Parallel PCF has a Unique Extensional Model*

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Abstract

We show that the continuous function model is the unique extensional (but not necessarily pointwise ordered) model of the variant of the applied typed lambda calculus PCF that includes the "parallel or" operation.

1 Introduction

Several extensional models of the applied typed lambda calculus PCF are known to exist, including:

(i) The continuous function model, which is order-extensional (pointwise ordered) but not equationally fully abstract [Plo]. (A model is equationally fully abstract when terms are identified in the model exactly when they are operationally equivalent.)

(ii) The stable function model, which is neither order-extensional nor equationally fully abstract [Ber][BCL].

(iii) The terminal object of the category of equationally fully abstract, extensional models, which is inequationally fully abstract and order-extensional [Mil][Sto2]. (A model is inequationally fully abstract iff one term is less than another in the model exactly when the first is operationally less defined than the second.)

(iv) The initial object of the above category, which is neither inequationally fully abstract nor order-extensional [Sto2].

In contrast, the only known extensional model of *parallel PCF*, i.e., PCF augmented with the "parallel or" operation, is the continuous function model, which is inequationally fully abstract and order-extensional [Plo]. In fact, a result of Plotkin/Milner/Berry's shows that this model is the unique inequationally fully abstract, extensional model of parallel PCF [Ber][BCL][Mil][Plo]. But does parallel PCF have extensional models that are not inequationally fully abstract or not even equationally fully abstract? What about (necessarily non-inequationally fully abstract) extensional models that are not order-extensional? The purpose of this paper is to answer these questions in the negative.

Some of the techniques used in our proof that the continuous function model is the unique extensional model of parallel PCF are similar to the ones used by Plotkin in his proof of the definability theorem for parallel PCF (see lemma 4.5 of [Plo]).

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2 Preliminaries

The reader is assumed to be familiar with such standard domain-theoretic concepts as complete partial orders (cpo's), continuous functions, and ω-algebraic, strongly algebraic and consistently complete cpo's. Given cpo's X and Y, we write $X \underset{c}{\rightarrow} Y$ for the cpo of all continuous functions from X to Y, ordered pointwise.

Familiarity with algebras and ordered algebras over many-sorted signatures is also assumed (see [Sto1] for an introduction to this material). Signatures contain distinguished constants Ω_s at each sort s, which stand for divergence and are interpreted as least elements in ordered algebras. We use uppercase script letters ($\mathcal{A}$, $\mathcal{B}$, etc.) to denote algebras and the corresponding italic letters (A, B, etc.) to stand for their carriers. The initial ordered algebra consists of the initial (term) algebra with the "Ω-match" ordering: one term is less than another when the second can be formed by replacing occurrences of Ω in the first by terms. An ordered algebra is called *complete* iff its carrier is a cpo and operations are continuous, and an *order-isomorphism* over complete ordered algebras is a homomorphism that is a surjective order-embedding on the underlying cpo's.

For technical simplicity, we have chosen to work with a combinatory logic version of parallel PCF with a single ground type ι, whose intended interpretation is the natural numbers. From the viewpoint of the conditional and parallel or operations, non-zero and zero are interpreted as true and false, respectively.

The syntax of parallel PCF is specified by a signature, the sorts of which consist of parallel PCF's types. The set of *sorts* S is least such that

(i) $\iota \in S$, and

(ii) $s_1 \rightarrow s_2 \in S$ if $s_1 \in S$ and $s_2 \in S$.

As usual, we let $\rightarrow$ associate to the right. Define ι^n, for $n \in \omega$, by: $\iota^0 = \iota$ and $\iota^{n+1} = \iota \rightarrow \iota^n$. The *signature* Σ over S has the following operators:

(i) Ω_s of type s,

(ii) K_{s_1,s_2} of type $(s_1 \rightarrow s_2 \rightarrow s_1)$,

(iii) S_{s_1,s_2,s_3} of type $((s_1 \rightarrow s_2 \rightarrow s_3) \rightarrow (s_1 \rightarrow s_2) \rightarrow s_1 \rightarrow s_3)$,

(iv) Y_s of type $((s \rightarrow s) \rightarrow s)$,

(v) n of type ι, for $n \in \omega$,

(vi) $Succ$ and $Pred$ of type $(\iota \rightarrow \iota)$,

(vii) If_s of type $(\iota \rightarrow s \rightarrow s \rightarrow s)$,

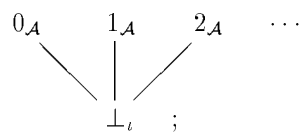
(viii) POr of type (ι^2),

(ix) $\cdot_{s_1,s_2}$ of type $(s_1 \rightarrow s_2) \times s_1 \rightarrow s_2$,

where the compound sorts are parenthesized in order to avoid confusion. Thus $\cdot$ (application) is a binary operator, and all of the other operators are nullary. In keeping with standard practice, we usually abbreviate $M \cdot N$ to $M\,N$, and let application associate to the left.

