Stable and Sequential Functions on Scott domains

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Abstract

The search for a general semantic characterization of sequential functions is motivated by the full abstraction problem for sequential programming languages such as PCF. We present here some new developments towards such a theory of sequentiality. We give a general definition of sequential functions on Scott domains, characterized by means of a generalized form of topology, based on sequential open sets. Our notion of sequential function coincides with the Kahn-Plotkin notion of sequential function when restricted to distributive concrete domains, and considerably expands the class of domains for which sequential functions may be defined.

We show that the sequential functions between two dI-domains, ordered stably, form a dI-domain. The analogous property fails for Kahn-Plotkin sequential functions. Our category of dI-domains and sequential functions is not cartesian closed, because application is not sequential. We attribute this to certain operational assumptions underlying our notion of sequentiality.

We show that the Scott domains satisfying a certain "finite meet" property are closed under the pointwise-ordered stable function space, so that we obtain a new stable model based on the pointwise order. We discuss some issues arising in the search for a class of domains closed under the pointwise-ordered sequential function space.

We discuss the relationship between our ideas and the full abstraction problem for PCF, and indicate directions for further development.

1 Introduction

The full abstraction problem for sequential programming languages such as PCF [Plo77, BCL85] has motivated the search for a characterization of sequential functions. Despite the (relatively) long history of the problem, there is currently no satisfactory definition of sequential functions between domains, and no known natural (*i.e.*, language-independent) sequential extensional semantic model. The first definitions of sequentiality, given by Milner [Mil77] and by Vuillemin [Vui73], were limited to functions on products of flat domains. Sazonov's definition of sequential function [Saz75] is also of limited scope. Kahn and Plotkin [KP78] introduced concrete data structures and concrete domains, and gave a definition of sequential function between concrete domains. However, the sequential functions between two concrete domains do not form a concrete domain (under both the pointwise and stable orders). Berry introduced dI-domains, stable functions and the stable ordering [Ber78]; the stable functions between two dI-domains, ordered stably, form a dI-domain. The stable functions do not provide the desired notion of sequential functions, since some stable functions are not sequential. Berry and Curien [BC82, Cur86] defined sequential algorithms between concrete domains, and obtained a sequential intensional model from which one may recover the Kahn-Plotkin sequential functions by taking an extensional quotient. More recently, Bucciarelli and Ehrhard [BE91] introduced a notion of strongly stable functions between qualitative domains equipped with a coherence structure (QDC's), and obtained a definition of sequential function between QDC's that generalizes the Kahn-Plotkin definition. The sequential functions between two QDC's, ordered stably, form a QDC. These notions of sequentiality are not applicable to Scott domains in general, and none of them yields a solution to the PCF full abstraction problem. Like other authors, for instance, Bucciarelli and Ehrhard [BE91], we feel that the problems concerning the semantic characterization of sequentiality are interesting in their own right, independently of the full abstraction problem.

We present here some progress towards a general theory of sequentiality, using a generalized topological approach. We first expand on recent work by Zhang [Zha89] and Lamarche [Lam91], and show that the *stable functions* are the continuous functions for a certain generalized topology, based on *stable open sets*, and that an appropriate order on the stable open sets induces the stable ordering on stable functions. We then define *sequential functions* as the continuous functions for another generalized topology. The key idea involves the designation of *sequential open sets* of a domain, intuitively intended to correspond to properties of elements of the domain that may be computed sequentially. Our notion of sequentiality is a conservative extension of the Kahn-Plotkin notion of sequentiality. We show that dI-domains are closed under the stably-ordered sequential function space. The category of dI-domains and sequential functions has finite products, but fails to be cartesian closed because application is not a sequential function. This reflects an operational assumption that seems to be inherent also in the Kahn-Plotkin setting: an attempt to compute an incremental piece of information about a proper value in a function space may diverge. The failure of cartesian closure does not mean that the search for an extensional sequential semantic model is doomed; closure under function space is sufficient to provide an *applicative structure* [Mit90], which is certainly enough to form an extensional model, and may perhaps, with further refinements, lead to a fully abstract model.

The desire to (eventually) achieve full abstraction suggests that we use the pointwise order on a function space, because it corresponds to the operational pre-order on terms in the standard operational semantics of PCF. As a step in this direction, we show that the Scott domains satisfying a simple *finite meet* property are closed under the pointwise-ordered stable function space. Since flat domains have the finite meet property, and the property is preserved by the relevant function

space construction, this will yield a pointwise-ordered stable extensional applicative structure. We hope that this will pave the way towards a similar construction of a pointwise-ordered sequential functions model. In the meantime, we expand on some aspects of the pointwise-ordered sequential function space. We identify some relevant properties, and obtain closure in a restricted form of the sequential function space, where the target domain is flat.

We intersperse the development of our ideas with some previous work by other authors, in order to emphasize the close analogues provided by the generalized topological approach that we take. We hope that this improves the presentation, and have taken care to supply relevant references to previously known definitions and results.

1.1 Preliminaries

A *Scott domain* is a directed-complete, bounded-complete, ω-algebraic poset with a least element. A subset X of a poset D is *directed* iff it is non-empty and every pair of elements of X has an upper bound in X. A subset X is *bounded* iff it has an upper bound in D. A poset D is *directed-complete* iff all directed subsets of D have a least upper bound (lub), and *bounded-complete* iff all bounded subsets have a least upper bound. An element $d \in D$ is *isolated* (or *finite*, or *compact*) iff for all directed subsets X of D, if d is below the lub of X then d is below some member of X. A poset D is *algebraic* iff every element is the lub of its isolated approximations, and *ω-algebraic* if in addition D has countably many isolated elements. We write D_{fin} for the set of isolated elements of D. We will use the word "consistent" as a synonym for "bounded", and we will write $d \Uparrow d'$ to indicate that d and d' are bounded. We use "meet" as a synonym for greatest lower bound, or glb, and "join" as a synonym for lub, but mainly use them when taking the glb or lub of a pair or a finite set. We usually use the symbol $\leq$ for the order relation of a poset, and $\bigvee$, $\bigwedge$ for the corresponding lub and glb operations, but we may resort to other families of symbols in order to introduce several orderings on the same set.

Every non-empty subset of a bounded-complete poset has a glb. The (binary) meet operation $\wedge$ of a Scott domain is continuous: If X_1 and X_2 are directed sets then

$$(\bigvee X_1) \wedge (\bigvee X_2) = \bigvee \{x_1 \wedge x_2 \mid x_1 \in X_1 \;\&\; x_2 \in X_2\}\,.$$

(A general definition of continuity will be given later.)

A *dI-domain* is a distributive Scott domain with property (I). A poset has property (I) iff every isolated element dominates finitely many elements. A bounded-complete poset is *distributive* iff for every element x and all consistent pairs of elements x_1 and x_2, the following equation holds:

$$x \wedge (x_1 \vee x_2) = (x \wedge x_1) \vee (x \wedge x_2).$$

The isolated elements of a dI-domain are down-closed: If $x \leq y$ in a dI-domain and y is isolated then x is isolated as well.

A subset X of a poset is *down-directed* iff every pair of elements of X has a lower bound in X. A subset X of a poset is a *chain* iff every pair of elements of X is comparable. We say that y *covers* x or that x is *covered* by y, and write $x \prec\!\!\!- y$, iff $x < y$ and the set $\{z \mid x < z \;\&\; z < y\}$ is empty.

We define an up-closure operation: For every element $x \in D$ let $\mathsf{up}(x) = \{z \in D \mid x \leq z\}$. For a subset u of D let $\mathsf{up}(u) = \{z \in D \mid \exists x \in u\,.\, x \leq z\}$. We say that a set u is up-closed iff $u = \mathsf{up}(u)$. A down-closure operation down is defined dually.

2 Topological Definitions

We employ a generalized topological approach to give some of our definitions. We recall here the well known fact that the continuous functions and the pointwise ordering have topological characterizations based on the Scott topology [Sco72, GHK+80]. Following on work by Zhang [Zha89] and Lamarche [Lam91] we show that the stable functions and the stable ordering are obtained as the continuous functions for a generalized topological framework. Lamarche [Lam91] has also given a generalized topological characterization of linear functions.

A generalized topological framework Ω assigns to each domain D a family ΩD of subsets of D, called Ω-open sets, ordered by an order $\leq^{\Omega}$. ΩD is a proper topology if the order used is inclusion, $\emptyset$ and D are Ω-open, and Ω-open sets are closed under arbitrary unions and finite intersections — this may be expressed by saying that ΩD is a sub-frame of the powerset lattice of D, ordered by inclusion (see for instance [Vic89]). Our generalization arises by relaxing the sub-frame requirement, and/or choosing an ordering on Ω-open sets different from set inclusion, and relaxing the (derived) requirement that the poset of Ω-opens form a complete lattice.

We define the Ω-continuous functions from D to E to be the functions f such that the inverse image $f^{-1}(q)$ of every $q \in \Omega E$ is in ΩD. We will order these functions by $f \leq^{\Omega} g$ iff for every $q \in \Omega E$, $f^{-1}(q) \leq^{\Omega} g^{-1}(q)$; the order $\leq^{\Omega}$ on the Ω-continuous functions is said to be induced by the order $\leq^{\Omega}$ on the Ω-opens. Different orders on Ω-opens will naturally induce different orders on the Ω-continuous functions. The usual notion of continuous functions is the one induced by the Scott topology, and we reserve the unqualified term of continuous functions for these Scott continuous functions.

Given a generalized topological framework Ω on a class of domains, we obviously obtain a category whose objects are the domains in the class and whose morphisms are the Ω-continuous functions: the identity function is always Ω-continuous, and function composition preserves Ω-continuity. We will mainly be interested here in establishing that a given class of domains is closed under function space.

Let the domain `Two` have elements $\{\bot, \top\}$, ordered by $\bot \leq \top$. If Ω assigns to `Two` the Sierpinski space structure, *i.e.*, if $\Omega(\texttt{Two}) = \{\emptyset, \{\top\}, \{\bot, \top\}\}$, ordered by $\emptyset \leq^{\Omega} \{\top\} \leq^{\Omega} \{\bot, \top\}$, then it is easy to see that for any domain D, $(\Omega D, \leq^{\Omega})$ is isomorphic to the Ω-continuous function space from D to `Two`, ordered by $\leq^{\Omega}$. This is because $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\{\bot, \top\}) = D$ for any function f from D to `Two`, so that $p \mapsto (\lambda x \in D \,.\, x \in p \to \top, \bot)$ and $f \mapsto f^{-1}(\{\top\})$ are the order-isomorphisms between the two posets. This means that a necessary condition for a class of domains to be closed under the Ω-continuous function space is that it be closed under the generalized topology Ω, *i.e.*, that $(\Omega D, \leq^{\Omega})$ should belong to the class whenever D does. This condition is not sufficient, in general.

We say that a family of subsets of D has the *T0 separation property* iff for every two distinct points of D there exists a subset in the family that includes one but not both of the points. More formally, $O \subseteq \mathcal{P}D$ has T0 separation iff for every $x, y \in D$, $x = y$ iff $\{p \in O \mid x \in p\} = \{p \in O \mid y \in p\}$.

3 Continuous, Stable and Sequential Functions

In this section we define continuous, stable and sequential functions between Scott domains, induced by differing notions of open sets: Scott opens, stable opens and sequential opens, respectively. Scott opens and continuous functions are well known [Sco72, GHK+80], and may be considered classical by now. Stable functions were introduced by Berry [Ber78], and our presentation here generalizes

Zhang's presentation of stable functions between dI-domains [Zha89].

We take D and E to be generic Scott domains, unless stated otherwise. We present a number of examples in the running text. In addition, an appendix presents all Scott opens, stable opens and sequential opens of an example domain.

3.1 Scott Opens and Continuous Functions

Definition 3.1 A set $p \subseteq D$ is *Scott open* iff it is up-closed and has the *Scott property*: for every directed set X, if $\bigvee X \in p$ then $x \in p$ for some $x \in X$. In an algebraic poset, $p \subseteq D$ is Scott open iff $p = \mathsf{up}(p \cap D_{\text{fin}})$, that is, the Scott opens are up-closed sets determined by their isolated elements. Write $\mathsf{Sc}\, D$ for the set of Scott opens of D. •

We state here some easy results about Scott opens: They define a topology, known as the Scott topology. In other words, Scott opens are closed under arbitrary union and finite intersection. For every $x \in D_{\text{fin}}$, $\mathsf{up}(x)$ is Scott open. For every Scott open p, $p = \bigcup \{\mathsf{up}(x) \mid x \in p \cap D_{\text{fin}}\}$.

Proposition 3.2 *The Scott opens of a domain D have T0 separation.*

Proof: For any two distinct points, there must, by algebraicity, exist an isolated approximation to one which does not approximate the other. But if x is such an isolated approximation then $\mathsf{up}(x)$ is a Scott open that contains one of the points, but not the other. ∎

Definition 3.3 A function $f : D \to E$ is Scott continuous, or just *continuous*, iff the inverse image of every Scott open is Scott open, *i.e.*, $f^{-1}(q) \in \mathsf{Sc}\, D$ for every $q \in \mathsf{Sc}\, E$. It is well known that a function $f : D \to E$ is continuous iff it is monotone and preserves directed lubs. •

Proposition 3.4 *A function $f : D \to E$ is continuous iff $f^{-1}(\mathsf{up}(y))$ is Scott open for every $y \in E_{\text{fin}}$. For every Scott open q of E,*

$$f^{-1}(q) = \bigcup \{f^{-1}(\mathsf{up}(y)) \mid y \in q \cap E_{\text{fin}}\}\,.$$

Proof: If f is continuous then $f^{-1}(\mathsf{up}(y))$ is Scott open, since $\mathsf{up}(y)$ is Scott open. For the converse, let q be a Scott open of E. Then $q = \bigcup \{\mathsf{up}(y) \mid y \in q \cap E_{\text{fin}}\}$, and we have

$$\begin{aligned} f^{-1}(q) &= \{x \mid f(x) \in q\} \\ &= \{x \mid f(x) \in \bigcup \{\mathsf{up}(y) \mid y \in q \cap E_{\text{fin}}\}\} \\ &= \{x \mid \exists y \in q \cap E_{\text{fin}}\,.\, f(x) \in \mathsf{up}(y)\} \\ &= \bigcup \{\{x \mid f(x) \in \mathsf{up}(y)\} \mid y \in q \cap E_{\text{fin}}\} \\ &= \bigcup \{f^{-1}(\mathsf{up}(y)) \mid y \in q \cap E_{\text{fin}}\}\,. \end{aligned}$$

But if $f^{-1}(\mathsf{up}(y))$ is Scott open for every $y \in E_{\text{fin}}$ then $f^{-1}(q)$ is Scott open, since Scott opens are closed under arbitrary union, so that f is continuous. ∎

3.2 Stable Opens and Stable Functions

Stable functions were defined by Berry [Ber78]. Zhang [Zha89] gave a generalized topological characterization of the stable functions on dI-domains, using what he called *stable neighborhoods*. We use the term *stable opens* rather than stable neighborhoods, and present here a generalization to Scott domains. Lamarche [Lam91] also gives a generalized topological characterization of stable functions on dI-domains (and on L-domains).

Definition 3.5 A set $p \subseteq D$ is *stable* iff it is closed under consistent meets, *i.e.*, if $x_1, x_2 \in p$ and $x_1 \Uparrow x_2$ then $x_1 \wedge x_2 \in p$. If p is Scott open and stable then it is said to be *stable open*. Write $\mathsf{St}\, D$ for the set of stable opens of D. •

It is easy to see that closure under (binary) consistent meets is equivalent to closure under meets of finite consistent sets.

Proposition 3.6 *For every $x \in D_{\mathrm{fin}}$, $\mathsf{up}(x)$ is stable open. The stable opens of D have T0 separation.*

Proof: $\mathsf{up}(x)$ is Scott open, and closed under all meets. T0 separation follows by proof of proposition 3.2 ∎

The requirement of closure under consistent meets may be weakened:

Proposition 3.7 *For every $p \subseteq D$, p is stable open iff it is Scott open and it is closed under meets of consistent isolated points.*

Proof: If p is stable open then it is Scott open and it is closed under all consistent meets.

For the converse, take p to be Scott open, and let $y_1, y_2 \in p$ be a consistent pair of elements. Since p is Scott open, there exists a pair of isolated elements $x_1, x_2 \in p$ approximating y_1 and y_2, respectively. Since y_1 and y_2 are consistent, so are x_1 and x_2; and since p is closed under meets of consistent isolated elements, $x_1 \wedge x_2 \in p$. It follows that $y_1 \wedge y_2 \in p$, since $x_1 \wedge x_2 \leq y_1 \wedge y_2$, and p is up-closed. ∎

Example 3.8 For an example of a non-stable Scott open, take the Scott open set

$$\mathsf{up}\,\{(\top, \bot), (\bot, \top)\} \subseteq \mathtt{Two} \times \mathtt{Two},$$

which does not contain $(\bot, \bot) = (\top, \bot) \wedge (\bot, \top)$, a consistent meet. •

Definition 3.9 A function $f : D \to E$ is stable continuous, or just *stable*, iff the inverse image of every stable open is stable open, *i.e.*, $f^{-1}(q) \in \mathsf{St}\, D$ for every $q \in \mathsf{St}\, E$. •

There are several alternative formulations of stable functions:

Proposition 3.10 *For every function $f : D \to E$, the following are equivalent:*

(1) f is stable.

(2) f is continuous and preserves consistent meets, i.e., if $x_1 \Uparrow x_2$ then

$$f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2).$$

(3) f is continuous and for every stable set q of E, $f^{-1}(q)$ is a stable set of D.

(Recall: a stable set is not necessarily stable open or Scott open.)

(4) f is continuous and, for every $d \in D$ and $e \leq f(d)$, the set $\{d' \in D \mid d' \leq d \;\&\; e \leq f(d')\}$ is down-directed.

