Y is not compact, the two classes have a non-empty intersection but neither is contained in the other:



Busy but not strict.



Strict but not busy.

□

[12] This is equivalent but not identical to the original definition in [5].

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