A *model* $\mathcal{A}$ of parallel PCF is a complete ordered algebra such that the following conditions hold:

(i) A_ι is the flat cpo



;

(ii) For all $a_1 \in A_{s_1}$ and $a_2 \in A_{s_2}$, $K_{s_1,s_2}\, a_1\, a_2 = a_1$;

(iii) For all $a_1 \in A_{s_1 \to s_2 \to s_3}$, $a_2 \in A_{s_1 \to s_2}$ and $a_3 \in A_{s_1}$, $S_{s_1,s_2,s_3}\, a_1\, a_2\, a_3 = a_1\, a_3\, (a_2\, a_3)$;

(iv) For all $a \in A_{s \to s}$, $Y_s\, a$ is the least fixed point of the continuous function over A_s that a represents;

(v) For all $a \in A_\iota$, $Succ\, a$ is equal to $\perp$, if $a = \perp$, and is equal to $a + 1$, if $a \in \omega$;

(vi) For all $a \in A_\iota$, $Pred\, a$ is equal to $\perp$, if $a = \perp$, is equal to 0, if $a = 0$, and is equal to $a - 1$, if $a \in \omega - \{0\}$;

(vii) For all $a_1 \in A_\iota$ and $a_2, a_3 \in A_s$, $If_s\, a_1\, a_2\, a_3$ is equal to $\perp$, if $a_1 = \perp$, is equal to a_2, if $a_1 \in \omega - \{0\}$, and is equal to a_3, if $a_1 = 0$;

(viii) For all $a_1, a_2 \in A_\iota$, $POr\, a_1\, a_2$ is equal to 1, if either $a_1 \in \omega - \{0\}$ or $a_2 \in \omega - \{0\}$, is equal to 0, if $a_1 = 0$ and $a_2 = 0$, and is equal to $\perp$, otherwise;

(ix) For all $a_1, a_2 \in A_{s_1 \to s_2}$, if $a_1\, a = a_2\, a$, for all $a \in A_{s_1}$, then $a_1 = a_2$.

We require that a model $\mathcal{A}$ be a complete ordered algebra so that each A_s is a cpo with least element Ω_s and the application operations $\cdot_{s_1,s_2}: A_{s_1 \to s_2} \times A_{s_1} \to A_{s_2}$ are continuous. Condition (iv) says that the recursion constants Y_s are least fixed point operations, and conditions (ii) and (iii) require that models be combinatory algebras. Condition (i) says that A_ι is the flat cpo of natural numbers, and conditions (v)–(viii) require that the operations on A_ι behave as expected. Finally, condition (ix) says that models are *extensional* and has been included in the definition of model since we have no need to consider non-extensional models in the sequel.

Application is left-strict in all models $\mathcal{A}$ since $\perp_{s_1 \to s_2} \sqsubseteq_{s_1 \to s_2} K_{s_2,s_1} \perp_{s_2}$, and thus $\perp_{s_1 \to s_2} a \sqsubseteq_{s_2} K_{s_2,s_1} \perp_{s_2} a = \perp_{s_2}$, for all $a \in A_{s_1}$. It is easy to see that if $a_1, a_2 \in A_{s_1 \to \cdots \to s_m \to s'}$ for $m \geq 0$, then $a_1 = a_2$ iff $a_1\, x_1\, \cdots\, x_m = a_2\, x_1\, \cdots\, x_m$ for all $x_i \in A_{s_i}$, $1 \leq i \leq m$.

The *continuous function model* is the unique model $\mathcal{A}$ such that $A_\iota = \omega_\perp$, $A_{s_1 \to s_2} = A_{s_1} \xrightarrow{c} A_{s_2}$ for all $s_1, s_2 \in S$, application is function application and $n_\mathcal{A} = n$ for all $n \in \omega$.