Proof: First note that if f is stable then it is continuous, by proposition 3.4. We therefore assume that f is a continuous function, and show the remaining equivalences.

(1)⇒(2) Assume that f is stable, and let x_1 and x_2 be a consistent pair of elements. By monotonicity, $f(x_1 \wedge x_2) \leq f(x_1) \wedge f(x_2)$. Let q be a stable open that contains $f(x_1) \wedge f(x_2)$. By stability, $f^{-1}(q)$ is stable open, and since $x_1, x_2 \in f^{-1}(q)$, it must be the case that $x_1 \wedge x_2 \in f^{-1}(q)$, *i.e.*, $f(x_1 \wedge x_2) \in q$. Since there exists no stable open that separates the two points, it follows that $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$.

(2)⇒(3) Assume that f preserves consistent meets, and let q be a stable set of E and $x_1, x_2 \in f^{-1}(q)$ be a consistent pair of elements. Then $f(x_1), f(x_2) \in q$ is a consistent pair, so that $f(x_1) \wedge f(x_2) \in q$ by stability of q. Since f preserves consistent meets, $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$. Hence $x_1 \wedge x_2 \in f^{-1}(q)$, and $f^{-1}(q)$ is a stable set.

(3)⇒(1) If f is continuous and $f^{-1}(q)$ is stable whenever q is stable, then, for every stable open q of E, which by definition is Scott open and stable, $f^{-1}(q)$ is both Scott open and stable, hence stable open, and f is stable.

(3)⇒(4) Assume that the inverse image map of f preserves Scott opens and stable sets, and let $e \leq f(d)$ and $p = \{d' \leq d \mid e \leq f(d')\} = f^{-1}(\mathsf{up}(e)) \cap \mathsf{down}(d)$. p is non-empty, since it has d as a (greatest) member. Since $\mathsf{up}(e)$ is a stable set, so is $f^{-1}(\mathsf{up}(e))$; therefore, if $x_1, x_2 \in p$ then $x_1 \wedge x_2 \in p$, and p is down-directed.

(4)⇒(2) Assume that $\{d' \leq d \mid e \leq f(d')\}$ is down-directed for every $e \leq f(d)$, and let $x_1, x_2 \in f^{-1}(q)$ have an upper bound x. Take $d = x$ and $e = f(x_1) \wedge f(x_2)$ in the above characterization of f. There must therefore exist some lower bound x' of x_1 and x_2 such that $f(x_1) \wedge f(x_2) \leq f(x') \leq f(x_1 \wedge x_2)$, and we have $f(x_1) \wedge f(x_2) = f(x_1 \wedge x_2)$. ∎

These alternative definitions of stable functions show that it is possible to decouple (Scott) continuity of a function and (a "pure" notion of) stability.

Alternative definition (4) specializes in dI-domains to the usual "minimum point" definition of stable functions: f is stable iff it is continuous and for every $e \leq f(d)$ the set $\{d' \leq d \mid e \leq f(d)\}$ has a least point. Note that in a dI-domain every down-directed set of isolated elements must have a least element. This is not true in more general settings, and, in fact, the minimum point definition and alternative definition (2), also referred to as defining *conditionally multiplicative* functions [Ber78], may diverge outside the scope of dI-domains. We resolve this mismatch by adopting the conditionally multiplicative variant as the correct notion of stability, and generalizing the minimum point definition so that the mismatch is removed. This approach is needed so that certain PCF-definable functions fit the definition of stability, even though they do not always possess "minimum points". We illustrate this situation in example 5.24, later in the development.

3.3 Relativization, Lobes, Covers and Indices

In order to obtain a notion of isolatedness or finite information relative to a given point in a domain, we now introduce an operation of "relativizing" a domain to an arbitrary element: The relativization of a domain D to an element $x \in D$ is the domain $\mathsf{up}_D(x)$, consisting of the up-closure of x in D, ordered by the restriction to $\mathsf{up}(x)$ of D's order. We usually drop the subscript, and write $\mathsf{up}(x)$.

Proposition 3.11 *For a Scott domain D and $x \in D$, $\mathsf{up}(x)$ is a Scott domain, and*

$$\mathsf{up}(x)_{\mathrm{fin}} = \{x \vee y \mid y \in D_{\mathrm{fin}} \;\&\; x \Uparrow y\} .$$

If D is a dI-domain then $\mathsf{up}(x)$ is a dI-domain.

Proof: Details omitted. ∎

Proposition 3.12 *For a domain D and $x \in D$, if p is Scott open (respectively, stable open) then $r = p \cap \mathsf{up}(x)$ is a Scott open (respectively, stable open) of $\mathsf{up}(x)$.*

Proof: Let p be a Scott open, and $r = p \cap \mathsf{up}(x)$. r is certainly up-closed. To see that r has the Scott property, note that if $z' \in r$ then there exists an isolated $z \in p$ with $z \leq z'$, and hence $x \Uparrow z$, so that $x \vee z \in r$. But $x \vee z \leq z'$, and $x \vee z$ is isolated in $\mathsf{up}(x)$, by 3.11.

If p is stable open then r is Scott open, by the above, and, in addition, it is closed under consistent meets, since both p and $\mathsf{up}(x)$ are. ∎

Proposition 3.13 *If r is a Scott open of $\mathsf{up}(x)$ then there exists a Scott open p such that $r = p \cap \mathsf{up}(x)$.*

Proof: If r is a Scott open of $\mathsf{up}(x)$ then let $p = \mathsf{up}\,\{z \in D_{\mathrm{fin}} \mid z \Uparrow x \;\&\; z \vee x \in r\}$. p is certainly Scott open. The inclusion $p \cap \mathsf{up}(x) \subseteq r$ is evident from the definition. For the reverse inclusion, choose $y' \in r$. There must exist $y \in r \cap \mathsf{up}(x)_{\mathrm{fin}}$ such that $y \leq y'$. By proposition 3.11, $y = x \vee z$ for some $z \in D_{\mathrm{fin}}$. But now, by definition, $z \in p$, so that $y' \in p$. ∎

We look now at the structure of stable opens. Every stable open may be decomposed into a disjoint union of lobes, as follows.

Definition 3.14 Define a *lobe* to be a Scott open that is down-directed. Thus, every lobe is stable open.

For every stable open p, there is an equivalence relation on p that identifies two points of p iff they have a lower bound in p. We call the equivalence classes of this relation the *lobes* of p, and denote by $\mathsf{lobes}(p)$ the set of lobes of p.

It is easy to see that every such equivalence class of a stable open p is a down-directed Scott open, so that the use of the term "lobe" is justified. •

As indicated above, in a dI-domain every down-directed set of isolated elements has a least element, so that every lobe has a least element. Stable opens of a dI-domain are therefore up-closures of pairwise inconsistent sets of isolated elements, coinciding with the notion of stable neighborhoods of dI-domains defined by Zhang. This is not necessarily the case in more general domains. Example 5.24 exhibits a lobe that has no least point.

Definition 3.15 A *cover* of $x \in D$ is a stable open r of $\mathsf{up}(x)$ such that $x < y$ for every $y \in r$ and $\Delta(x, r) = \emptyset$, where

$$\Delta(x, r) = \{z \mid x < z \;\&\; \exists r' \in \mathsf{lobes}(r)\,.\,\forall y \in r'\,.\,z < y\}\,.$$

Write $\mathsf{I}(x)$ for the set of covers of x.

An equivalent definition of a cover is obtained by noting that a stable open r of $\mathsf{up}(x)$ is a cover of x iff for every lobe r' of r, either r' has a least element y and y covers x, or r' has no least element and $x = \bigwedge r'$.

For $x \in D$ and a set $s \subseteq D$, let $\mathsf{I}(x, s)$ be the set of *indices* of s at x, defined to be covers r of x such that $s \cap \mathsf{up}(x) \subseteq r$, that is,

$$\mathsf{I}(x, s) = \{r \in \mathsf{I}(x) \mid s \cap \mathsf{up}(x) \subseteq r\}\,.$$

•

Note that we use relativization here: A cover r is taken to be stable open relative to x, since it is defined by elements that are isolated with respect to to x, but not necessarily isolated in D.

We have trivially

- $\Delta(x, \emptyset) = \emptyset$, for every x.
- $\Delta(x, r) = \bigcup \{\Delta(x, r') \mid r' \in \mathsf{lobes}(r)\}$, for every x and stable open r of $\mathsf{up}(x)$.
- $\emptyset \in \mathsf{I}(x, \emptyset)$, for every x.
- $\mathsf{I}(x, s) = \mathsf{I}(x, s \cap \mathsf{up}(x))$, for every x and s.

An arbitrary set s may be seen as presenting a choice between several alternative states of information, its elements. If s is stable, it may be seen as presenting a choice between its lobes. The existence of an index $r \in \mathsf{I}(x, s)$ indicates that the choice implicit in s may be decomposed, with the index r serving as a first step from x towards making the decision implicit in s. If the current state of information is represented as the point x, a cover of x represents an atomic increase in information content. The requirement that $\Delta(x, r)$ be empty for a r to be a cover of x conveys the intuition that there are no elements "between" x and r, hence the atomicity of the increase in the information content. This generalizes the covering relation between elements of a domain – note that $x \prec y$ iff $\mathsf{up}(y)$ is a cover of x.

A cover r of x provides a way of locally decomposing the domain at x into a flat domain, with x as the least element and the lobes of r as the proper elements. Covers may be used to reason about the progress of computation, and they can be seen as generalizations of the notion of cell in a concrete data structure, or as an abstract notion of "argument position" for a function on a domain.

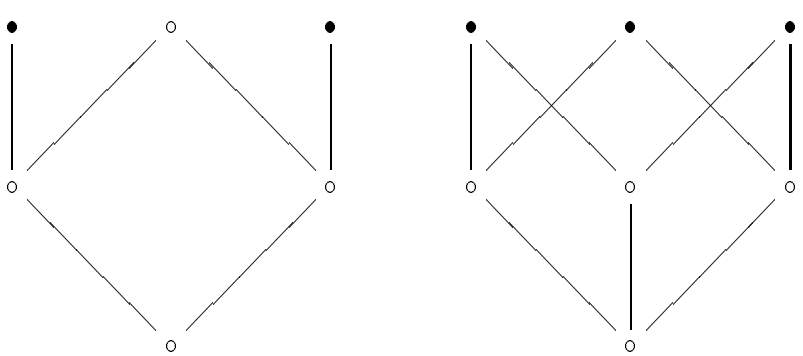
In a dI-domain every lobe has a least element, so that r is a cover of x iff $r = \mathsf{up}(u)$ where u is a set with pairwise inconsistent elements, and each element of u covers x.

To see better the relevance of covers to decomposition of domains, we look at covers in a product of domains.

Proposition 3.16 *A cover of $(x_1, x_2) \in D_1 \times D_2$ is either of the form $r_1 \times \mathsf{up}(x_2)$ for some cover r_1 of x_1, or $\mathsf{up}(x_1) \times r_2$ for some cover r_2 of x_2. Conversely, every cover r_1 of x_1 defines a cover $r_1 \times \mathsf{up}(x_2)$ of (x_1, x_2), and symmetrically for every cover r_2 of x_2.*

Proof: Details omitted. ∎

Example 3.17 In each of the two Hasse diagrams shown we present a domain and a stable open of the domain, consisting of the shaded points. These stable opens do not have an index at bottom, since they are not contained in any cover of bottom.



Take $\mathtt{Bool}$ to be the flat domain of booleans, with $\bot \leq \mathtt{tt}, \mathtt{ff}$. Another example of a stable open with no index at bottom is the stable open set

$$\mathsf{up}\,\{(\mathtt{tt},\mathtt{ff},\bot),(\bot,\mathtt{tt},\mathtt{ff}),(\mathtt{ff},\bot,\mathtt{tt})\} \subseteq \mathtt{Bool} \times \mathtt{Bool} \times \mathtt{Bool}.$$

Note that $(\bot,\bot,\bot)$ has three covers in $\mathtt{Bool} \times \mathtt{Bool} \times \mathtt{Bool}$,

$$\mathsf{up}\{(\mathtt{tt},\bot,\bot),(\mathtt{ff},\bot,\bot)\},$$
$$\mathsf{up}\{(\bot,\mathtt{tt},\bot),(\bot,\mathtt{ff},\bot)\},$$
$$\mathsf{up}\{(\bot,\bot,\mathtt{tt}),(\bot,\bot,\mathtt{ff})\},$$

corresponding to the three components of the product. •

3.4 Sequential Opens

Definition 3.18 A set $p \subseteq D$ is *sequential at* $x \in D$ iff either $x \in p$, or $x \notin p$ and for every finite $s \subseteq p$, $\mathsf{I}(x,s) \neq \emptyset$. If p is sequential at every $x \in D_{\mathrm{fin}}$ then it is said to be *sequential*. A *sequential open* is a stable open that is sequential[1]. Write $\mathsf{Sq}\,D$ for the set of sequential opens of D. •

Definition 3.19 A set s is *critical* iff it is finite, non-empty, and has no index at its meet, *i.e.*, $\mathsf{I}(\wedge s, s) = \emptyset$. •

We can give an alternative, direct, characterization of sequential opens in terms of closure under critical meets. The sets described in example 3.17 are critical. Also note that if a finite set s has a least element, *i.e.*, $\wedge s \in s$, and, in particular, if s is a singleton, then $\mathsf{I}(\wedge s, s) = \emptyset$ and s is critical.

Proposition 3.20 *If $x < y$ then $\mathsf{I}(x, \mathsf{up}(y)) \neq \emptyset$.*

Proof: Let c be a maximal chain in $\Delta(x, \mathsf{up}(y)) \cap \mathsf{up}(x)_{\mathrm{fin}}$ (maximal with respect to inclusion), and take $r = \mathsf{up}(c \cup \{y\})$. It follows that $\Delta(x,r) = \emptyset$, or else the maximality of c would be contradicted, using algebraicity of $\mathsf{up}(x)$. Moreover, r is easily seen to be a stable open of $\mathsf{up}(x)$, so that $r \in \mathsf{I}(x, \mathsf{up}(y))$. ∎

Corollary 3.21 *If $\mathsf{I}(x,s) = \emptyset$ then $x = \wedge(s \cap \mathsf{up}(x))$.*

Proof: $s \cap \mathsf{up}(x)$ must be non-empty, or else $\emptyset \in \mathsf{I}(x,s)$. Clearly $x \leq \wedge(s \cap \mathsf{up}(x))$. If $x < \wedge(s \cap \mathsf{up}(x))$ then there exists an index for $\mathsf{up}(\wedge(s \cap \mathsf{up}(x)))$ at x, which is also an index for s at x, contradicting the assumption that $\mathsf{I}(x,s) = \emptyset$. Hence $x = \wedge(s \cap \mathsf{up}(x))$. ∎

Proposition 3.22 *A set p is sequential at every isolated point iff it is closed under isolated critical meets, that is, if $s \subseteq p$ is a critical set with $\wedge s$ isolated then $\wedge s \in p$.*

A set p is sequential open iff it is stable open and closed under isolated critical meets.

---

[1] We remark that in restricted classes of domains, such as concrete domains, dI-domains, and FM-domains (to be introduced later), we will be able to give an equivalent and slightly weaker definition of sequential opens — a sequential open is a Scott open that is sequential — and we will be able to prove that every sequential open is in fact stable open.

Proof: For p sequential at every isolated point and $s \subseteq p$ a critical set with isolated meet, if $\wedge s \notin p$ then p would fail to be sequential at $\wedge s$, since $\mathsf{I}(\wedge s, s) = \emptyset$. Hence $\wedge s \in p$, and p is closed under isolated critical meets.

For p closed under isolated critical meets and x isolated, if there exists a finite set $s \subseteq p$ such that $\mathsf{I}(x, s) = \emptyset$ then $x = \wedge(s \cap \mathsf{up}(x))$ and $\mathsf{I}(x, s \cap \mathsf{up}(x)) = \emptyset$; hence $s \cap \mathsf{up}(x)$ is a critical set with isolated meet, and $x \in p$, so that p is sequential at x. If no such set s exists then p is sequential at x whether or not $x \in p$. ∎

Proposition 3.23 *For every $x \in D_{\text{fin}}$, $\mathsf{up}(x)$ is sequential open. The sequential opens of D have T0 separation.*

Proof: $\mathsf{up}(x)$ is stable open, and closed under all meets. T0 separation follows by proof of proposition 3.2 ∎

The stable open sets presented in example 3.17 are not sequential, since they do not have an index at bottom, and, equivalently, they are not closed under isolated critical meets.

3.5 Sequential Functions

Definition 3.24 A function $f : D \to E$ is sequential continuous, or just *sequential*, iff the inverse image of every sequential open is sequential open, *i.e.*, $f^{-1}(q) \in \mathsf{Sq}\, D$ for every $q \in \mathsf{Sq}\, E$. •

Sequential functions are continuous:

Proposition 3.25 *A sequential function $f : D \to E$ is continuous.*

Proof: Immediate corollary of proposition 3.4. ∎

A reasonable requirement of any proposed definition of sequential functions is that the sequential functions be stable. It is known, for instance, that the functions in the fully abstract model of PCF are stable [Ber78, theorem 4.5.4]. Our definition passes this test.