Next, we define several combinators that will be required below. We confuse use and mention for these combinators: given a combinator C, we also write C for its denotation in any model that may be at hand.

For $s \in S$, we write I_s for the term

$$S_{s,s \to s,s}\, K_{s,s \to s}\, K_{s,s}$$

of sort $s \to s$. I is the identity operation in all models.

We code lambda abstractions in terms of the S, K and I combinators, in the standard way.

For $s \in S$, define approximations Y_s^n to Y_s of sort $(s \to s) \to s$ by

$$Y_s^0 = \Omega_{(s \to s) \to s}, \qquad Y_s^{n+1} = S_{s \to s,s,s}\, I_{s \to s}\, Y_s^n,$$

so that Y_s^n is an ω-chain in the initial ordered algebra, and thus in all models.

Following [Mil][Ber][BCL], we can define syntactic projections Ψ_s^n of sort $s \to s$, for all $n \in \omega$ and $s \in S$, by

$$\Psi_\iota^n = Y_{\iota \to \iota}^n\, F, \qquad \Psi_{s_1 \to s_2}^n = \lambda xy.\, \Psi_{s_2}^n(x(\Psi_{s_1}^n\, y)),$$

where F of sort $(\iota \to \iota) \to \iota \to \iota$ is

$$\begin{array}{l} \lambda xy.\, If\ y \\ \qquad (Succ(x(Pred\ y))) \\ \qquad 0. \end{array}$$

Expanding the abstractions, one can see that the Ψ_s^n form an ω-chain in the initial ordered algebra. Furthermore, in any model $\mathcal{A}$, the Ψ_s^n represent an ω-chain of continuous projections (retractions less than the identity function) with finite image whose lub is the identify function. Thus each A_s is a strongly algebraic (SFP) cpo whose set of isolated elements is $\bigcup_{n\in\omega} A_s^n$, where we write A_s^n for the subposet of A_s whose elements are $\{\, \Psi_s^n\, a \mid a \in A_s \,\}$. Clearly $A_\iota^n = \{\perp, 0, 1, \ldots, n-1\}$, $A_s^0 = \{\perp_s\}$ and $A_s^n \subseteq A_s^m$ if $n \leq m$. It is useful to note that for all $a \in A_{s_1 \to \cdots \to s_m \to s'}$, $a \in A^n_{s_1 \to \cdots \to s_m \to s'}$ iff

$$a\, x_1\, \cdots\, x_m = \Psi^n (a(\Psi^n\, x_1) \cdots (\Psi^n\, x_m))$$

for all $x_i \in A_{s_i}$. Given sorts s_1 and s_2, we define an ω-chain

$$\psi^n_{s_1,s_2} \in (A_{s_1} \xrightarrow[c]{} A_{s_2}) \xrightarrow[c]{} (A_{s_1} \xrightarrow[c]{} A_{s_2})$$

of continuous projections with finite image whose lub is the identity function by

$$\psi^n_{s_1,s_2}\, f\, x = \Psi^n_{s_2} \cdot (f(\Psi^n_{s_1} \cdot x)).$$

This demonstrates that $A_{s_1} \xrightarrow[c]{} A_{s_2}$ is also strongly algebraic.

Although POr is not isolated in any model, it is uniformly interdefinable in all models with $\Psi^2\, POr$, which is isolated.

Let the equality test Eq of sort ι^2 be

$$\begin{aligned} Y(\lambda zxy.\, \mathit{If}\ x\ & \\ & (\mathit{If}\ y\, (z(\mathit{Pred}\ x)(\mathit{Pred}\ y))\, 0) \\ & (\mathit{Not}\ y)), \end{aligned}$$

where Not of sort $\iota \to \iota$ is $\lambda x.\, \mathit{If}\ x\, 0\, 1$.

For $n \in \omega$, define operators And_n of sort ι^n by: $\mathit{And}_0 = 1$ and

$$\mathit{And}_{n+1} = \lambda x y_1 \cdots y_n.\, \mathit{If}\ x\, (\mathit{And}_n\, y_1\ \cdots\ y_n)\, 0.$$

From [Sto3], we know that we can define the parallel if operator, PIf, of sort ι^3 by $\mathit{PIf} = Y_{\iota^4}\, H\, 0$, where H of sort $\iota^4 \to \iota^4$ is defined by

$$\begin{aligned} H = \lambda fwxyz.\, \mathit{If}\ (POr\ & (PAnd\, (Eq\, y\, w)\, (Eq\, z\, w)) \\ & (PAnd\ x \qquad\ (Eq\, y\, w)) \\ & (PAnd\, (Not\ x)\ \ (Eq\, z\, w))) \\ w & \\ (f\, (Succ\ w)\, x\, y\, z)&. \end{aligned}$$

Here, we have extended POr to three arguments in the obvious way, and $PAnd$ of sort ι^2 is the "parallel and" operation, dual to POr:

$$PAnd = \lambda xy.\, Not\, (POr\, (Not\ x)\, (Not\ y)).$$

Then, for all models $\mathcal{A}$ and $a_1, a_2, a_3 \in A_\iota$, $\mathit{PIf}\ a_1\, a_2\, a_3$ is equal to $a_2 \sqcap a_3$, if $a_1 = \perp$, is equal to a_2, if $a_1 \in \omega - \{0\}$, and is equal to a_3, if $a_1 = 0$.

3 Standard Models

Let $\leq$ be the least reflexive relation over S such that

$$s' \leq s_1 \rightarrow s_2 \text{ if } s' \leq s_1 \text{ or } s' \leq s_2.$$

Then $\leq$ is a partial ordering, and we read $s \leq s'$ as s is a *subsort* of s'.

Let $s \in S$. A model $\mathcal{A}$ is

(i) *s-order-extensional* iff for all $s_1, s_2 \in S$ such that $s_1 \rightarrow s_2 \leq s$ and $a_1, a_2 \in A_{s_1 \rightarrow s_2}$,

$$a_1 \sqsubseteq a_2 \text{ iff } a_1\, x \sqsubseteq a_2\, x \text{ for all } x \in A_{s_1}; \text{ and}$$

(ii) *s-standard* iff it is s-order-extensional and for all $s_1, s_2 \in S$ such that $s_1 \rightarrow s_2 \leq s$ and $f \in A_{s_1} \xrightarrow[c]{} A_{s_2}$, there is a (unique) $a \in A_{s_1 \rightarrow s_2}$ such that $a \cdot x = f\, x$ for all $x \in A_{s_1}$.

We say that a model is *order-extensional* (respectively, *standard*) iff it is s-order-extensional (respectively, s-standard) for all $s \in S$.

By the above definitions, all models are ι-order-extensional and ι-standard, and if $\mathcal{A}$ is s-order-extensional (respectively, s-standard) and $s' \leq s$, then $\mathcal{A}$ is s'-order-extensional (respectively, s'-standard). It is easy to see that the continuous function model is standard.

Proposition 3.1 *If $\mathcal{A}$ and $\mathcal{B}$ are standard models, then there is a unique order-isomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Routine. The order-isomorphism can be defined by recursion on S. $\Box$

Lemma 3.2 (Milner) *If $\mathcal{A}$ is an s-order-extensional model, then A_s is a Scott domain, i.e., a consistently complete, ω-algebraic cpo.*

Proof. First, define glb operators Inf_s of sort $s \rightarrow s \rightarrow s$ by

$$\begin{aligned} Inf_\iota &= \lambda xy.\, If\,(Eq\, x\, y)\, x\, \Omega, \\ Inf_{s_1 \rightarrow s_2} &= \lambda xyz.\, Inf_{s_2}(x\, z)(y\, z), \end{aligned}$$

and show by induction on $s \in S$ that if $\mathcal{A}$ is s-order-extensional, then $Inf_s\, a_1\, a_2$ is the glb of a_1 and a_2 for all $a_1, a_2 \in A_s$. The result then follows from the fact that strongly algebraic cpo's with binary glb's have lub's of all consistent pairs, i.e., are consistently complete. See [Mil] (or lemma 5.5 of [Sto2]). $\Box$

Lemma 3.3 *Suppose that $\mathcal{A}$ is an s_1- and s_2-standard model such that*

(i) *for all isolated $a_1, a_2 \in A_{s_1 \rightarrow s_2}$, $a_1 \sqsubseteq a_2$ iff $a_1\, x \sqsubseteq a_2\, x$ for all $x \in A_{s_1}$; and*

(ii) *for all isolated $f \in A_{s_1} \xrightarrow[c]{} A_{s_2}$, there is an isolated $a \in A_{s_1 \rightarrow s_2}$ such that $a \cdot x = f\, x$ for all $x \in A_{s_1}$.*

Then $\mathcal{A}$ is $s_1 \rightarrow s_2$-standard.