Proposition 3.26 *A sequential function $f : D \to E$ is stable.*

Proof: Continuity has been established above. We show that f preserves consistent meets. Let x_1 and x_2 be a consistent pair of elements of D, and let q be a sequential open that contains $f(x_1) \wedge f(x_2)$. By sequentiality of f, $f^{-1}(q)$ is sequential open, hence stable open, and since $f(x_1), f(x_2) \in q$, $x_1, x_2 \in f^{-1}(q)$, so that $x_1 \wedge x_2 \in f^{-1}(q)$, and $f(x_1 \wedge x_2) \in q$. But if the collections of sequential opens containing $f(x_1 \wedge x_2)$ and $f(x_1) \wedge f(x_2)$ coincide, then, by T0 separation, these two elements must be equal. It follows that f preserves consistent meets, and is thus a stable function. ∎

Example 3.27 Let $\mathtt{sor} : \mathtt{Bool}^2 \to \mathtt{Bool}$ be the doubly-strict-or function, defined as the least continuous function such that

$\mathtt{sor}(\mathtt{tt},\mathtt{tt}) = \mathtt{tt}$
$\mathtt{sor}(\mathtt{tt},\mathtt{ff}) = \mathtt{tt}$
$\mathtt{sor}(\mathtt{ff},\mathtt{tt}) = \mathtt{tt}$
$\mathtt{sor}(\mathtt{ff},\mathtt{ff}) = \mathtt{ff}.$

This function is sequential (and stable). The inverse image of the sequential open set $\{\mathtt{tt}\}$ is the set $p = \{(\mathtt{tt},\mathtt{tt}),(\mathtt{tt},\mathtt{ff}),(\mathtt{ff},\mathtt{tt})\}$. It is easy to check that p is a sequential open. For instance, at $(\bot,\bot)$ there are two covers that contain p: $\mathsf{up}\,\{(\mathtt{tt},\bot),(\mathtt{ff},\bot)\}$ and $\mathsf{up}\,\{(\bot,\mathtt{tt}),(\bot,\mathtt{ff})\}$. These two indices at $(\bot,\bot)$ correspond to the fact that this function is strict in both arguments.

The left-strict-or function lor is also sequential, and has a single index $\mathsf{up}\,\{(\mathtt{tt},\bot),(\mathtt{ff},\bot)\}$ for $\mathsf{lor}^{-1}(\{\mathtt{tt}\})$ at $(\bot,\bot)$.

Let $\mathsf{por} : \mathtt{Bool}^2 \to \mathtt{Bool}$ be the parallel-or function, defined to be the least continuous function such that

$$\begin{aligned}\mathsf{por}(\mathtt{tt},\bot) &= \mathtt{tt}\\ \mathsf{por}(\bot,\mathtt{tt}) &= \mathtt{tt}\\ \mathsf{por}(\mathtt{ff},\mathtt{ff}) &= \mathtt{ff}.\end{aligned}$$

Parallel-or is neither stable nor sequential, since the inverse image of $\{\mathtt{tt}\}$ is neither stable open, nor sequential open. The fact that there is no index for $\mathsf{por}^{-1}(\{\mathtt{tt}\})$ at $(\bot,\bot)$ indicates that por is not strict in either of its arguments.

Let $\mathsf{gf} : \mathtt{Bool}^3 \to \mathtt{Bool}$ be (a variation on) Gustave's function (due to Berry), *i.e.*, the least continuous function such that

$$\begin{aligned}\mathsf{gf}(\mathtt{tt},\mathtt{ff},\bot) &= \mathtt{tt}\\ \mathsf{gf}(\bot,\mathtt{tt},\mathtt{ff}) &= \mathtt{tt}\\ \mathsf{gf}(\mathtt{ff},\bot,\mathtt{tt}) &= \mathtt{tt}\\ \mathsf{gf}(\mathtt{ff},\mathtt{ff},\mathtt{ff}) &= \mathtt{ff}.\end{aligned}$$

This function is stable but not sequential. Even though the set

$$\mathsf{gf}^{-1}(\{\mathtt{tt}\}) = \mathsf{up}\,\{(\mathtt{tt},\mathtt{ff},\bot),(\mathtt{ff},\bot,\mathtt{tt}),(\bot,\mathtt{tt},\mathtt{ff})\}$$

is stable open, it is not sequential open, since it has no index at $(\bot,\bot,\bot)$, corresponding to the fact that gf is not strict in any of its arguments. •

3.6 Products

As we have pointed out, the generalized topological approach taken in defining the sequential functions ensures that for any class $\mathcal{C}$ of Scott domains, we obtain a category of $\mathcal{C}$-domains and sequential functions. We show now that, if the class $\mathcal{C}$ is closed under poset product (*i.e.*, cartesian product with componentwise ordering), then the category obtained is cartesian. In fact, all we need to show is that the projection functions are sequential, so that a categorical product may be given by the poset product, together with the projections. If the projection functions are sequential then they are certainly stable, so that a category obtained from such a class $\mathcal{C}$ of domains and stable functions will also be cartesian. Most of the classes of domains under consideration here are closed under poset product, including Scott domains, dI-domains and concrete domains.

Proposition 3.28 *For Scott domains D_1 and D_2,*

$$\begin{aligned}\mathsf{Sc}(D_1 \times D_2) &= \{p_1 \times p_2 \mid p_1 \in \mathsf{Sc}\,D_1 \;\&\; p_2 \in \mathsf{Sc}\,D_2\}\\ \mathsf{St}(D_1 \times D_2) &\supseteq \{p_1 \times p_2 \mid p_1 \in \mathsf{St}\,D_1 \;\&\; p_2 \in \mathsf{St}\,D_2\}\\ \mathsf{Sq}(D_1 \times D_2) &\supseteq \{p_1 \times p_2 \mid p_1 \in \mathsf{Sq}\,D_1 \;\&\; p_2 \in \mathsf{Sq}\,D_2\}\end{aligned}$$

Proof: Covers in a product of domains are characterized in proposition 3.16.

$p_1 \times p_2$ is easily seen to be a Scott open of $D_1 \times D_2$ if both p_1 and p_2 are. If both are closed under consistent meets then so is $p_1 \times p_2$. If $(x_1, x_2) \notin p_1 \times p_2$ then $x_i \notin p_i$ for $i = 1$ and/or $i = 2$. Let s be a finite subset of $p_1 \times p_2$, and assume without loss of generality that $s \subseteq \mathsf{up}(x_1, x_2)$, and that $x_1 \notin p_1$. Then $r_1 \times \mathsf{up}(x_2) \in \mathsf{I}((x_1, x_2), s)$ for every $r_1 \in \mathsf{I}(x_1, \pi_1(s))$. Therefore $p_1 \times p_2$ is sequential open if both p_1 and p_2 are.

If p is a Scott open of $D_1 \times D_2$ then $\pi_i(p) = \{x_i \mid (x_1, x_2) \in p\}$ is easily seen to be Scott open, for $i = 1, 2$. This cannot be carried over to stable opens and sequential opens, since the projection loses too much information. ∎

For the Scott topology we simply obtain the product topology, but this is not the case in the stable or sequential cases. To see that not every stable or sequential open of a product may be decomposed into a product of stable or sequential opens of the components, consider

$$p = \mathsf{up}\,\{((\mathtt{tt}, \bot), \mathtt{tt}), ((\bot, \mathtt{tt}), \mathtt{ff})\}\,.$$

While p is stable and sequential, $\pi_1(p) = \mathsf{up}\,\{(\mathtt{tt}, \bot), (\bot, \mathtt{tt})\}$ is neither stable nor sequential.

Proposition 3.29 *The projection functions $\pi_i : D_1 \times D_2 \to D_i$, $i = 1, 2$, are sequential.*

Proof: Let q be a sequential open of D_1. Then ${\pi_1}^{-1}(q) = q \times D_2 = q \times \mathsf{up}(\bot)$. But $q \times \mathsf{up}(\bot)$ is sequential open, by proposition 3.28, so that π_1 is sequential. ∎

3.7 Relationship to Kahn-Plotkin Sequential Functions

Our definition generalizes the Kahn-Plotkin definition of sequential functions on distributive concrete domains. We present here a sketch of the proof. For full definitions, notation and an exposition of concrete domains, concrete data structures and Kahn-Plotkin sequential functions see [KP78, Cur86, BG90].

We blur the distinction between concrete domains and concrete data structures, so that we may regard each element of a concrete domain as a state of a concrete data structure. We only consider distributive concrete domains; this is convenient, since distributive concrete domains are dI-domains. Among the implications, every lobe of a distributive concrete domain has a least element.

Not surprisingly, there is a close correspondence between covers and accessible cells, and between sequential opens and (finite sets of) cells. Moreover, indices in distributive concrete domains are very well behaved, in that they may be conveniently combined, to obtain indices for arbitrary sets or at arbitrary points.

Proposition 3.30 *In a distributive concrete domain D,*

(1) For every non-empty cover r of x there exists a unique cell c accessible from x and filled in all elements of r.

For every cell c accessible from x, the set of states above x that fill c is a cover of x. This is also the case if c is filled with only a subset of its permissible values.

(2) For every Scott open p and $x \notin p$, every finite subset s of p has an index at x iff p itself has an index at x.

(3) For every sequential open p the set C of cells that are filled in all elements of p is finite and, if $p \neq \emptyset$ and $p \neq \mathsf{up}(\bot)$, C is non-empty.

For every finite set of cells C, the set of states that fill all cells in C is sequential open. This is also the case if we restrict the values filling each of the cells to a subset of the permissible values.

(4) A Scott open p is sequential at every isolated point iff it is sequential at every point.

(5) A finite set s is critical iff there is no cell accessible from its meet that is filled in all elements of the set.

Proof:

(1) In a dI-domain, a cover of x is of the form $\mathsf{up}(u)$ for some set u of pairwise inconsistent elements that cover x. But if y covers x in a distributive concrete domain then y increments x by filling exactly one cell c, accessible from x, and if elements of u are pairwise inconsistent then they must all fill the same cell c.

(2) Without loss of generality, assume that $p \subseteq \mathsf{up}(x)$. If p is Scott open then $p = \mathsf{up}(u)$ for some set u of isolated elements. For $u' \subseteq u$ let $G(u')$ be the set of cells filled in each element of u'. If every finite $s \subseteq p$ has an index at x then $G(u')$ is non-empty for every finite $u' \subseteq u$, and $G(u')$ is finite, because isolated elements may only have finitely many cells filled. But $G(u'_1 \cup u'_2) = G(u'_1) \cap G(u'_2)$, so that the family $\Gamma = \{G(u') \mid u' \subseteq_{\mathrm{fin}} u\}$ is a directed family, under reverse set inclusion, of finite non-empty sets. Hence the limit $G(u) = \bigcap \Gamma$ is non-empty, that is, there exists a cell c filled in all elements of p. An index for p is obtained by choosing a cell c' that precedes c and is accessible from x, and taking the set of all states above x that have c' filled.

(3) C is finite since p is defined by its isolated elements, each of which has finitely many filled cells. If p is non-empty and $\bot \notin p$ then p has an index at $\bot$ (by sequentiality and (2)), which implies that C is non-empty.

Conversely, if C is a finite set of cells then the set p of states that fill all cells in C is Scott open (since C is finite), and if $x \notin p$ then $p \cap \mathsf{up}(x)$, if non-empty, has an index at x, obtained from some cell c' accessible in x that precedes some $c \in C$ that is not filled in x.

(4) Follows from finiteness of enablings.

(5) Follows from (1). ∎

Proposition 3.31 *For distributive concrete domains D and E, a function $f : D \to E$ is sequential iff it is sequential in the Kahn-Plotkin sense.*

Proof: The function f is KP-sequential iff it is continuous and for every state x of D, either no cell is accessible from x, or, for every cell c' accessible from $f(x)$ there exists a cell c accessible from x, such that if $y \supseteq x$ and c' is filled in $f(y)$ then c is filled in y. Such a cell c, if it exists, is called an index of sequentiality of f at x for c'.

If f is a sequential function, x is a state from which at least one cell is accessible, and c' is accessible from $f(x)$, then the inverse image by f of the sequential open q of all states filling c' is sequential open, hence there exists some cover r of x such that $f^{-1}(q) \cap \mathsf{up}(x) \subseteq r$. If r

is non-empty then it determines a cell c accessible from x that is filled in all elements of r, thus c is an index of sequentiality. If r is empty, any accessible cell c may be chosen.

If f is KP-sequential and q is a sequential open of E, let x be any state not in $f^{-1}(q)$, *i.e.*, such that $f(x)$ is not in q. Since q is sequential open, there must be some cover r' of $f(x)$ such that $q \cap \mathsf{up}(f(x)) \subseteq r'$, and if r' is non-empty there must be some cell c' accessible from $f(x)$ that is filled in all states of r'. By KP-sequentiality of f, if there is some cell accessible from x, then there must exist an index of sequentiality c accessible from x, and it is easy to verify that the cover r of x containing all supersets of x that fill c satisfies $f^{-1}(q) \cap \mathsf{up}(x) \subseteq r$. If r' is empty or if no cells are accessible from x (and thus x has no super-states) then $f^{-1}(q) \cap \mathsf{up}(x) = \emptyset$, and r may be chosen to be the empty set. ∎

4 Function Spaces

We turn now to the question of closure of a class of domains under function space constructions. Of course, the function space must be equipped with an ordering. We present first the well known pointwise ordering. Since application fails to be stable and sequential under the pointwise order, we turn next to the stable ordering and restate some of the known results about stable functions under the pointwise ordering. We also show that the stable ordering is induced by an appropriate order on stable opens, following [Zha89]. We then show that dI-domains are closed under the stably-ordered sequential function space, but application still fails to be sequential, so that we obtain an applicative structure, but not a cartesian closed category.

4.1 The Pointwise Order

It is well known that the continuous function space $D \to^{\mathrm{ct}} E$ between two Scott domains D and E, equipped with the pointwise ordering, is itself a Scott domain. Function application may be shown to be a continuous function in this setting, so that the category of Scott domains and continuous functions is cartesian closed.

Definition 4.1 The pointwise ordering is induced on the continuous functions by the inclusion ordering on Scott opens. For continuous functions $f, g : D \to E$, we define $f \leq^{\mathrm{p}} g$ iff $f^{-1}(q) \subseteq g^{-1}(q)$ for every Scott open q of E. It is well known that this induced order coincides with the direct definition of the pointwise order, *i.e.*, $f \leq^{\mathrm{p}} g$ iff $f(x) \leq g(x)$ for every $x \in D$. •

Proposition 4.2 *For continuous functions $f, g : D \to E$, $f \leq^{\mathrm{p}} g$ iff $f^{-1}(\mathsf{up}(y)) \subseteq g^{-1}(\mathsf{up}(y))$ for every $y \in E_{\mathrm{fin}}$.*

Proof: If $f \leq^{\mathrm{p}} g$ then the desired result is an immediate specialization of the definition. Conversely, assume that $f^{-1}(\mathsf{up}(y)) \subseteq g^{-1}(\mathsf{up}(y))$ for every $y \in E_{\mathrm{fin}}$, and let q be a Scott open of E. Use proposition 3.4 to obtain:

$$\begin{aligned} f^{-1}(q) &= \bigcup \{ f^{-1}(\mathsf{up}(y)) \mid y \in q \cap E_{\mathrm{fin}} \} \\ &\subseteq \bigcup \{ g^{-1}(\mathsf{up}(y)) \mid y \in q \cap E_{\mathrm{fin}} \} \\ &= g^{-1}(q). \end{aligned}$$

∎

Another well known fact is that directed lubs and bounded lubs in the pointwise ordering on continuous functions on Scott domains are taken pointwise; we write $\bigvee^{\mathrm{p}} F$ for the pointwise lub of a family F of functions, defined by

$$(\bigvee^{\mathrm{p}} F)(x) = \bigvee \{f(x) \mid f \in F\}\,.$$

We prove this for the directed case:

Proposition 4.3 *For every (pointwise) directed family F of continuous functions from D to E, the pointwise lub $\bigvee^{\mathrm{p}} F$ is continuous and is the lub of F in the continuous function space $D \to^{\mathrm{ct}} E$, and, for every Scott open q of E,*

$$(\bigvee^{\mathrm{p}} F)^{-1}(q) = \bigcup \{f^{-1}(q) \mid f \in F\}\,.$$

Proof: Since F is (pointwise) directed, $\{f(x) \mid f \in F\}$ is directed for every x. Let q be a Scott open of E. By the Scott property of Scott opens we obtain:

$$\begin{array}{rcl}(\bigvee^{\mathrm{p}} F)^{-1}(q) & = & \{x \mid (\bigvee^{\mathrm{p}} F)(x) \in q\} \\ & = & \{x \mid \bigvee \{f(x) \mid f \in F\} \in q\} \\ & = & \{x \mid \exists f \in F\,.\, f(x) \in q\} \\ & = & \bigcup \{\{x \mid f(x) \in q\} \mid f \in F\} \\ & = & \bigcup \{f^{-1}(q) \mid f \in F\}\,.\end{array}$$

Since Scott opens are closed under arbitrary unions, $(\bigvee^{\mathrm{p}} F)^{-1}(q)$ is a Scott open of D, so that $\bigvee^{\mathrm{p}} F$ is continuous. It is easy to check that every upper bound of F dominates $\bigvee^{\mathrm{p}} F$. ∎

We show now that sequential functions and stable functions are also closed under pointwise lubs of directed families.

Proposition 4.4 *The union $\bigcup P$ of a (set inclusion) directed family P of stable opens is stable open.*

The union $\bigcup P$ of a (set inclusion) directed family P of sequential opens is sequential open.

Proof: If P is a (set inclusion) directed family of stable opens then, for every consistent pair $x_1, x_2 \in \bigcup P$ there exists $p \in P$ such that $x_1, x_2 \in p$, by directedness of P. Since p is stable open, $x_1 \wedge x_2 \in p$, so that $x_1 \wedge x_2 \in \bigcup P$, and we have closure of $\bigcup P$ under consistent meets. Since Scott opens are closed under arbitrary unions, it follows that $\bigcup P$ is Scott open, and we may conclude that $\bigcup P$ is stable open.

Let P be a (set inclusion) directed family of sequential opens, and let $x \notin \bigcup P$. For every finite $s \subseteq \bigcup P$ there exists $p \in P$ such that $s \subseteq p$, by directedness of P and finiteness of s. Since p is sequential at x and $x \notin p$, s must have an index at x. Therefore $\bigcup P$ is sequential at every x. $\bigcup P$ is stable open by the first part of the proposition, and we may conclude that $\bigcup P$ is sequential open. ∎

Proposition 4.5 *The pointwise lub $\bigvee^{\mathrm{p}} F$ of a (pointwise) directed family F of sequential functions from D to E is a sequential function.*

The pointwise lub $\bigvee^{\mathrm{p}} F$ of a (pointwise) directed family F of stable functions from D to E is a stable function.

Proof: Let F be a directed set of sequential functions, and let q be a sequential open. By proposition 4.3,

$$(\bigvee^{\mathrm{p}} F)^{-1}(q) = \bigcup \{f^{-1}(q) \mid f \in F\}\,.$$

But $\{f^{-1}(q) \mid f \in F\}$ is a directed family of sequential opens, so that $(\bigvee^{\mathrm{p}} F)^{-1}(q)$ is sequential open, by proposition 4.4. Hence $\bigvee^{\mathrm{p}} F$ is a sequential function, and it is clear that $\bigvee^{\mathrm{p}} F$ is the least sequential upper bound of F.

The same reasoning applies in the stable case. ∎

4.2 Stable Functions under the Stable Order

If we want to consider stable functions as our morphisms, and obtain a cartesian closed structure, the pointwise ordering is not adequate, since function application is not stable when functions are ordered pointwise. For instance, application does not preserve the meet of the pair

$$((\lambda x\,.\,x), \top), ((\lambda x\,.\,\top), \bot) \in (\mathtt{Two} \to \mathtt{Two}) \times \mathtt{Two},$$

although the pair has $((\lambda x\,.\,\top), \top)$ as an upper bound in the pointwise order.

Berry [Ber78] introduced the stable ordering on stable functions, and has shown that function application is stable when functions are ordered stably, that dI-domains are closed under the stably-ordered stable function space, and that the category of dI-domains and stable functions is cartesian closed. Zhang [Zha89] has shown that the stable ordering on functions between dI-domains is induced by a minimal-elements-inclusion ordering among stable neighborhoods; in our formulation this ordering is given by a lobe inclusion ordering among stable opens. We present here a generalization of Zhang's result to Scott domains.

Definition 4.6 Define the *lobe inclusion* order on stable opens $p_1, p_2 \in \mathsf{St}\,D$ by $p_1 \leq^{\mathrm{s}} p_2$ iff $\mathsf{lobes}(p_1) \subseteq \mathsf{lobes}(p_2)$. Define the *stable order* on stable functions $f, g : D \to E$ by $f \leq^{\mathrm{s}} g$ iff for every $q \in \mathsf{St}\,E$, $f^{-1}(q) \leq^{\mathrm{s}} g^{-1}(q)$. We write $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ for the stably-ordered stable function space. •

Our definition coincides with the conventional direct definitions of the stable ordering.

Proposition 4.7 *For any stable functions $f, g : D \to E$, the following are equivalent:*

(1) $f \leq^{\mathrm{s}} g$.

(2) $f \leq^{\mathrm{p}} g$ *and* $f(x) = g(x) \wedge f(y)$ *whenever* $x \leq y$.

(3) $f \leq^{\mathrm{p}} g$ *and* $f(x) \wedge g(y) = g(x) \wedge f(y)$ *whenever* $x \Uparrow y$.

(4) $f \leq^{\mathrm{p}} g$ *and, for every* $d \in D$ *and* $e \leq f(d)$,

$$\{d' \leq d \mid e \leq f(d')\} = \{d' \leq d \mid e \leq g(d')\}\,.$$

Proof: First, note that if $f \leq^{\mathrm{s}} g$ then $f \leq^{\mathrm{p}} g$. This follows from proposition 4.2, since for every $y \in E_{\mathrm{fin}}$, $f^{-1}(\mathsf{up}(y)) \leq^{\mathrm{s}} g^{-1}(\mathsf{up}(y))$, so that $f^{-1}(\mathsf{up}(y)) \subseteq g^{-1}(\mathsf{up}(y))$. We therefore assume without loss of generality that $f \leq^{\mathrm{p}} g$.

(1)⇒(2) If $f \leq^{\mathrm{s}} g$ and $x \leq y$, let q be a stable open that contains $g(x) \wedge f(y)$. Then $g(x), f(y) \in q$, and $x \in g^{-1}(q)$, $y \in f^{-1}(q)$. Let r' be the lobe of $f^{-1}(q)$ that contains y. Since $f^{-1}(q) \leq^{\mathrm{s}} g^{-1}(q)$, r' is also a lobe of $g^{-1}(q)$, and since $x \leq y$, it must be the case that $x \in r'$. Therefore $f(x) \in q$, for every stable open q that contains $g(x) \wedge f(y)$, and by monotonicity every stable open q that contains $f(x)$ also contains $g(x) \wedge f(y)$. By T0 separation conclude that $f(x) = g(x) \wedge f(y)$.

(2)⇒(1) Assume (2), and let q be a stable open of E. Since $f \leq^{\mathrm{p}} g$, $f^{-1}(q) \leq g^{-1}(q)$. By stability of f and g, both $f^{-1}(q)$ and $g^{-1}(q)$ are stable opens. Clearly, every lobe of $f^{-1}(q)$ is contained in some lobe of $g^{-1}(q)$.

Let $r \in \mathsf{lobes}\, f^{-1}(q)$ and $r' \in \mathsf{lobes}\, g^{-1}(q)$ such that $r \subseteq r'$, and let $x \in r$, $z \in r'$. By definition of lobes, $x \wedge z \in r'$, so that $g(x \wedge z) \in q$. But since $f(x) \in q$, we have that $g(x \wedge z)$ and $f(x)$ are consistent elements of q (bounded by $g(x)$), so that $g(x \wedge z) \wedge f(x) \in q$. By (2), $f(x \wedge z) = g(x \wedge z) \wedge f(x)$, so that $x \wedge z \in r$, and $z \in r$. It follows that $r = r'$, and therefore $\mathsf{lobes}(f^{-1}(q)) \subseteq \mathsf{lobes}(g^{-1}(q))$.

(2)⇒(3) Assume (2), and let $x \Uparrow y$. If z is an upper bound of x and y we have

$$\begin{aligned} g(y) \wedge f(x) &= g(y) \wedge (f(z) \wedge g(x)) \\ &= (g(y) \wedge f(z)) \wedge g(x) \\ &= f(y) \wedge g(x). \end{aligned}$$

(3)⇒(2) Assume (3), *i.e.*, if $x \Uparrow y$ then $f(x) \wedge g(y) = g(x) \wedge f(y)$. But if $x \leq y$ this specializes to $f(x) = g(x) \wedge f(y)$, since $f(x) \leq f(y) \leq g(y)$.

(2)⇒(4) Assume (2), so that, for every $d' \leq d$, $f(d') = g(d') \wedge f(d)$. Now, if $e \leq f(d)$ then $e \leq f(d')$ iff $e \leq g(d')$.

(4)⇒(2) Assume (4), and let $x \leq y$. Clearly, $f(x) \leq g(x) \wedge f(y)$. Let $d = y$ and $e = g(x) \wedge f(y)$. But by (4), since $x \leq d$ and $e \leq g(x)$, it must be the case that $e \leq f(x)$. It follows that $f(x) = g(x) \wedge f(y)$. ∎

Alternative (4) specializes in dI-domains to: $f \leq^{\mathrm{s}} g$ iff $f \leq^{\mathrm{p}} g$ and, for every $d \in D$ and $e \leq f(d)$,

$$\min\{d' \leq d \mid e \leq f(d')\} = \min\{d' \leq d \mid e \leq g(d')\},$$

which is the usual direct definition of the stable ordering using the "minimum point" approach.

Proposition 4.8 *The stably ordered function space $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ between two dI-domains D and E is a dI-domain. Directed lubs and bounded lubs are taken pointwise.*

The category **dI-st** *of dI-domains and stable functions is cartesian closed. An exponentiation of dI-domains D and E is given by the stably-ordered stable function space $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$, together with function application.*

Proof: Refer to [Ber78, theorem 4.4.2]. ∎

4.3 Sequential Functions under the Stable Order

We now impose the stable ordering on sequential functions, and show that dI-domains are closed under the stably-ordered sequential function space. This will follow as an easy corollary of the down-closure, under the stable ordering, of sequential functions in the stable function space. This result generalizes the down-closure of Kahn-Plotkin sequential functions in the stable function space (see [BC82, proposition 3.4.4] and [Cur86, proposition 2.4.7]). We write $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ for the stably-ordered sequential function space between domains D and E.

Proposition 4.9 *The sequential opens of every domain are a down-closed subset of the stably-ordered stable opens. That is, if p is stable open, p' is sequential open, and $p \leq^{\mathrm{s}} p'$, then p is sequential open.*

Proof: If $p \leq^{\mathrm{s}} p'$ then $p \subseteq p'$. Let $x \in D_{\mathrm{fin}}$. If $x \in p$ then p is sequential at x. If $x \notin p'$ then p is easily seen to be sequential at x, since p' is. If $x \in p' \setminus p$ then $p \cap \mathsf{up}(x) = \emptyset$, since every lobe of p is a lobe of p', and it follows that p is sequential at x. Hence p is sequential open. ∎

Corollary 4.10 *The sequential functions between domains D and E are down-closed in the stably-ordered stable function space $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$. That is, if $f, g : D \to E$ are stable functions with $f \leq^{\mathrm{s}} g$ and g sequential then f is also sequential.*

Proof: For every sequential open q of E, $f^{-1}(q) \leq^{\mathrm{s}} g^{-1}(q)$. Since g is sequential, $g^{-1}(q)$ is a sequential open, and by down-closure of the sequential opens in the stably-ordered stable opens, $f^{-1}(q)$ is sequential open as well, so that f is a sequential function. ∎

Proposition 4.11 *For all domains D and E, the isolated elements of the stably-ordered sequential function space $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ are the isolated elements of the stably-ordered stable function space $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ that are also sequential.*

Proof: Let f be a sequential function. If f is isolated in $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ then it is clearly isolated in $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$, by definition of isolated elements. If f is not isolated in $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ then it is not in the directed set F of its isolated stable approximations in $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$, but f is the lub of F, $f = \bigvee^{\mathrm{p}} F$, by algebraicity of $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$ and proposition 4.5. Since f is sequential all elements of F are sequential, so that $f = \bigvee^{\mathrm{p}} F$ is the lub of F in $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$, again relying on proposition 4.5. But $f \notin F$, so that f is not isolated in $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$. ∎

Proposition 4.12 *dI-domains are closed under the stably-ordered sequential function space.*

Proof: Let D and E be dI-domains.

$(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ is directed-complete as a corollary of proposition 4.5. Of course, every family of sequential functions that is directed under the stable ordering is also directed under the pointwise ordering.

If F is a family of sequential functions with a sequential function f as an upper bound in the stable order, then $\bigvee^{\mathrm{p}} F$ is known to be a stable function, and it is the lub of F in $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$, so that $\bigvee^{\mathrm{p}} F \leq^{\mathrm{s}} f$. Therefore, by down-closure, $\bigvee^{\mathrm{p}} F$ must be a sequential function, and $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ is bounded-complete.

Distributivity of $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ is inherited directly from $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$, since the meets and joins of sequential functions coincide in the two spaces. Similarly, ω-algebraicity and property (I) for $(D \to^{\mathrm{sq}} E, \leq^{\mathrm{s}})$ are inherited from $(D \to^{\mathrm{st}} E, \leq^{\mathrm{s}})$, by proposition 4.11. ∎

4.4 Application is not Sequential

Consider now the category **dI-sq** of dI-domains and sequential functions. We know that this category is cartesian, and we know that dI-domains are closed under the stably-ordered sequential function space, which is a natural candidate for an exponentiation object in the category. Is **dI-sq** cartesian closed?

Since **dI-sq** is a sub-category of **dI-st**, and some simple additional requirements hold, it follows by a lemma of Berry and Curien concerning cartesian closure of continuous functions categories [BC82, lemma 3.1.2] that **dI-sq** must employ the stable order if it is to be cartesian closed. More precisely, if **dI-sq** is a ccc then its exponentiation is given by the stably-ordered sequential function space, together with function application. Here is a proof of one of the technical requirements of their lemma:

Proposition 4.13 *For any sequential functions $f, g : D \to E$ such that $f \leq^{\mathrm{s}} g$, the function $h : \mathtt{Two} \times D \to E$, defined by $h(\bot, x) = f(x)$ and $h(\top, x) = g(x)$, for all $x \in D$, is sequential.*

Proof: For q a sequential open of E,

$$\begin{aligned} h^{-1}(q) &= (\{\bot\} \times f^{-1}(q)) \cup (\{\top\} \times g^{-1}(q)) \\ &= (\{\bot, \top\} \times f^{-1}(q)) \cup (\{\top\} \times g^{-1}(q)). \end{aligned}$$

Note that $f^{-1}(q) \subseteq g^{-1}(q)$, since $f^{-1}(q) \leq^{\mathrm{s}} g^{-1}(q)$.

$h^{-1}(q)$ is Scott open, since it is the union of two Scott opens. To establish sequentiality, choose an isolated $(\alpha, x) \notin h^{-1}(q)$, and a finite subset $s \subseteq h^{-1}(q)$. Let $s_2 = \{z \mid (\zeta, z) \in s\}$; clearly, $s_2 \subseteq g^{-1}(q)$. Without loss of generality we assume $s \subseteq \mathsf{up}(\alpha, x)$, so that $s_2 \subseteq \mathsf{up}(x)$.

If $x \notin g^{-1}(q)$, there exists $r \in \mathsf{I}(x, s_2)$, since $g^{-1}(q)$ is sequential at x, so that $\mathsf{up}(\alpha) \times r \in \mathsf{I}((\alpha, x), s)$.

If $x \in g^{-1}(q)$ then $\alpha = \bot$ and $x \notin f^{-1}(q)$ (since $(\alpha, x) \notin h^{-1}(q)$). We show that the cover $\mathsf{up}(\top, x)$ of $(\bot, x)$ contains s: if $(\bot, y) \in s$ then $x \leq y$ and $y \in f^{-1}(q)$. But this would contradict $f^{-1}(q) \leq^{\mathrm{s}} g^{-1}(q)$, since $x \in (g^{-1}(q) \cap \mathsf{down}(y)) \setminus (f^{-1}(q) \cap \mathsf{down}(y))$, that is, the lobes of $g^{-1}(q)$ and $f^{-1}(q)$ that contain y are not equal. Therefore all elements of s must have $\top$ as their first component, and are thus contained in the cover $\mathsf{up}(\top, x)$. ∎

All that is left to verify is that function application is a sequential function. This is not the case, however. Function application fails to be sequential under the stable ordering, so that, by Berry and Curien's above-mentioned lemma, **dI-sq** is not cartesian closed. As a counter example to the sequentiality of application consider the following.

Example 4.14 Let $\mathsf{gf}_1, \mathsf{gf}_2, \mathsf{gf}_3 : \mathtt{Bool}^3 \to \mathtt{Bool}$ be the least continuous functions defined so that they each map $(\mathtt{ff}, \mathtt{ff}, \mathtt{ff})$ to $\mathtt{ff}$, and in addition

$$\begin{aligned} \mathsf{gf}_1(\mathtt{tt}, \mathtt{ff}, \bot) &= \mathtt{tt} \\ \mathsf{gf}_2(\bot, \mathtt{tt}, \mathtt{ff}) &= \mathtt{tt} \\ \mathsf{gf}_3(\mathtt{ff}, \bot, \mathtt{tt}) &= \mathtt{tt}. \end{aligned}$$

They are easily seen to be sequential. Let their pairwise lubs be $\mathsf{gf}_{1,2} = \mathsf{gf}_1 \vee^{\mathrm{p}} \mathsf{gf}_2$, $\mathsf{gf}_{1,3} = \mathsf{gf}_1 \vee^{\mathrm{p}} \mathsf{gf}_3$, and $\mathsf{gf}_{2,3} = \mathsf{gf}_2 \vee^{\mathrm{p}} \mathsf{gf}_3$; these lubs are sequential as well. The lub $\mathsf{gf} = \mathsf{gf}_1 \vee^{\mathrm{p}} \mathsf{gf}_2 \vee^{\mathrm{p}} \mathsf{gf}_3$ is of course stable but not sequential, as shown before.

In fact, the set $\{\mathtt{gf}_1, \mathtt{gf}_2, \mathtt{gf}_3\}$ is an example of a pairwise consistent set of sequential functions that does not possess a sequential lub (under both the pointwise and stable orderings; the stable ordering among these functions coincides with the pointwise ordering). This serves to illustrate that concrete domains — which are coherent, *i.e.*, such that every pairwise consistent set has a lub — are not closed under the stably-ordered sequential function space [BC82]. The reason we insist on $(\mathtt{ff}, \mathtt{ff}, \mathtt{ff})$ being mapped to $\mathtt{ff}$ is so that the constant true function is removed as a potential lub.