Proof. We begin by showing that $\mathcal{A}$ is $s_1 \rightarrow s_2$-order-extensional. Suppose that $a_1, a_2 \in A_{s_1 \rightarrow s_2}$ and $a_1\, x \sqsubseteq a_2\, x$ for all $x \in A_{s_1}$. Suppose toward a contradiction that $a_1 \not\sqsubseteq a_2$. Then, since $a_1 = \bigsqcup_{n \in \omega} \Psi^n\, a_1$, there exists a least $n \in \omega$ such that $\Psi^n\, a_1 \not\sqsubseteq a_2$. Then, for all $x \in A^n_{s_1}$,

$$\Psi^n\, a_1\, x \sqsubseteq a_1\, x \sqsubseteq a_2\, x = \bigsqcup_{m \in \omega} (\Psi^m\, a_2\, x),$$

and, since $\Psi^n\, a_1\, x$ is isolated, there is a least $m_x \geq n$ such that $\Psi^n\, a_1\, x \sqsubseteq \Psi^{m_x}\, a_2\, x$. But $A^n_{s_1}$ is finite, and thus we can let l be the greatest m_x such that $x \in A^n_{s_1}$. Clearly, $l \geq n$ and $\Psi^n\, a_1\, x \sqsubseteq \Psi^{m_x}\, a_2\, x \sqsubseteq \Psi^l\, a_2\, x$ for all $x \in A^n_{s_1}$, and thus

$$\begin{aligned}\Psi^n\, a_1\, x &= \Psi^n\, a_1\, (\Psi^n\, x) \\ &\sqsubseteq \Psi^l\, a_2\, (\Psi^n\, x) \\ &\sqsubseteq \Psi^l\, a_2\, (\Psi^l\, x) \\ &= \Psi^l\, a_2\, x\end{aligned}$$

for all $x \in A_{s_1}$. But then (i) implies that $\Psi^n\, a_1 \sqsubseteq \Psi^l\, a_2$, in contradiction to the fact that $\Psi^n\, a_1 \not\sqsubseteq a_2$.

Now, suppose that $f \in A_{s_1} \xrightarrow[c]{} A_{s_2}$. By (ii) and the $s_1 \to s_2$-order-extensionality of $\mathcal{A}$, there is an ω-chain $a_n \in A_{s_1 \to s_2}$ of isolated elements such that $a_n \cdot x = \psi^n_{s_1,s_2}\, f\, x$ for all $x \in A_{s_1}$. But then it is easy to see that $(\bigsqcup_{n\in\omega} a_n) \cdot x = f\, x$ for all $x \in A_{s_1}$. $\square$

Lemma 3.4 *If $\mathcal{A}$ is an $s_1 \to s_2$-standard model, then two elements $a_1, a_2 \in A_{s_1 \to s_2}$ are inconsistent iff there is an $x \in A_{s_1}$ such that $a_1\, x$ and $a_2\, x$ are inconsistent.*

Proof. The "if" direction is obvious. For the "only if" direction, suppose toward a contradiction that there is no such x. Then we can define an $h \in A_{s_1} \xrightarrow[c]{} A_{s_2}$ by $h\, x = (a_1 \cdot x) \sqcup (a_2 \cdot x)$, since A_{s_2} is consistently complete (lemma 3.2). Furthermore, there is an $a' \in A_{s_1 \to s_2}$ such that $a' \cdot x = h\, x$ for all $x \in A_{s_1}$, since $\mathcal{A}$ is $s_1 \to s_2$-standard. But a' is the lub of a_1 and a_2 since $\mathcal{A}$ is $s_1 \to s_2$-order-extensional—a contradiction. $\square$

Lemma 3.5 *If $\mathcal{A}$ is an s-order-extensional model, then for all isolated elements $a \in A_s$, there is a unique $[a] \in A_{s\to\iota}$ such that*

$$[a]x = \begin{cases} 1 & \textit{if } x \sqsupseteq a, \\ \bot & \textit{if } x \not\sqsupseteq a. \end{cases}$$

Proof. By induction on S. For the base case ι, we can define $[\bot_\iota] = \lambda x.\, 1$ and

$$[n] = \lambda x.\, \mathit{If}\, (Eq\; n\; x)\, 1\, \Omega, \text{ for } n \in \omega.$$