Now return to application. Let $\mathtt{app} : (\mathtt{Bool}^3 \to \mathtt{Bool}) \times \mathtt{Bool}^3 \to \mathtt{Bool}$ be the application function, and consider $p = \mathtt{app}^{-1}(\{\mathtt{tt}\})$ at $x = (\mathtt{gf}_1, \bot, \bot, \bot)$. By the decomposition of covers for products, every cover r of x must be in one of the following forms:

$$\begin{aligned} r &= r_1 \times \mathtt{up}(\bot) \times \mathtt{up}(\bot) \times \mathtt{up}(\bot) \\ r &= \mathtt{up}(\mathtt{gf}_1) \times r_2 \times \mathtt{up}(\bot) \times \mathtt{up}(\bot) \\ r &= \mathtt{up}(\mathtt{gf}_1) \times \mathtt{up}(\bot) \times r_2 \times \mathtt{up}(\bot) \\ r &= \mathtt{up}(\mathtt{gf}_1) \times \mathtt{up}(\bot) \times \mathtt{up}(\bot) \times r_2, \end{aligned}$$

where r_1 is some cover of $\mathtt{gf}_1$, or r_2 is some cover of $\bot$ in $\mathtt{Bool}$. In the first case, the element $(\mathtt{gf}_1, \mathtt{tt}, \mathtt{ff}, \bot)$ of $p \cap \mathtt{up}(x)$ is not in r. In each of the other cases it is also possible to find elements of $p \cap \mathtt{up}(x)$ that are not contained in r, since for each argument position there is a function above $\mathtt{gf}_1$ that is non-strict in that argument. Hence $\mathsf{I}(x, p)$ is empty, p is not sequential open, and $\mathtt{app}$ is not a sequential function. •

Intuitively, application is not sequential in this case, since when we know only that the function is at least $\mathtt{gf}_1$ but have no information about its arguments there is no way to determine which of the function being applied or its three arguments needs to be evaluated further. This lack of sequentiality of application would seem to be inherent in the nature of application, under the assumption that an attempt to increase the information known about the function to be applied may diverge, even when we know that the function itself is not everywhere diverging. We say more about the failure of application to be sequential in the concluding discussion.

5 Function Spaces: The Pointwise Order Revisited

Even though the category **dI-sq** is not cartesian closed, we have shown closure of dI-domains under the stably-ordered sequential function space. We continue to investigate closure of classes of domains under function spaces, and leave aside for the time being the question of application's stability or sequentiality. As we will argue in the conclusion, a sequential applicative structure, *i.e.*, a class of domains closed under the sequential function space, may provide an adequate first-order notion of sequentiality, as a basis for a notion of higher-order sequentiality that will address better the sequentiality of application.

However, the stable ordering is not adequate; the pointwise ordering is needed for a fully abstract model because of the nature of the operational preorder on terms. Approaches to overcoming this problem have included bi-domains and bi-CDSs [Ber78, BCL85, Cur86], using both the stable and pointwise orderings in the construction. But recall that the stable order was used in order to make application a stable function and thereby achieve a cartesian closed category. If we decide to look instead for an applicative structure, we no longer require that application be stable (although of course application must still make sense as a set-theoretic operation). We therefore advocate the direct use of the pointwise order, and we would like to find a class of domains closed under the pointwise-ordered stable (or sequential) function space. We may employ a bi-ordered construction

at some later point of our investigation, but it seems sensible at first to deal with the pointwise order separately.

All function spaces in this section are ordered pointwise. We write $D \to^{\mathrm{st}} E$ for the pointwise-ordered stable function space between domains D and E, and $D \to^{\mathrm{sq}} E$ for the corresponding pointwise-ordered sequential function space.

5.1 FM-domains

It is now time to introduce the long-awaited FM-domains.

Definition 5.1 We say that a Scott domain has the *finite meet* property, (FM) for short, iff the meet of each pair of isolated elements is isolated[2]. We refer to Scott domains with property (FM) as *FM-domains.* •

Since isolated elements are down-closed in a dI-domain, it follows that dI-domains are FM-domains. The converse is not generally true, and it is easy to see that FM-domains are a proper intermediate notion, between Scott domains and dI-domains. As an important special case, flat domains have property (FM). It is also straightforward to check that FM-domains are closed under poset product.

We carry out the rest of the development using FM-domains, unless stated otherwise.

Property (FM) will be essential for finding a class of domains closed under the pointwise-ordered stable function space, but first we show how it enables us to simplify some of the definitions.

We can give another simplification of the definition of stable opens, similar to proposition 3.7.

Proposition 5.2 *For every $p \subseteq D$, p is stable open iff it is Scott open and it is closed under isolated consistent meets, i.e., for every consistent pair $x_1, x_2 \in p$ with $x_1 \wedge x_2$ isolated, $x_1 \wedge x_2 \in p$.*

Proof: If p is stable open then it is Scott open and it is closed under all consistent meets.

For the converse, take p to be Scott open, and let $y_1, y_2 \in p$ be a consistent pair of elements. Since p is Scott open, there exists a pair of isolated elements $x_1, x_2 \in p$ approximating y_1 and y_2, respectively. x_1 and x_2 are necessarily consistent. Moreover, $x_1 \wedge x_2$ is isolated, by property (FM), so that $x_1 \wedge x_2 \in p$, and $y_1 \wedge y_2 \in p$, by up-closure. ∎

As promised, we show now that the definition of sequential opens may be simplified in FM-domains.

Proposition 5.3 *For every Scott open p,*

(1) If s is critical, $s \subseteq \mathsf{up}(t)$, and $\wedge s = \wedge t$ for some finite t, then t is critical.

(2) If $s \subseteq p$ is a finite set with $\wedge s$ isolated then there exists a finite $t \subseteq p \cap D_{\mathrm{fin}}$ such that $\wedge s = \wedge t$ and $s \subseteq \mathsf{up}(t)$.

(3) If $s \subseteq p$ is a critical set with $\wedge s$ isolated then there exists a critical $t \subseteq p \cap D_{\mathrm{fin}}$ with $s \subseteq \mathsf{up}(t)$ and $\wedge s = \wedge t$.

Proof: (3) follows from (1) and (2).

[2]We really should talk of an "isolated meet" property to be consistent with our usage here. The term "arithmetic" is used in [GHK+80] for a poset having this property.

(1) If r is an index of t at $\wedge t$ then it is an index of s at $\wedge s = \wedge t$, a contradiction.

(2) Since p is Scott open, there exists some $t' \subseteq p \cap D_{\text{fin}}$ such that $s \subseteq \mathsf{up}(t')$. Clearly $\wedge t' \leq \wedge s$. One may assume without loss of generality that t' is finite. Now let $t = \{x \vee (\wedge s) \mid x \in t' \;\&\; x \Uparrow \wedge s\}$, and it is easy to see that t is as desired. ∎

Proposition 5.4 *The following are equivalent in an FM-domain:*

(1) p is sequential open.

(2) p is Scott open and is sequential at every isolated point.

(3) p is Scott open and is closed under isolated critical meets.

(4) p is Scott open and is closed under critical meets of isolated elements.

Proof: For every Scott open p, (2) and (3) are equivalent, by proposition 3.22.

To show that (1) and (3) are equivalent we only need to show that if p is Scott open and is closed under isolated critical meets then it is closed under isolated consistent meets, using proposition 5.2. But if x_1 and x_2 are a consistent pair of elements with an isolated meet then $\{x_1, x_2\}$ is a critical set with an isolated meet, so that closure under isolated critical meets implies closure under isolated consistent meets.

By property (FM), every critical set of isolated elements has an isolated meet, so that (3) implies (4). To see that (4) implies (3), let $s \subseteq p$ be a critical set with $\wedge s$ isolated. Then, by proposition 5.3, there exists a critical set $t \subseteq p \cap D_{\text{fin}}$ with $\wedge s = \wedge t$. Now, if p is closed under critical meets of isolated elements then $\wedge t \in p$, so that $\wedge s \in p$. Therefore p is closed under isolated critical meets. ∎

We show now that property (FM) is preserved by the continuous function space, so that FM-domains and continuous functions are a sub-ccc of the ccc of Scott domains and continuous functions. In order to prove this, we give explicit representations for the isolated elements of the continuous function space.

It is well known that the meet in the continuous function space is taken pointwise:

Proposition 5.5 *The pointwise meet $f_1 \wedge^{\mathrm{p}} f_2$ of two continuous functions $f_1, f_2 : D \to E$, defined for every x by*

$$(f_1 \wedge^{\mathrm{p}} f_2)(x) = f_1(x) \wedge f_2(x)$$

is continuous, and it is the meet of f_1 and f_2 in $D \to^{\mathrm{ct}} E$.

Proof: Let X be a directed set. The pointwise meet $f_1 \wedge^{\mathrm{p}} f_2$ is clearly monotone, so that

$$\bigvee \{(f_1 \wedge^{\mathrm{p}} f_2)(x) \mid x \in X\} \leq (f_1 \wedge^{\mathrm{p}} f_2)(\bigvee X).$$

For the reverse, use continuity of the meet in E and continuity of f_1 and f_2:

$$\begin{aligned}
(f_1 \wedge^{\mathrm{p}} f_2)(\bigvee X) &= f_1(\bigvee X) \wedge f_2(\bigvee X) \\
&= (\bigvee \{f_1(x) \mid x \in X\}) \wedge (\bigvee \{f_2(x) \mid x \in X\}) \\
&= \bigvee \{f_1(x_1) \wedge f_2(x_2) \mid x_1, x_2 \in X\} \\
&\leq \bigvee \{f_1(x) \wedge f_2(x) \mid x \in X\} \\
&= \bigvee \{(f_1 \wedge^{\mathrm{p}} f_2)(x) \mid x \in X\}\,.
\end{aligned}$$

Hence, the pointwise meet is continuous. ∎

Definition 5.6 The *threshold function* $[x, y] : D \to E$ (also called a *one-step function*) is defined for $x \in D_{\text{fin}}$ and $y \in E_{\text{fin}}$ by

$$[x, y](z) = \begin{cases} y & x \leq z \\ \bot & x \not\leq z. \end{cases}$$

The use of the notation $[x, y]$ will always imply that x and y are isolated in their respective domains. •

A threshold function is easily seen to be continuous, stable and sequential. Every continuous function $f : D \to E$ is the lub of the threshold functions below it; note that $[x, y] \leq^{\text{P}} f$ iff $y \leq f(x)$, so that

$$f = \bigvee^{\text{P}} \{[x, y] \mid x \in D_{\text{fin}} \,\&\, y \in E_{\text{fin}} \,\&\, y \leq f(x)\}\,.$$

The lub of an upper-bounded set ρ of threshold functions is given by

$$(\bigvee^{\text{P}} \rho)(z) = \bigvee \{y \mid \exists x \leq z \,.\, [x, y] \in \rho\}\,.$$

This is well defined since the lub is taken over a bounded set (with upper bound $h(z)$, for any upper bound h of ρ).

Definition 5.7 Define a *representation* of f to be a set ρ of threshold functions such that $f = \bigvee^{\text{P}} \rho$, and a *directed representation* of f to be a representation of f such that for every z the set $\{y \mid \exists x \leq z \,.\, [x, y] \in \rho\}$ is directed. For every continuous function f, f has a directed representation, by taking all threshold functions approximating f.

A *step function* $f : D \to E$ is the lub of a finite set of threshold functions. The step functions are the isolated elements of the continuous function space. Note that a step function may dominate infinitely many threshold functions, but is the lub of a finite subset thereof. In other words, step functions are functions that have a finite representation.

For a set ρ of threshold functions, we use the abbreviations $\pi_1(\rho) = \{x \mid \exists y \,.\, [x, y] \in \rho\}$ and $\pi_2(\rho) = \{y \mid \exists x \,.\, [x, y] \in \rho\}$. •

The characterization of the finite elements of the continuous function space by means of threshold functions is well known, see for instance [Plo77]. We need the new notion of directed representations, however, in order to show preservation of (FM) by the continuous function space. Technically, the directedness of the representation will permit us to use the continuity of the meet operation.

Proposition 5.8 *If f is a step function then f has a finite directed representation.*

Proof: Let ρ be a finite representation of a step function f, and let

$$\rho_0 = \{[\bigvee \pi_1(\rho'), \bigvee \pi_2(\rho')] \mid \rho' \subseteq \rho \,\&\, \Uparrow \pi_1(\rho')\}\,,$$

ρ_0 is well defined, since $\pi_2(\rho')$ is upper-bounded by $f(\bigvee \pi_1(\rho'))$, ρ' is finite, and isolated elements are closed under finite lubs, so that we have defined valid threshold functions.

For every z,

$$\begin{aligned} (\bigvee^{\text{P}} \rho_0)(z) &= \bigvee \{\bigvee \pi_2(\rho') \mid \rho' \subseteq \rho \,\&\, \bigvee \pi_1(\rho') \leq z\} \\ &= \bigvee \{y \mid \exists \rho' \subseteq \rho \,.\, \forall x \,.\, [x, y] \in \rho' \Rightarrow x \leq z\} \\ &\leq \bigvee \{y \mid [x, y] \in \rho \,\&\, x \leq z\} \\ &= (\bigvee^{\text{P}} \rho)(z), \end{aligned}$$

so that $f = \bigvee^{\mathrm{p}} \rho = \bigvee^{\mathrm{p}} \rho_0$, and ρ_0 is a representation of f.

Finally, ρ_0 is finite, and it can be seen to be a directed representation: for every z,

$$\{y \mid \exists x \leq z \,.\, [x,y] \in \rho_0\} = \{\bigvee \pi_2(\rho') \mid \rho' \subseteq \rho \;\&\; \bigvee \pi_1(\rho') \leq z\}\,,$$

is a directed set, since, if, for $i = 1,2$, $\rho'_i \subseteq \rho$ is such that $\bigvee \pi_1(\rho'_i) \leq z$ then $\bigvee \pi_1(\rho'_1 \cup \rho'_2) \leq z$, so that the elements $\bigvee \pi_2(\rho'_1), \bigvee \pi_2(\rho'_2)$ of the set have an upper bound $\bigvee \pi_2(\rho'_1 \cup \rho'_2)$. ∎

Proposition 5.9 *The continuous function space $D \to^{\mathrm{ct}} E$ between two FM-domains D and E has property (FM).*

Proof: Let f_1, f_2 be two isolated elements of $D \to^{\mathrm{ct}} E$ with finite directed representations ρ_1, ρ_2, respectively. Let

$$\rho = \{[x_1 \vee x_2, y_1 \wedge y_2] \mid [x_1,y_1] \in \rho_1 \;\&\; [x_2,y_2] \in \rho_2 \;\&\; x_1 \Uparrow x_2\}\,.$$

Every element of ρ is a threshold function, since isolated elements are closed under meet, by property (FM), and under join. Moreover, since the elements of ρ are dominated by f_1 and f_2 (for instance, $[x_1 \vee x_2, y_1 \wedge y_2] \leq^{\mathrm{p}} [x_1,y_1] \leq^{\mathrm{p}} f_1$), $\bigvee^{\mathrm{p}} \rho$ is well defined, and $\bigvee^{\mathrm{p}} \rho \leq^{\mathrm{p}} f_1 \wedge^{\mathrm{p}} f_2$. In order to establish equality we rely on continuity of meet and directedness of the representations, and obtain

$$\begin{array}{rcl}
(f_1 \wedge^{\mathrm{p}} f_2)(z) & = & f_1(z) \wedge f_2(z) \\
& = & (\bigvee \{y_1 \mid \exists x_1 \leq z \,.\, [x_1,y_1] \in \rho_1\}) \wedge (\bigvee \{y_2 \mid \exists x_2 \leq z \,.\, [x_2,y_2] \in \rho_2\}) \\
& = & \bigvee \{y_1 \wedge y_2 \mid \exists x_1, x_2 \leq z \,.\, [x_1,y_1] \in \rho_1 \;\&\; [x_2,y_2] \in \rho_2\} \\
& = & \bigvee \{y \mid \exists x \leq z \,.\, [x,y] \in \rho\} \\
& = & (\bigvee^{\mathrm{p}} \rho)(z).
\end{array}$$

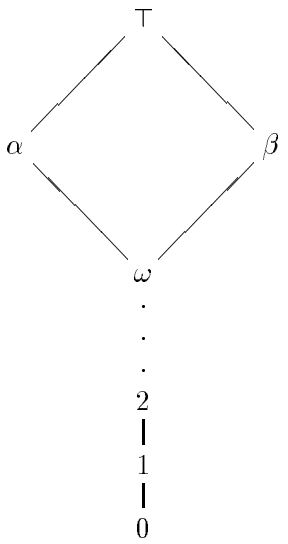
Therefore $f_1 \wedge^{\mathrm{p}} f_2 = \bigvee^{\mathrm{p}} \rho$, and $f_1 \wedge^{\mathrm{p}} f_2$ has a finite representation and is isolated. ∎

5.2 Stable Functions under the Pointwise Order

We have already shown, in proposition 4.5, that the stable function space between two Scott domains is directed-complete. We complete here the proof that FM-domains are closed under the stable function space. This improves on a result that the pointwise-ordered stable function space between dI-domains is a Scott domain, a corollary of [Ber78, theorem 4.5.3].

We restrict ourselves to FM-domains, because the poset of stable opens, ordered by inclusion, is not bounded-complete for general Scott domains. (Recall that a necessary condition for closure of a family of domains under a function space is closure under the corresponding generalized topology.)

Example 5.10 For a counter-example to bounded-completeness of stable opens, consider the following Scott domain, where ω is the limit of an infinite ascending chain, and all other elements are isolated. The domain lacks (FM) since ω is the meet of the isolated elements α and β. The stable opens $\mathsf{up}(\alpha)$ and $\mathsf{up}(\beta)$ are upper-bounded under inclusion, but have no lub: $\mathsf{up}(\omega)$ is not stable open, since ω is not isolated.



•

Property (FM) rules out the counter-example, and lets us define an operation to obtain the least stable open (with respect to inclusion) that contains a given Scott open set. We define first a general notation for iterated closure operators.