Now, suppose that the result holds for $s_1, s_2 \in S$ and that $\mathcal{A}$ is $s_1 \to s_2$-order-extensional. If $a \in A_{s_1 \to s_2}$ is isolated, then $a \in A^n_{s_1 \to s_2}$ for some $n \in \omega$. Let $u_1, \ldots, u_m$ be an enumeration of $A^n_{s_1}$, so that $a\, u_i \in A^n_{s_2}$ for all $1 \leq i \leq m$. We can then define

$$[a] = \lambda x.\, And_m\, ([a\, u_1](x\, u_1)) \cdots ([a\, u_m](x\, u_m)),$$

since $\mathcal{A}$ is s_2-order-extensional. If $x \sqsupseteq a$, then $x\, u_i \sqsupseteq a\, u_i$ for all i, and thus $[a]x = 1$. Alternatively, if $x \not\sqsupseteq a$, then there exists a $y \in A_{s_1}$ such that $x\, y \not\sqsupseteq a\, y$. But then $x(\Psi^n\, y) \not\sqsupseteq a(\Psi^n\, y)$, since otherwise we would have

$$x\, y \sqsupseteq x(\Psi^n\, y) \sqsupseteq a(\Psi^n\, y) = a\, y.$$

Thus $[a]x = \bot$. $\square$

Lemma 3.6 *If $\mathcal{A}$ is an s-standard model, then for all inconsistent pairs of isolated elements $a_1, a_2 \in A_s$, there is a unique $[a_1, a_2] \in A_{s\to\iota}$ such that*

$$[a_1, a_2]x = \begin{cases} 1 & \textit{if } x \sqsupseteq a_1, \\ 0 & \textit{if } x \sqsupseteq a_2, \\ \bot & \textit{if } x \not\sqsupseteq a_1 \textit{ and } x \not\sqsupseteq a_2. \end{cases}$$

Proof. By induction on S. For the base case ι, define

$$[n,m] = \lambda x.\, \mathit{If}\,(Eq\; x\; n)\; 1\; (\mathit{If}\,(Eq\; x\; m)\; 0\; \Omega),$$

for all $n, m \in \omega$ such that $n \neq m$. Now, suppose that the result holds for $s_1, s_2 \in S$ and that $\mathcal{A}$ is $s_1 \to s_2$-standard. If $a_1, a_2 \in A_{s_1 \to s_2}$ is an inconsistent pair of isolated elements, then $a_1, a_2 \in A^n_{s_1 \to s_2}$ for some $n \in \omega$. Lemma 3.4 tells us that there exists an $x \in A_{s_1}$ such that $a_1\, x, a_2\, x \in A^n_{s_2}$ are inconsistent. Thus we can define

$$[a_1, a_2] = \lambda h.\, \mathit{If}\,([a_1\, x, a_2\, x](h\, x))\; ([a_1]h)\; (\mathit{If}\,([a_2]h)\; 0\; \Omega),$$

by lemma 3.5 and the inductive hypothesis. $\Box$

As the reader may have noticed, we have made no use of parallel or so far, and thus everything that we have proved will also hold for models of ordinary, sequential PCF. The proof of the next lemma does require the existence of parallel or, however, and that lemma and the following theorem and corollary do not hold for models of sequential PCF.

Given a model $\mathcal{A}$, $n \in \omega$ and $s_1, \ldots, s_m \in S$ for $m \geq 1$, we define a poset $\mathbf{A}^n_{s_1,\ldots,s_m}$ as follows. Its elements are the partial functions f from $A^n_{s_1} \times \cdots \times A^n_{s_m}$ to A^n_ι that are *consistent* in the sense that if $\langle\langle a_1, \ldots, a_m\rangle, l\rangle \in f$ and $\langle\langle a'_1, \ldots, a'_m\rangle, l'\rangle \in f$ for $l, l' \in \omega$, then either $l = l'$ or there is an i such that a_i and a'_i are inconsistent in $A^n_{s_i}$ (or, equivalently, in A_{s_i}). The elements of $\mathbf{A}^n_{s_1,\ldots,s_m}$ are ordered by: $f \leq g$ iff $\mathit{dom}\, f = \mathit{dom}\, g$ and $f\langle a_1, \ldots, a_m\rangle \sqsubseteq g\langle a_1, \ldots, a_m\rangle$ whenever $\langle a_1, \ldots, a_m\rangle \in \mathit{dom}\, f$. If $f \in \mathbf{A}^n_{s_1,\ldots,s_m}$ and $\langle a_1, \ldots, a_m\rangle \in \mathit{dom}\, f$, then we write $f \setminus \langle a_1, \ldots, a_m\rangle$ for the element $f|(\mathit{dom}\, f) - \{\langle a_1, \ldots, a_m\rangle\}$ of $\mathbf{A}^n_{s_1,\ldots,s_m}$.