Definition 5.11 We use the following notation for iterated versions of a function (or functional) $\mathcal{F} : S \to S$ defined on some set S:

$$\begin{array}{rcll} \mathcal{F}^0 & = & \lambda x \in S \,.\, x & \\ \mathcal{F}^{n+1} & = & \lambda x \in S \,.\, \mathcal{F}(\mathcal{F}^n(x)) & = \quad \mathcal{F} \circ \mathcal{F}^n \\ \mathcal{F}^* & = & \lambda x \in S \,.\, \bigvee \{\mathcal{F}^n(x) \mid n \geq 0\} & = \quad \bigvee\nolimits^{\mathrm{p}} \{\mathcal{F}^n \mid n \geq 0\} \,. \end{array}$$

•

Definition 5.12 For a Scott open set p, define the *stable closure* of p to be $\mathsf{stc}^*(p)$, where

$$\mathsf{stc}(p) = \mathsf{up}\,\{x_1 \wedge x_2 \mid x_1, x_2 \in p \;\&\; x_1 \Uparrow x_2\} \,.$$

•

Proposition 5.13 *For every Scott open p of an FM-domain D, $\mathsf{stc}(p)$ is Scott open, $p \subseteq \mathsf{stc}(p)$, and $\mathsf{stc}^*(p)$ is the least stable open that contains p.*

Proof: Property (FM) is necessary to show that $\mathsf{stc}(p)$ and $\mathsf{stc}^*(p)$ are determined by their isolated elements. It is easy to check that $\mathsf{stc}^*(p)$ is closed under consistent meets. Finally, it is easy to see that $\mathsf{stc}^*(p)$ is least among stable opens containing p. ∎

Corollary 5.14 *The stable opens of an FM-domain, ordered by inclusion, are bounded-complete, and a complete lattice, with $\mathsf{stc}^*(\bigcup P)$ as the lub of a family P of stable opens.*

The lub operation $\mathsf{stc}^*(\bigcup P)$ was defined by Zhang [Zha89, proposition 8.1.1] for dI-domains.

We move now to the general case – the stable function space. The stable closure operation can be generalized to stable functions.

Definition 5.15 For a function $f : D \to E$ dominated by some stable function, define the *stable closure* of f to be $\mathsf{stc}^*(f)$, where

$$\mathsf{stc}(f) = \lambda x \in D \,.\, \bigvee \{f(z_1) \wedge f(z_2) \mid z_1, z_2 \in D_{\text{fin}} \,\&\, z_1 \Uparrow z_2 \,\&\, z_1 \wedge z_2 \leq x\}\,.$$

•

Proposition 5.16 *Let $f : D \to E$ be a continuous function and let h be a stable function such that $f \leq^{\text{p}} h$. Then $\mathsf{stc}(f)$ is a well defined continuous function, $f \leq^{\text{p}} \mathsf{stc}(f) \leq^{\text{p}} h$, and $\mathsf{stc}^*(f)$ is well defined and is the least stable function that dominates f.*

Proof: For every x and $z_1 \Uparrow z_2$ such that $z_1 \wedge z_2 \leq x$,

$$\begin{array}{rcl} f(z_1) \wedge f(z_2) & \leq & h(z_1) \wedge h(z_2) \\ & = & h(z_1 \wedge z_2) \\ & \leq & h(x), \end{array}$$

since $f \leq^{\text{p}} h$ and h is stable, so that $\mathsf{stc}(f)(x)$ is the lub of a bounded set in E, and thus $\mathsf{stc}(f)$ is well defined. Moreover, $\mathsf{stc}(f)(x) \leq h(x)$.

Monotonicity of $\mathsf{stc}(f)$ is immediate. If X is a directed set we rely on property (FM) to obtain continuity:

$$\begin{array}{rcl} \mathsf{stc}(f)(\bigvee X) & = & \bigvee \{f(z_1) \wedge f(z_2) \mid z_1, z_2 \in D_{\text{fin}} \,\&\, z_1 \Uparrow z_2 \,\&\, z_1 \wedge z_2 \leq \bigvee X\} \\ & = & \bigvee \{f(z_1) \wedge f(z_2) \mid z_1, z_2 \in D_{\text{fin}} \,\&\, z_1 \Uparrow z_2 \,\&\, \exists x \in X \,.\, z_1 \wedge z_2 \leq x\} \\ & = & \bigvee \{\bigvee \{f(z_1) \wedge f(z_2) \mid z_1, z_2 \in D_{\text{fin}} \,\&\, z_1 \Uparrow z_2 \,\&\, z_1 \wedge z_2 \leq x\} \mid x \in X\} \\ & = & \bigvee \{\mathsf{stc}(f)(x) \mid x \in X\}\,. \end{array}$$

To see that $f(x) \leq \mathsf{stc}(f)(x)$ for isolated x take $z_1 = z_2 = x$; for non-isolated x rely on continuity.

Now turn to $\mathsf{stc}^*(f)$. It is well defined and continuous, since it is the pointwise lub of an increasing chain in the domain $D \to^{\text{ct}} E$. To see that $\mathsf{stc}^*(f)$ is stable, let x_1 and x_2 be a pair of consistent elements. By monotonicity,

$$\mathsf{stc}^*(f)(x_1 \wedge x_2) \leq \mathsf{stc}^*(f)(x_1) \wedge \mathsf{stc}^*(f)(x_2).$$

For the reverse direction, we first use the continuity of meet and of f to show that

$$\begin{array}{rcl} f(x_1) \wedge f(x_2) & = & f(\bigvee \{z_1 \in D_{\text{fin}} \mid z \leq x_1\}) \wedge f(\bigvee \{z_2 \in D_{\text{fin}} \mid z_2 \leq x_2\}) \\ & = & \bigvee \{f(z_1) \wedge f(z_2) \mid z_1 \leq x_1 \,\&\, z_2 \leq x_2 \,\&\, z_1, z_2 \in D_{\text{fin}}\} \\ & \leq & \mathsf{stc}(f)(x_1 \wedge x_2). \end{array}$$

Now recall that stc is inflationary, *i.e.*, $f \leq^{\text{p}} \mathsf{stc}(f)$, and again use continuity of meet to obtain:

$$\begin{array}{rcl} \mathsf{stc}^*(f)(x_1) \wedge \mathsf{stc}^*(f)(x_2) & = & (\bigvee^{\text{p}} \{\mathsf{stc}^{n_1}(f)(x_1) \mid n_1 \geq 0\}) \wedge (\bigvee^{\text{p}} \{\mathsf{stc}^{n_2}(f)(x_2) \mid n_2 \geq 0\}) \\ & = & \bigvee \{\mathsf{stc}^{n_1}(f)(x_1) \wedge \mathsf{stc}^{n_2}(f)(x_2) \mid n_1, n_2 \geq 0\} \\ & = & \bigvee \{\mathsf{stc}^{n}(f)(x_1) \wedge \mathsf{stc}^{n}(f)(x_2) \mid n \geq 0\} \\ & \leq & \bigvee \{\mathsf{stc}^{n+1}(f)(x_1 \wedge x_2) \mid n \geq 0\} \\ & = & \mathsf{stc}^*(f)(x_1 \wedge x_2). \end{array}$$

Finally, $\mathsf{stc}^*(f)$ is pointwise below h, and since h is chosen arbitrarily among the stable functions dominating f, $\mathsf{stc}^*(f)$ is the least stable function dominating f. ∎

Corollary 5.17 *The stable function space between FM-domains is bounded-complete. The lub of a bounded set F in the stable function space is* $\mathsf{stc}^*(\bigvee^{\mathrm{p}} F)$.

Remark: by proposition 4.5, if F is directed then $\bigvee^{\mathrm{p}} F$ is stable, and $\bigvee^{\mathrm{p}} F = \mathsf{stc}^*(\bigvee^{\mathrm{p}} F)$.
We can give an alternative formulation of stc, in terms of threshold functions:

Proposition 5.18 *For a continuous function $f : D \to E$ with a directed representation ρ, $\mathsf{stc}(f)$ has a representation*

$$\mathsf{stc}(\rho) = \{[x_1 \wedge x_2, y_1 \wedge y_2] \mid [x_1, y_1], [x_2, y_2] \in \rho \;\&\; x_1 \Uparrow x_2\}\,.$$

Proof: For every z,

$$\begin{array}{rcl}
\mathsf{stc}(f)(z) & = & \bigvee \{f(x_1) \wedge f(x_2) \mid x_1, x_2 \in D_{\mathrm{fin}} \;\&\; x_1 \Uparrow x_2 \;\&\; x_1 \wedge x_2 \leq z\} \\
& = & \bigvee \{(\bigvee \{y_1 \mid \exists x_1' \leq x_1 \,.\, [x_1', y_1] \in \rho\}) \wedge (\bigvee \{y_2 \mid \exists x_2' \leq x_2 \,.\, [x_2', y_2] \in \rho\}) \mid \\
& & \quad x_1, x_2 \in D_{\mathrm{fin}} \;\&\; x_1 \Uparrow x_2 \;\&\; x_1 \wedge x_2 \leq z\} \\
& = & \bigvee \{\bigvee \{y_1 \wedge y_2 \mid \exists x_1' \leq x_1, x_2' \leq x_2 \,.\, [x_1', y_1], [x_2', y_2] \in \rho\} \mid \\
& & \quad x_1, x_2 \in D_{\mathrm{fin}} \;\&\; x_1 \Uparrow x_2 \;\&\; x_1 \wedge x_2 \leq z\} \\
& = & \bigvee \{y_1 \wedge y_2 \mid \exists x_1', x_2' \,.\, [x_1', y_1], [x_2', y_2] \in \rho \;\&\; x_1' \Uparrow x_2' \;\&\; x_1' \wedge x_2' \leq z\} \\
& = & \bigvee \{y_1 \wedge y_2 \mid \exists x_1', x_2' \,.\, [x_1' \wedge x_1', y_1 \wedge y_1] \in \mathsf{stc}(\rho) \;\&\; x_1' \wedge x_2' \leq z\} \\
& = & \bigvee \{y \mid \exists x \leq z \,.\, [x, y] \in \mathsf{stc}(\rho)\} \\
& = & (\bigvee^{\mathrm{p}} \mathsf{stc}(\rho))(z).
\end{array}$$

It follows that $\mathsf{stc}(f) = \bigvee^{\mathrm{p}} \mathsf{stc}(\rho)$. ∎

Proposition 5.19 *If f is isolated in $D \to^{\mathrm{ct}} E$ then $\mathsf{stc}(f)$ and $\mathsf{stc}^*(f)$ are isolated as well, and $\mathsf{stc}^*(f) = \mathsf{stc}^N(f)$ for some N.*

Proof: If f is isolated in $D \to^{\mathrm{ct}} E$ then f has a finite directed representation ρ, and, by 5.18, $\mathsf{stc}(f)$ has finite representation $\mathsf{stc}(\rho)$, so that $\mathsf{stc}(f)$ also is isolated, as is $\mathsf{stc}^n(f)$ for every n.

Consider now the operation $\rho \mapsto \mathsf{stc}(\rho)$ defined by proposition 5.18. If we start out with a finite ρ then repeated application of this operation must eventually result in a finite fixpoint, since we always remain within the finite set

$$\{[\wedge\, \pi_1(\rho'), \wedge\, \pi_2(\rho')] \mid \rho' \subseteq \rho\}\,.$$

Therefore there exists N such that for every $n \geq N$, $\mathsf{stc}^n(f) = \mathsf{stc}^N(f)$, and $\mathsf{stc}^*(f) = \mathsf{stc}^N(f)$, an isolated function. ∎

Proposition 5.20 *The isolated elements of $D \to^{\mathrm{st}} E$ are the isolated elements of $D \to^{\mathrm{ct}} E$ that are stable.*

Proof: Let $f : D \to E$ be a stable function. If f is isolated in $D \to^{\mathrm{ct}} E$ then it is certainly isolated in $D \to^{\mathrm{st}} E$. If f is not isolated in $D \to^{\mathrm{ct}} E$ let F be the set of finite approximations to f in $D \to^{\mathrm{ct}} E$. Of course, $f \notin F$. Now consider $\mathsf{stc}^*(F) = \{\mathsf{stc}^*(f') \mid f' \in F\}$. This is clearly a directed set of $D \to^{\mathrm{st}} E$, and $f = \bigvee^{\mathrm{p}} \mathsf{stc}^*(F)$. However, $f \notin \mathsf{stc}^*(F)$, since stc^* preserves isolatedness of elements in F and f is not isolated in $D \to^{\mathrm{ct}} E$; therefore f is not isolated in $D \to^{\mathrm{st}} E$. ∎

Proposition 5.21 *The pointwise meet of two stable functions is stable.*

Proof: If $f_1, f_2 : D \to E$ are stable functions then their pointwise meet is continuous by proposition 5.5. If x_1 and x_2 are two consistent elements, then

$$\begin{aligned}(f_1 \wedge^{\mathrm{p}} f_2)(x_1 \wedge x_2) &= f_1(x_1 \wedge x_2) \wedge f_2(x_1 \wedge x_2) \\ &= f_1(x_1) \wedge f_1(x_2) \wedge f_2(x_1) \wedge f_2(x_2) \\ &= f_1(x_1) \wedge f_2(x_1) \wedge f_1(x_2) \wedge f_2(x_2) \\ &= (f_1 \wedge^{\mathrm{p}} f_2)(x_1) \wedge (f_1 \wedge^{\mathrm{p}} f_2)(x_2),\end{aligned}$$

and the pointwise meet is stable. ∎

Corollary 5.22 *For any FM-domains D and E, $D \to^{\mathrm{st}} E$ has property (FM).*

Proof: If f_1 and f_2 are two isolated elements of $D \to^{\mathrm{st}} E$ then they are isolated elements of $D \to^{\mathrm{ct}} E$, so that $f_1 \wedge^{\mathrm{p}} f_2$ is isolated in $D \to^{\mathrm{ct}} E$. But $f_1 \wedge^{\mathrm{p}} f_2$ is stable, so that it is isolated in $D \to^{\mathrm{st}} E$. ∎

Proposition 5.23 *For any FM-domains D and E, $D \to^{\mathrm{st}} E$ is an FM-domain.*

Proof: directed-completeness follows from proposition 4.5. Bounded completeness and property (FM) have been established. ω-algebraicity follows easily from the explicit characterization of isolated elements of $D \to^{\mathrm{st}} E$. ∎

Example 5.24 We have yet to motivate the generalization of the "minimum point" definition of the stable functions when the pointwise order is used, see proposition 3.10 and the discussion following it.

For every set s of naturals, let $\phi_s : \mathtt{Nat} \to^{\mathrm{st}} \mathtt{Bool}$ be the strict function that maps all integers in s to $\mathtt{tt}$, and all others to $\bot$. It is easy to see that ϕ_s is stable (and sequential), and $\phi_s = \bigvee^{\mathrm{p}} \{[n, \mathtt{tt}] \mid n \in s\}$. It is easy to see that ϕ_s is isolated in the pointwise-ordered function space iff s is finite. $\phi_\omega : \mathtt{Nat} \to^{\mathrm{st}} \mathtt{Bool}$ is the strict function that maps all naturals to $\mathtt{tt}$, and is not isolated.

Now let $\Psi : ((\mathtt{Nat} \to^{\mathrm{st}} \mathtt{Bool}) \to^{\mathrm{st}} \mathtt{Bool}) \to \mathtt{Bool}$ be the functional that applies its argument to ϕ_ω; $\Psi = \lambda\Phi\,.\,\Phi(\phi_\omega)$. It is easy to verify that $p_0 = \Psi^{-1}(\{\mathtt{tt}\})$ is given by

$$p_0 = \mathtt{up}\,\{[\phi_s, \mathtt{tt}] \mid s \text{ a finite subset of } \mathtt{Nat}\}\,,$$

and since $s_1 \subseteq s_2$ implies $\phi_{s_1} \leq^{\mathrm{p}} \phi_{s_2}$ which implies $[\phi_{s_2}, \mathtt{tt}] \leq^{\mathrm{p}} [\phi_{s_1}, \mathtt{tt}]$, p_0 is down-directed. Moreover, p_0 has no least element, since ϕ_ω is not isolated. Thus, p_0 is an example of a lobe that has no least element.

Finally, Ψ is not stable according to the "minimum point" definition of stability, since p_0 has no least element. But Ψ should be considered stable, however, since it is definable in PCF. Using self-explanatory shorthands, let

$$\begin{aligned}M &\equiv \lambda F : (\mathtt{Nat} \to \mathtt{Bool}) \to \mathtt{Bool}\,.\,FN \\ N &\equiv Y(\lambda f : \mathtt{Nat} \to \mathtt{Bool}\,.\,\lambda n : \mathtt{Nat}\,.\,\mathtt{if}\ 0 = n\ \mathtt{then\ tt\ else}\ f(n-1)).\end{aligned}$$

When the function spaces are ordered pointwise, N should be interpreted as ϕ_ω, and M should be interpreted as Ψ. •

6 Sequential Functions under the Pointwise Order

We now consider the sequential functions under the pointwise order. Of course, a necessary condition for closure under the pointwise-ordered sequential function space is that sequential opens form a domain under inclusion. This is not the case for Scott domains in general, as illustrated by example 5.10. It turns out, however, that the sequential opens of an FM-domain are bounded-complete (and hence a complete lattice). We show this by defining a sequential closure operation for Scott opens, analogous to the stable closure operation.

Definition 6.1 For a Scott open set p, define the *sequential closure* of p to be $\mathsf{sqc}^*(p)$, where

$$\mathsf{sqc}(p) = \mathsf{up}\,\{\wedge s \mid s \subseteq p \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \ s \text{ is a critical set}\}\,.$$

•

Proposition 6.2 *For every Scott open p of an FM-domain D, $\mathsf{sqc}(p)$ is Scott open, $p \subseteq \mathsf{sqc}(p)$, and $\mathsf{sqc}^*(p)$ is the least sequential open that contains p.*

Proof: $\mathsf{sqc}(p)$ is determined by its isolated elements, by construction. $p \subseteq \mathsf{sqc}(p)$, since every singleton is a critical set. It is easy to check that $\mathsf{sqc}^*(p)$ is closed under isolated critical meets, so that it is sequential open (proposition 5.4). Finally, it is easy to see that $\mathsf{sqc}^*(p)$ is least among sequential opens containing p, since every point added to p by the closure process must also be present in any other sequential open that contains p. ∎

Corollary 6.3 *The sequential opens of an FM-domain, ordered by inclusion, are bounded-complete, and a complete lattice, with $\mathsf{sqc}^*(\bigcup P)$ as the lub of a family P of sequential opens.*

6.1 Upwards Motion of Covers: Property (U)

We would like to continue the analogy with the development in the stable case, and define a sequential closure operation for continuous functions dominated by sequential functions, in order to establish bounded-completeness of the sequential function space. Some preliminaries are needed, however, in order to obtain suitable analogues for some of the building blocks of the proofs in the stable case. We now impose on our domains a property (U) that talks about "upwards motion" of covers, and enables us to obtain simple and uniform characterizations of sequential opens, critical sets, and sequential functions.