We say that an $f \in \mathbf{A}^n_{s_1,\ldots,s_m}$ *represents* an $a \in A^n_{s_1 \to \cdots \to s_m \to \iota}$ iff for all $\langle x_1, \ldots, x_m\rangle \in A^n_{s_1} \times \cdots \times A^n_{s_m}$ and $l \in A^n_\iota - \{\bot\}$, $a\, x_1\, \cdots\, x_m = l$ iff there exists a $\langle x'_1, \ldots, x'_m\rangle \in \mathit{dom}\, f$ such that $x'_i \sqsubseteq x_i$ for all i and $f\langle x'_1, \ldots, x'_m\rangle = l$. It is easy to see that any $f \in \mathbf{A}^n_{s_1,\ldots,s_m}$ represents at most one $a \in A^n_{s_1 \to \cdots \to s_m \to \iota}$, since if a_1 and a_2 are represented by f, then

$$a_1\, x_1\, \cdots\, x_m = a_2\, x_1\, \cdots\, x_m$$

for all $x_i \in A^n_{s_i}$, and thus $a_1 = a_2$.

Lemma 3.7 *Suppose $n \in \omega$, $s_1, \ldots, s_m \in S$ for $m \geq 1$ and $\mathcal{A}$ is a model that is s_i-standard for all i. Then for all $f \in \mathbf{A}^n_{s_1,\ldots,s_m}$, there exists a unique $\widehat{f} \in A^n_{s_1 \to \cdots \to s_m \to \iota}$ that is represented by f. Furthermore, if $f \leq g$, then $\widehat{f} \sqsubseteq \widehat{g}$, for all $f, g \in \mathbf{A}^n_{s_1,\ldots,s_m}$.*

Proof. If $n = 0$, then both elements of $\mathbf{A}^n_{s_1,\ldots,s_m}$ represent the single element of $A^n_{s_1 \to \cdots \to s_m \to \iota}$, $\bot$. So, assume that $n \geq 1$.

We show by induction on $k \in \omega$ that for all $f, g \in \mathbf{A}^n_{s_1,\ldots,s_m}$, if $|f| = k$ and $f \leq g$, then $\widehat{f}$ and $\widehat{g}$ exist and $\widehat{f} \sqsubseteq \widehat{g}$. The base case $k = 0$ is obvious since $\emptyset$ represents $\bot$. So, assume that the result holds for k, $|f| = k + 1$ and $f \leq g$. There are two cases to consider.

(i) Suppose that there exist $\langle a_1, \ldots, a_m \rangle \in dom\, f$ and $\langle a'_1, \ldots, a'_m \rangle \in dom\, f$ such that a_i and a'_i are inconsistent for some i. Define

$$F = f \setminus \langle a_1, \ldots, a_m \rangle, \qquad G = g \setminus \langle a_1, \ldots, a_m \rangle$$

and

$$F' = f \setminus \langle a'_1, \ldots, a'_m \rangle, \qquad G' = g \setminus \langle a'_1, \ldots, a'_m \rangle,$$

so that, by the inductive hypothesis, $\widehat{F}$, $\widehat{F'}$, $\widehat{G}$ and $\widehat{G'}$ exist, $\widehat{F} \sqsubseteq \widehat{G}$ and $\widehat{F'} \sqsubseteq \widehat{G'}$. Then, we can define

$$\widehat{f} = H\, \widehat{F}\, \widehat{F'}, \qquad \widehat{g} = H\, \widehat{G}\, \widehat{G'},$$

where

$$\begin{aligned} H = \lambda y y' x_1 \cdots x_m.\ & \mathit{PIf}\, ([a_i, a'_i] x_i) \\ & (y'\ x_1\ \cdots\ x_m) \\ & (y\ \ x_1\ \cdots\ x_m). \end{aligned}$$

Then $\widehat{f} \sqsubseteq \widehat{g}$ since application is monotonic, and it is straightforward to check that $\widehat{f}$ and $\widehat{g}$ have the required properties.