Definition 6.4 We say that a domain has property (U) iff

(U) For every $x \leq x'$ and every cover r of x such that $x' \notin r$, $r \cap \mathsf{up}(x')$ is a cover of x'.

•

This property states, roughly, that covers may be "moved upwards". The analogue for concrete domains would be to say that if a cell c is enabled (and accessible) from x and is not filled in $x' \geq x$ then c is enabled (and accessible) from x'. This is, in fact, the case in all concrete domains.

By proposition 3.12, if r is a cover of x and $x' \notin r$ then $r' = r \cap \mathsf{up}(x')$ is a stable open of $\mathsf{up}(x')$, and $x' \notin r'$, so that property (U) amounts to a guarantee that $\Delta(x', r')$ is empty.

Property (U) may be stated in terms of single lobes:

Proposition 6.5 *A domain has property (U) iff for every $x \leq x'$ and single lobe cover r of x such that $x' \notin r$, $r \cap \mathsf{up}(x')$ is a single lobe cover of x'.*

Proof: Easy consequence of the decomposition $\Delta(x,r) = \bigcup\{\Delta(x,r_0) \mid r_0 \in \mathsf{lobes}(r)\}$. Of course, $r \cap \mathsf{up}(x)$ is a single lobe whenever r is a single lobe. ∎

Upwards motion of covers provides upwards motion of indices.

Proposition 6.6 *In a domain with (U), if $s' \subseteq \mathsf{up}(s)$, $r \in \mathsf{I}(x,s)$, $x \leq x'$ and $x' \notin r$ then $r \cap \mathsf{up}(x') \in \mathsf{I}(x',s')$.*

Proof: By (U), $r \cap \mathsf{up}(x')$ is a cover of x'. Moreover,

$$s' \cap \mathsf{up}(x') \subseteq (s \cap \mathsf{up}(x)) \cap \mathsf{up}(x') \subseteq r \cap \mathsf{up}(x'),$$

so that $r \cap \mathsf{up}(x')$ is an index of s' at x'. ∎

Corollary 6.7 *In a domain with (U), if $s' \subseteq \mathsf{up}(s)$, $r \in \mathsf{I}(\wedge s, s)$, and $\wedge s' \notin r$ then $r \cap \mathsf{up}(\wedge s') \in \mathsf{I}(\wedge s', s')$.*

Proof: Note that if $s' \subseteq \mathsf{up}(s)$ then $\wedge s \leq \wedge s'$. ∎

Upwards motion of covers and indices provides for "downwards motion" of criticality. Compare this with proposition 5.3 for FM-domains.

Definition 6.8 We write $s \sqsubseteq s'$ for the Egli-Milner order on the sets s and s', *i.e.*, $s \sqsubseteq s'$ iff $s \subseteq \mathsf{down}(s')$ and $s' \subseteq \mathsf{up}(s)$. •

Proposition 6.9 *In a domain with (U), for every critical set s' and finite set s, if $s \sqsubseteq s'$ then s is critical.*

Proof: Let s be a finite set with $s \sqsubseteq s'$. Surely $\wedge s \leq \wedge s'$.

If s is not critical then there exists an index $r \in \mathsf{I}(\wedge s, s)$. If $\wedge s' \notin r$ then, by (U), $r \cap \mathsf{up}(\wedge s') \in \mathsf{I}(\wedge s', s')$, and s' is not critical, a contradiction. If $\wedge s' \in r$ then let r_0 be the lobe of r that contains $\wedge s'$. But then r_0 contains s', and necessarily all elements of s, since every element of s is consistent with $\wedge s'$. Since r_0 is down-directed and s is finite, s must have a lower bound in r_0, and thus $\wedge s \in r_0$, a contradiction. ∎

6.2 Sequential Functions on U-domains

Let U-domains be FM-domains with property (U). We now show that our definitions may be improved in this class of domains. We can give a more uniform definition of the sequential closure operation. Moreover, sequential opens may be characterized as being closed under critical meets, and critical sets may be characterized as sets that are not separable from their meets by sequential opens.

Proposition 6.10 *In a U-domain, for every Scott open p, $\mathsf{sqc}(p) = \mathsf{sqc}'(p) = \mathsf{sqc}''(p)$, where*

$$\begin{aligned} \mathsf{sqc}'(p) &= \mathsf{up}\,\{\wedge s \mid s \subseteq p \;\&\; s \text{ is a critical set}\} \\ \mathsf{sqc}''(p) &= \mathsf{up}\,\{\wedge s \mid s \subseteq p \cap D_{\mathrm{fin}} \;\&\; s \text{ is a critical set}\}\,. \end{aligned}$$

Proof: By property (FM), the meet of a finite set of isolated elements is isolated. Therefore $\mathsf{sqc}''(p) \subseteq \mathsf{sqc}(p) \subseteq \mathsf{sqc}'(p)$. To show that $\mathsf{sqc}'(p) \subseteq \mathsf{sqc}''(p)$, note that if s' is a critical set contained in p then, by the Scott property, there exists a finite set $s \subseteq p$ of isolated elements, with $s \subseteq \mathsf{down}(s')$ and $s' \subseteq \mathsf{up}(s)$, *i.e.*, $s \sqsubseteq s'$, which must be critical, by proposition 6.9, so that $\wedge s \in \mathsf{sqc}''(p)$, and $\wedge s' \in \mathsf{sqc}''(p)$ by up-closure. ∎

We may now extend proposition 5.4.

Proposition 6.11 *In a U-domain, the following are equivalent,*

(1) p is a sequential open.

(2) p is Scott open and is sequential at every $x \in D$.

(3) p is Scott open and is closed under critical meets, i.e., if $s \subseteq p$ is critical then $\wedge s \in p$.

Proof: Each of (2) and (3) trivially imply (1), by proposition 5.4.

(1)⇒(2) If p is sequential open then it is sequential at all isolated $x \in D$, *i.e.*, for every $x \in D_{\mathrm{fin}}$ and finite $s \subseteq p$, $\mathsf{I}(x, s)$ is non-empty.

Now, consider an arbitrary $x' \notin p$ and a finite $s' \subseteq p$. If $x' < \wedge(s' \cap \mathsf{up}(x'))$ then $\mathsf{I}(x', s')$ is non-empty, by proposition 3.20. We may therefore, without loss of generality, consider solely the case where $s' \subseteq \mathsf{up}(x')$, and $x' = \wedge s'$.

By the Scott property there exists a finite set $s \subseteq p$ of isolated elements with $s' \subseteq \mathsf{up}(s)$. We may take s minimal so that $s \subseteq \mathsf{down}(s')$. Since $\wedge s \leq \wedge s'$ and $\wedge s' \notin p$, $\wedge s \notin p$. By (FM), $\wedge s$ is isolated, and since p is sequential at all isolated points, s is not critical. But, by proposition 6.9, this means that s' cannot be critical, and thus p is sequential at $x' = \wedge s'$.

(2)⇒(3) Assume that p is sequential at every x. Let s' be a finite subset of p. If $\wedge s' \notin p$ then $\mathsf{I}(\wedge s', s')$ is non-empty, since p is sequential at $\wedge s'$. Therefore s' is not critical, and p must necessarily be closed under all meets of critical sets. ∎

Proposition 6.12 *In a U-domain, a finite set s is critical iff every sequential open p that contains s also contains its meet $\wedge s$.*

Equivalently, a finite set s is not critical iff there exists a sequential open p that contains s but not $\wedge s$.

Proof: If s is a critical set then every sequential open that contains it contains its meet, by proposition 6.11.

Conversely, assume that s is finite, but not critical. Since s is not critical, there exists an index r of s at $\wedge s$. Let p be a Scott open of D such that $s \subseteq p \cap \mathsf{up}(\wedge s) \subseteq r$. If all elements of s are isolated then $p = \mathsf{up}(s)$ will do, and under more general circumstances the existence of such a p is assured by proposition 3.13[3].

For any such Scott open p, we show, by induction on n, that $r \in \mathsf{I}(\wedge s, \mathsf{sqc}^n(p))$. This is immediate for $n = 0$, since $p \cap \mathsf{up}(\wedge s) \subseteq r$. Now assume $r \in \mathsf{I}(\wedge s, \mathsf{sqc}^n(p))$, and thus

[3]We remark that this is the essential use of the (relative) Scott property of covers. Were it not for this use, we could equally well have defined covers of x to be up-closed stable sets r of $\mathrm{up}(x)$ with $x \notin r$ and $\Delta(x, r)$ empty. But then we would not have been able to use proposition 3.13 to obtain a Scott open p as desired here.

$\mathsf{sqc}^n(p) \cap \mathsf{up}(\wedge s) \subseteq r$. For every $x' \in \mathsf{sqc}^{n+1}(p) \cap \mathsf{up}(\wedge s)$ there exists a critical set $s' \subseteq \mathsf{sqc}^n(p)$ with $\wedge s \leq \wedge s' \leq x'$. But if $\wedge s' \notin r$ then, by (U), $r \cap \mathsf{up}(\wedge s') \in \mathsf{I}(\wedge s', s')$, contradicting criticality of s'; therefore $\wedge s' \in r$, and $\mathsf{sqc}^{n+1}(p) \cap \mathsf{up}(\wedge s) \subseteq r$.

It follows that $\mathsf{sqc}^*(p)$ is a sequential open that contains s, but not its meet. ∎

We are now able to give a characterization of sequential functions in terms of preservation of critical sets. We always assume here that the target domain is a U-domain, and the source domain is an FM-domain, and point out where we additionally assume property (U) for the source domain.

Proposition 6.13 *For a continuous function $f : D \to E$ between an FM-domain D and a U-domain E, the following are equivalent:*

(1) f is sequential.

(2) f preserves criticality and meets of critical sets with isolated meet.

That is, for every critical set s with isolated meet, $f(s)$ is a critical set, and $f(\wedge s) = \wedge f(s)$.

(3) f preserves criticality and meets of critical sets of isolated elements.

If, in addition, D is a U-domain, then the following is also equivalent to the above:

(4) f preserves criticality and meets of critical sets.

Proof: (4) clearly implies (2), and (2) implies (3), since, by property (FM), every finite set of isolated elements has an isolated meet.

(3)⇒(1) Assume that f preserves criticality and meets of critical sets of isolated elements. Let q be a sequential open of E, and $s \subseteq f^{-1}(q)$ a critical set of isolated elements, so that $f(s) \subseteq q$ is critical and $f(\wedge s) = \wedge f(s)$. By proposition 5.4, $f(\wedge s) = \wedge f(s) \in q$, so that $\wedge s \in f^{-1}(q)$, and q is closed under critical meets of isolated elements. Therefore $f^{-1}(q)$ is sequential open, by 5.4, and f is a sequential function.

(1)⇒(2) Assume that f is a sequential function, and let s be a critical set with isolated meet. Let q be a sequential open that contains $f(s)$. Therefore $f^{-1}(q)$ is a sequential open and $s \subseteq f^{-1}(q)$, and by proposition 5.4, $\wedge s \in f^{-1}(q)$, so that $f(\wedge s) \in q$. By monotonicity, $f(\wedge s) \leq \wedge f(s)$. It follows that every sequential open q of E that contains $f(s)$ also contains $\wedge f(s)$, and hence $f(s)$ is critical, proposition 6.12, and, moreover, a sequential open contains $f(\wedge s)$ iff it contains $\wedge f(s)$, so that, by T0 separation, $f(\wedge s) = \wedge f(s)$.

(1)⇒(4) We prove this implication under the additional assumption that D is a U-domain.

Assume that f is a sequential function, and let s be a critical set. Let q be a sequential open that contains $f(s)$. Therefore $s \subseteq f^{-1}(q)$, and by proposition 6.11, $\wedge s \in f^{-1}(q)$, so that $f(\wedge s) \in q$. By monotonicity, $f(\wedge s) \leq \wedge f(s)$. It follows that every sequential open q that contains $f(s)$ also contains $\wedge f(s)$, and hence $f(s)$ is critical, proposition 6.12, and, moreover, a sequential open contains $f(\wedge s)$ iff it contains $\wedge f(s)$, so that, by T0 separation, $f(\wedge s) = \wedge f(s)$. ∎

Every monotone function preserves consistency of a set, so that, when testing continuous functions for stability we need only ask whether they preserve meets of consistent sets. This is not true in general for critical sets, but a weak analogue does hold.

Proposition 6.14 *If a function $f : D \to E$ between an FM-domain D and a U-domain E is dominated by a sequential function, then f preserves criticality of critical sets with isolated meet. That is, $f(s)$ is critical whenever s is critical with isolated meet.*

If, in addition, D is a U-domain, then f preserves criticality of all critical sets.

Proof: Let h be a sequential function dominating f, and $s \subseteq D$ be a critical set. If s has an isolated meet, or if D is a U-domain, then, by proposition 6.13, $h(s)$ is critical. Since $f(s) \sqsubseteq h(s)$, $f(s)$ must itself be critical, by proposition 6.9. ∎

Example 6.15 Note that a function f dominated by a sequential function h need not preserve the critical meets themselves, unless f happens to be sequential. Consider for instance the function $\mathsf{gf} \wedge^{\mathrm{p}} [\bot, \mathtt{tt}]$, dominated by the sequential function $[\bot, \mathtt{tt}]$, that preserves criticality of the critical set $s_0 = \{(\mathtt{tt}, \mathtt{ff}, \bot), (\bot, \mathtt{tt}, \mathtt{ff}), (\mathtt{ff}, \bot, \mathtt{tt})\}$, but not its meet. The function gf itself is not dominated by a sequential function. It preserves criticality, but not the meet, of s_0, and preserves the meet, but not the criticality, of $s_0 \cup \{(\mathtt{ff}, \mathtt{ff}, \mathtt{ff})\}$. •

We are now able to define a sequential closure operation for continuous functions.

Definition 6.16 For a function $f : D \to E$ dominated by some sequential function, define the *sequential closure* of f to be $\mathsf{sqc}^*(f)$, where

$$\mathsf{sqc}(f) = \lambda x \in D \,.\, \bigvee \{\wedge f(s) \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq x\} \,.$$

•

Proposition 6.17 *If $f : D \to E$ is a continuous function between an FM-domain D and a U-domain E, dominated by some sequential function $h : D \to E$, then $\mathsf{sqc}(f)$ is a well defined continuous function, $f \leq^{\mathrm{p}} \mathsf{sqc}(f) \leq^{\mathrm{p}} h$, and $\mathsf{sqc}^*(f)$ is well defined and is the least sequential function that dominates f.*

Proof: For every x and critical s with isolated meet such that $\wedge s \leq x$,

$$\begin{array}{rcl} \wedge f(s) & \leq & \wedge h(s) \\ & = & h(\wedge s) \\ & \leq & h(x), \end{array}$$

since $f \leq^{\mathrm{p}} h$ and h is sequential, so that $\mathsf{sqc}(f)(x)$ is the lub of a bounded set in E. Thus $\mathsf{sqc}(f)$ is well defined. Moreover, $\mathsf{sqc}(f)(x) \leq h(x)$.

Monotonicity of $\mathsf{sqc}(f)$ is immediate. To show continuity we rely on isolatedness of meets used in the definition of sqc:

$$\begin{array}{rcl} \mathsf{sqc}(f)(\bigvee X) & = & \bigvee \{\wedge f(s) \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq \bigvee X\} \\ & = & \bigvee \{\wedge f(s) \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \exists x \in X \,.\, \wedge s \leq x\} \\ & = & \bigvee \{\bigvee \{\wedge f(s) \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq x\} \mid x \in X\} \\ & = & \bigvee \{\mathsf{sqc}(f)(x) \mid x \in X\} \,. \end{array}$$

To see that $f(x) \leq \mathsf{sqc}(f)(x)$ for isolated x take $s = \{x\}$ in the definition of $\mathsf{sqc}(f)(x)$; for non-isolated x rely on continuity.

Now turn to $\mathsf{sqc}^*(f)$. It is well defined and continuous, since it is the pointwise lub of an increasing chain in the domain $D \to^{\mathrm{ct}} E$.

In order to establish that $\mathsf{sqc}^*(f)$ is sequential we show that it preserves criticality and meets of critical sets with isolated meet. Criticality is automatically preserved, by proposition 6.14, since $\mathsf{sqc}^*(f)$ is dominated by the sequential function h. Now, if s' is a critical set with isolated meet then

$$\begin{array}{rcl}
\mathsf{sqc}^*(f)(\wedge s') & = & \bigvee^{\mathrm{p}} \{\mathsf{sqc}^n(f)(\wedge s') \mid n \geq 0\} \\
& = & \bigvee \{\wedge\, \mathsf{sqc}^n(f)(s) \mid n \geq 0 \;\&\; s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq \wedge s'\} \\
& \geq & \bigvee \{\wedge\, \mathsf{sqc}^n(f)(s') \mid n \geq 0\} \\
& = & \wedge\, \mathsf{sqc}^*(f)(s'),
\end{array}$$

using continuity of meet. Of course, by monotonicity, we have $\mathsf{sqc}^*(f)(\wedge s') \leq \wedge\, \mathsf{sqc}^*(f)(s')$, so that we have established preservation of isolated critical meets. Thus $\mathsf{sqc}^*(f)$ is a sequential function, by proposition 6.13.