(ii) Suppose that $\langle a_1, \ldots, a_m \rangle \in dom\, f$ and $\langle a'_1, \ldots, a'_m \rangle \in dom\, f$ implies that a_i and a'_i are consistent for all i. Then, there is an $l \in A^n_\iota - \{\bot\}$ such that $ran\, f \subseteq \{\bot, l\} \supseteq ran\, g$. (If $ran\, g = \{\bot\}$, then l can be any element of $A^n_\iota - \{\bot\}$.) Suppose that $\langle \langle a_1, \ldots, a_m \rangle, b \rangle \in f$ and $\langle \langle a_1, \ldots, a_m \rangle, b' \rangle \in g$, so that $b \sqsubseteq b'$. Define

$$f' = f \setminus \langle a_1, \ldots, a_m \rangle, \qquad g' = g \setminus \langle a_1, \ldots, a_m \rangle,$$

so that $\widehat{f'}$ and $\widehat{g'}$ exist and $\widehat{f'} \sqsubseteq \widehat{g'}$. Then we can define

$$\widehat{f} = H\, \widehat{f'}\, b, \qquad \widehat{g} = H\, \widehat{g'}\, b',$$

where

$$\begin{aligned} H = \lambda y z x_1 \cdots x_m.\ & \\ & Z\, (\mathit{If}\, (And_m\, ([a_1] x_1)\ \cdots\ ([a_m] x_m))\, z\, \Omega) \\ & \ (y\, x_1\ \cdots\ x_m) \end{aligned}$$

and Z of sort ι^2 stands for

$$\lambda x y.\ \mathit{If}\, (POr\, (Eq\, l\, x)\, (Eq\, l\, y))\, l\, \Omega.$$

Clearly $\widehat{f} \sqsubseteq \widehat{g}$, and it is straightforward to check that $\widehat{f}$ and $\widehat{g}$ have the required properties. □

Theorem 3.8 *All models are standard.*

Proof. We show that any model $\mathcal{A}$ is s-standard for all $s \in S$, by induction on S. The base case ι is immediate, so suppose that the result holds for sorts s_1 and $s' = s_2 \to \cdots \to s_m \to \iota$. We use lemma 3.3 to show that $\mathcal{A}$ is $s_1 \to s'$-standard.

(i) Suppose $a_1, a_2 \in A^n_{s_1 \to s'}$ for some $n \in \omega$ and $a_1\, x \sqsubseteq a_2\, x$ for all $x \in A_{s_1}$. Define total functions $f_1, f_2 \in \mathbf{A}^n_{s_1, \ldots, s_m}$ by $f_i \langle x_1, \ldots, x_m \rangle = a_i\, x_1\ \cdots\ x_m$. Then $f_1 \leq f_2$, and thus $\widehat{f_i}$ is represented by f_i for $i = 1, 2$ and $\widehat{f_1} \sqsubseteq \widehat{f_2}$, by lemma 3.7. But a_i is also represented by f_i for $i = 1, 2$, and thus

$$a_1 = \widehat{f_1} \sqsubseteq \widehat{f_2} = a_2,$$

as required.

(ii) If $f \in A_{s_1} \xrightarrow[c]{} A_{s'}$ is isolated, then $f = \psi^n_{s_1,s'}\, f$ for some $n \in \omega$, so that $f\,x = \Psi^n_{s'} \cdot (f(\Psi^n_{s_1} \cdot x))$ for all $x \in A_{s_1}$. Define a total function $f' \in \mathbf{A}^n_{s_1,\ldots,s_m}$ by $f'\langle x_1, \ldots, x_m\rangle = (f\,x_1) \cdot x_2 \cdot \cdots \cdot x_m$. Then $\widehat{f'} \in A^n_{s_1 \to s'}$ is represented by f' (lemma 3.7), and it is easy to show that

$$(f\,x_1) \cdot x_2 \cdot \cdots \cdot x_m = \widehat{f'} \cdot x_1 \cdot \cdots \cdot x_m$$

for all $x_i \in A^n_{s_i}$. But this implies that $f\,x = \widehat{f'} \cdot x$ for all $x \in A^n_{s_1}$, and thus that $f\,x = \widehat{f'} \cdot x$ for all $x \in A_{s_1}$, as required. $\Box$

Corollary 3.9 *The continuous function model is, up to order-isomorphism, the unique model.*

Proof. Immediate from theorem 3.8 and proposition 3.1. $\Box$

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