Finally, $\mathsf{sqc}^*(f)$ is pointwise below h, and since h is chosen arbitrarily among the sequential functions dominating f, $\mathsf{sqc}^*(f)$ is the least sequential function dominating f. ∎

As for sequential closure for Scott opens, the sequential closure for continuous functions may be given alternative characterizations that differ in the quantification over critical sets.

Proposition 6.18 *For a continuous function $f : D \to E$ between an FM-domain D and a U-domain E, $\mathsf{sqc}(f) = \mathsf{sqc}''(f)$. If, in addition, D is a U-domain, then $\mathsf{sqc}(f) = \mathsf{sqc}'(f)$, where*

$$\begin{array}{rcl}
\mathsf{sqc}'(f) & = & \lambda x \in D \,.\, \bigvee \{\wedge f(s) \mid s \textit{ critical } \;\&\; \wedge s \leq x\} \\
\mathsf{sqc}''(f) & = & \lambda x \in D \,.\, \bigvee \{\wedge f(s) \mid s \textit{ critical } \;\&\; s \subseteq D_{\mathrm{fin}} \;\&\; \wedge s \leq x\}\,.
\end{array}$$

Proof: The argument that shows that $\mathsf{sqc}(f)$ is well defined also shows that $\mathsf{sqc}''(f)$ is well defined, and likewise for $\mathsf{sqc}'(f)$ if D is a U-domain.

Let s be a critical set with $\wedge s \leq x$, and let $x_1, \ldots, x_n$ be the elements of s. By algebraicity, $x_i = \bigvee X_i$, where X_i is the directed set of isolated approximations to x_i. By continuity of f and of meet we have:

$$\begin{array}{rcl}
\wedge f(s) & = & f(x_1) \wedge \cdots \wedge f(x_n) \\
& = & f(\bigvee X_1) \wedge \cdots \wedge f(\bigvee X_n) \\
& = & (\bigvee f(X_1)) \wedge \cdots \wedge (\bigvee f(X_n)) \\
& = & \bigvee \{f(z_1) \wedge \cdots \wedge f(z_n) \mid \forall i \leq n \,.\, z_i \in X_i\}\,.
\end{array}$$

For every choice of $s_0 = \{z_1, \ldots, z_n\}$ with $z_i \in X_i$ for $i \leq n$, $s_0 \sqsubseteq s$ and $\wedge s_0 \leq \wedge s \leq x$.

- If s has an isolated meet then, for every such choice of s_0, $t_0 = \{z \vee (\wedge s) \mid z \in s_0\}$ is a finite set of isolated elements, with $\wedge s_0 \leq \wedge t_0 = \wedge s \leq x$ and $\wedge f(s_0) \leq \wedge f(t_0)$. By proposition 5.3 (for the Scott open D), t_0 is critical, and it follows that $\mathsf{sqc}'(f)(x) = \mathsf{sqc}(f)(x)$.
- If D is a U-domain then every such choice of s_0 is critical, by 6.9, and thus $\mathsf{sqc}''(f)(x) = \mathsf{sqc}'(f)(x)$.

∎

We are now able to assert bounded-completeness of the sequential function space.

Corollary 6.19 *The sequential function space from an FM-domain to a U-domain is bounded-complete. The lub of a bounded set F in the sequential function space is* $\mathsf{sqc}^*(\bigvee^{\mathrm{p}} F)$.

Directed-completeness has already been established, by proposition 4.5. Moreover, if F is a directed set in the sequential function space then $\bigvee^{\mathrm{p}} F$ is sequential, and $\bigvee^{\mathrm{p}} F = \mathsf{sqc}^*(\bigvee^{\mathrm{p}} F)$.

We can give an alternative formulation of sqc, in terms of threshold functions:

Proposition 6.20 *For a continuous function $f : D \to E$ between an FM-domain D and a U-domain E with a directed representation ρ,* $\mathsf{sqc}(f)$ *has a representation*

$$\mathsf{sqc}(\rho) = \{[\wedge\, \pi_1(\rho'), \wedge\, \pi_2(\rho')] \mid \rho' \subseteq_{\mathrm{fin}} \rho \;\&\; \pi_1(\rho') \textit{ is critical}\}\,.$$

Proof: For every z we obtain the following. We may use continuity of meet, since the representation is directed.

$$\begin{array}{rcl}
\mathsf{sqc}(f)(z) & = & \bigvee\{\wedge f(s) \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq z\} \\
& = & \bigvee\{\wedge\{f(x) \mid x \in s\} \mid s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq z\} \\
& = & \bigvee\{\wedge\{\bigvee\{y \mid \exists x' \leq x\,.\,[x', y] \in \rho\} \mid x \in s\} \mid \\
& & \qquad s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq z\} \\
& = & \bigvee\{\bigvee\{\wedge\, \pi_2(\iota(s)) \mid \iota : s \to \rho \;\&\; \forall x \in s\,.\,\pi_1(\iota(x)) \leq x\} \mid \\
& & \qquad s \text{ critical } \;\&\; \wedge s \in D_{\mathrm{fin}} \;\&\; \wedge s \leq z\} \\
& = & \bigvee\{\wedge\, \pi_2(\rho') \mid \rho' \subseteq_{\mathrm{fin}} \rho \;\&\; \pi_1(\rho') \text{ critical } \;\&\; \wedge\, \pi_1(\rho') \leq z\} \\
& = & \bigvee\{y \mid \exists x \leq z\,.\,[x, y] \in \mathsf{sqc}(\rho)\} \\
& = & (\bigvee^{\mathrm{p}} \mathsf{sqc}(\rho))(z).
\end{array}$$

Note that if s is critical with an isolated meet and $\iota : s \to \rho$ is such that $\pi_1(\iota(x)) \leq x$ for every $x \in s$ we may assume without loss of generality that $\wedge\, \pi_1(\iota(s)) = \wedge s$, since we can replace ι by $\iota' = \lambda x \in s\,.\,\iota(x) \vee (\wedge s)$, because $\wedge\, \pi_2(\iota(s)) \leq \wedge\, \pi_2(\iota'(s))$. Hence, by proposition 5.3, $\pi_1(\iota(s))$ is critical. Conversely, if $\rho' \subseteq_{\mathrm{fin}} \rho$ with $\pi_1(\rho')$ critical, then one may take $s = \pi_1(\rho')$ and $\iota(x)$ to be $[x, y]$ for some y such that $[x, y] \in \rho'$.

We therefore conclude that $\mathsf{sqc}(f) = \bigvee^{\mathrm{p}} \mathsf{sqc}(\rho)$. ∎

Proposition 6.21 *For an FM-domain D and a U-domain E, if f is isolated in $D \to^{\mathrm{ct}} E$ then* $\mathsf{sqc}(f)$ *and* $\mathsf{sqc}^*(f)$ *are isolated as well, and* $\mathsf{sqc}^*(f) = \mathsf{sqc}^N(f)$ *for some N.*

Proof: If f is isolated in $D \to^{\mathrm{ct}} E$ then f has a finite directed representation ρ, and, by 6.20, $\mathsf{sqc}(f)$ has finite representation $\mathsf{sqc}(\rho)$, so that $\mathsf{sqc}(f)$ is isolated as well, as is $\mathsf{sqc}^n(f)$ for every n.

Consider now the operation $\rho \mapsto \mathsf{sqc}(\rho)$ defined by proposition 6.20. If we start out with a finite ρ then repeated application of this operation must eventually result in a finite fixpoint, since we always remain within the finite set

$$\{[\wedge\, \pi_1(\rho'), \wedge\, \pi_2(\rho')] \mid \rho' \subseteq_{\mathrm{fin}} \rho\}\,.$$

Therefore there exists N such that for every $n \geq N$, $\mathsf{sqc}^n(f) = \mathsf{sqc}^N(f)$, and $\mathsf{sqc}^*(f) = \mathsf{sqc}^N(f)$, an isolated function. ∎

Proposition 6.22 *For an FM-domain D and a U-domain E, the isolated elements of $D \to^{\mathrm{sq}} E$ are the isolated elements of $D \to^{\mathrm{ct}} E$ that are sequential.*

Proof: Let $f : D \to E$ be a sequential function. If f is isolated in $D \to^{\mathrm{ct}} E$ then it is certainly isolated in $D \to^{\mathrm{sq}} E$. If f is not isolated in $D \to^{\mathrm{ct}} E$ let F be the set of finite approximations to f in $D \to^{\mathrm{ct}} E$. Of course, $f \notin F$. Now consider $\mathsf{sqc}^*(F) = \{\mathsf{sqc}^*(f') \mid f' \in F\}$. This is clearly a directed set of $D \to^{\mathrm{sq}} E$, and $f = \bigvee^{\mathrm{p}} \mathsf{sqc}^*(F)$. However, $f \notin \mathsf{sqc}^*(F)$, since sqc^* preserves isolatedness of elements in F and f is not isolated in $D \to^{\mathrm{ct}} E$; therefore f is not isolated in $D \to^{\mathrm{sq}} E$. ∎

Proposition 6.23 *The pointwise meet of two sequential functions between an FM-domain D and a U-domain E is sequential.*

Proof: If $f_1, f_2 : D \to E$ are sequential functions then their pointwise meet is continuous by proposition 5.5. Since $f_1 \wedge^{\mathrm{p}} f_2$ is a continuous function dominated by sequential functions, it must preserve criticality of critical sets with isolated meets, proposition 6.14. To show that it also preserves isolated critical meets, let s be a critical set with isolated meet. Since f_1 and f_2 are sequential they preserve isolated critical meets, and we have

$$\begin{aligned}(f_1 \wedge^{\mathrm{p}} f_2)(\wedge s) &= f_1(\wedge s) \wedge f_2(\wedge s) \\ &= (\wedge f_1(s)) \wedge (\wedge f_2(s)) \\ &= \wedge(f_1 \wedge^{\mathrm{p}} f_1)(s).\end{aligned}$$

Therefore $f_1 \wedge^{\mathrm{p}} f_2$ is sequential, proposition 6.13. ∎

Corollary 6.24 *For any FM-domain D and U-domain E, $D \to^{\mathrm{sq}} E$ has property (FM).*

Proof: If f_1 and f_2 are two isolated elements of $D \to^{\mathrm{sq}} E$ then they are isolated elements of $D \to^{\mathrm{ct}} E$, so that $f_1 \wedge^{\mathrm{p}} f_2$ is isolated in $D \to^{\mathrm{ct}} E$. But $f_1 \wedge^{\mathrm{p}} f_2$ is sequential, so that it is isolated in $D \to^{\mathrm{sq}} E$. ∎

Proposition 6.25 *For any FM-domain D and U-domain E, $D \to^{\mathrm{sq}} E$ is an FM-domain.*

Proof: directed-completeness follows from proposition 4.5. Bounded completeness and property (FM) have been established. ω-algebraicity follows easily from the explicit characterization of isolated elements of $D \to^{\mathrm{sq}} E$. ∎

We are still working on finding a class of domains closed under the pointwise-ordered sequential function space. We continue in the next section to refine property (U), in the hope of eventually attaining that goal. Meanwhile, we conclude this section by pointing out that we have already established such a closure for a restricted setting: FM-domains are closed under the pointwise-ordered sequential function space into a flat domain. Flat domains are especially important, since they serve to interpret PCF base types. For instance, every function in the continuous functions model of PCF has a flat target domain when taken in its maximally uncurried form. In the conclusion we point out a possible need to interpret PCF terms as maximally uncurried functions.

Corollary 6.26 *If D is an FM-domain and E is a flat domain, then the pointwise-ordered sequential function space from D to E is an FM-domain.*

Proof: It is easy to check that a flat domain is a U-domain. ∎

6.3 From Property (U) to Property (U^+)

We have established so far that the pointwise-ordered sequential function space between two U-domains is an FM-domain. An eventual goal is to show (a subclass of) U-domains closed under the pointwise-ordered sequential function space. We have yet to identify such a class. The class of U-domains is not satisfactory since they are not closed under product, as we shall show. Instead we strengthen property (U) to property (U^+), and show that U^+-domains are closed under the continuous function space, as well as products.

We have seen, in proposition 6.5, that property (U) may be expressed in terms of single lobe covers: A domain has property (U) iff for every $x \leq x'$ and r a single lobe cover of x such that $x' \notin r$, $r \cap \mathsf{up}(x')$ is a single lobe cover of x'.

Recall now that a lobe r of $\mathsf{up}(x)$ is a cover of x iff one of two cases holds: Either r has a least element, *i.e.*, $\bigwedge r \in r$ (and $r = \mathsf{up}(\bigwedge r)$), and $\bigwedge r$ covers x — call this a type I single lobe cover; or else r has no least element, *i.e.*, $\bigwedge r \notin r$, and $x = \bigwedge r$ — call this a type II single lobe cover. Therefore property (U) is equivalent to saying that, for every $x \leq x'$ and lobe r with $x' \notin r$, if r is a type I or type II single lobe cover of x then $r \cap \mathsf{up}(x')$ is a type I or type II single lobe cover of x'.

We may rule out one of the combinations by noting that if r is a type I single lobe cover of x then $r \cap \mathsf{up}(x') = \mathsf{up}((\bigwedge r) \vee x')$, so that $r \cap \mathsf{up}(x')$ cannot be of type II. Thus, a domain has property (U) iff for every $x \leq x'$ and lobe r with $x' \notin r$, if r is a type I single lobe cover of x then $r \cap \mathsf{up}(x')$ is a type I single lobe cover of x', and if r is a type II single lobe cover of x then $r \cap \mathsf{up}(x')$ is a type I or type II single lobe cover of x'. Property (U) may now be seen as the conjunction of two properties, (U_1) and (U_2). Informally, (U_1) states that if r is type I then so is $r \cap \mathsf{up}(x')$, and (U_2) states that if r is type II then $r \cap \mathsf{up}(x')$ is type I or type II.

We may rule out by decree another of the combinations, by strengthening property (U_2) to (U_2^+), informally stating that if r is type II then so is $r \cap \mathsf{up}(x')$. The conjunction of (U_1) and (U_2^+) gives us property (U^+), a strengthening of property (U).

More formally,

Definition 6.27 We say that a domain has property (U_1) iff

(U_1) For every x, x' and y, if $x \leq x'$, $x \prec y$, $x' \Uparrow y$ and $y \not\leq x'$, then $x' \prec x' \vee y$.

•

Property (U_1) is a generalization of property (C) [KP78, Cur86]: A domain is said to have property (C) iff if $x \prec y$, $x \prec z$, $y \neq z$ and $y \Uparrow z$, then $y \prec y \vee z$, for all *isolated* elements x, y and z. Concrete domains, event domains, and distributive domains all have property (C). (In fact, it is easy to check that distributivity implies property (U_1) as well, but we will not expand on that here.)

For every y and every set X, we use the notation

$$X \vee y = \{x \vee y \mid x \in X \;\&\; x \Uparrow y\}\,.$$

It is easy to see that for every y and every up-closed set X, $X \vee y = X \cap \mathsf{up}(y)$.

Definition 6.28 Define a *self-lobe* to be a lobe X of $\mathsf{up}(\bigwedge X)$. That is, a self-lobe is a down-directed up-closed set determined by elements isolated with respect to its glb. Note that this definition does not rule out $\bigwedge X \in X$, but this makes no difference, as we show in proposition 6.30. The proof of that proposition also makes it clear why we need to use self-lobes, rather than simply down-directed sets. •

Definition 6.29 We say that a domain has property (U_2) iff

(U_2) For every self-lobe X and $y \in \mathsf{down}(X)$, either

$$(\bigwedge X) \vee y = \bigwedge(X \vee y),$$

or there exists z such that

$$(\bigwedge X) \vee y \prec z \quad \& \quad X \vee y = \mathsf{up}(z).$$

We say that a domain has property (U_2^+) iff

(U_2^+) For every down-directed up-closed set X and $y \in \mathsf{down}(X)$,

$$(\bigwedge X) \vee y = \bigwedge(X \vee y).$$

•

Property (U_2^+) asserts that the binary join distributes over certain glbs. We chose here down-directed glbs, but could have also chosen glbs of self-lobes, *i.e.*, kept a relativized Scott property. We point this out in the relevant proofs. Property (U_2^+) trivially implies property (U_2).

Define the class of U^+-domains to be the class of FM-domains that have properties (U_1) and (U_2^+).

Proposition 6.30 *A domain has property (U) iff it has both property (U_1) and property (U_2).*

Proof: We formalize the informal arguments given above.

- Assume property (U).

  Let $x \leq x'$, and $x \prec y$ such that $y \Uparrow x'$ and $y \not\leq x'$. Therefore $r = \mathsf{up}(y)$ is a cover of x, and by (U), $r \cap \mathsf{up}(x') = \mathsf{up}(x' \vee y)$ is a cover of x', so that $x' \prec x' \vee y$. Therefore (U_1) holds.

  Let r be a self-lobe with $x = \bigwedge r$, and $z' \in \mathsf{down}(r)$. We let $x' = x \vee z'$. If $x' \in r$ then $r \vee z' = r \vee x' = r \cap \mathsf{up}(x') = \mathsf{up}(x')$, and thus $(\bigwedge r) \vee z' = x' = \bigwedge(r \vee z')$. If $x' \notin r$ then $x \notin r$, so that r is a cover of x, and, by (U), $r \cap \mathsf{up}(x')$ is a cover of x'. But this means that either $(\bigwedge r) \vee z' = x' = \bigwedge(r \vee x') = \bigwedge(r \vee z')$, or that there exists y such that $(\bigwedge r) \vee z' = x' \prec y$ and $r \vee x' = r \vee z' = \mathsf{up}(y)$.

  Note that property (U_2) must quantify over self-lobes, rather than just down-directed sets, precisely so that property (U) can be used here, *i.e.*, so that r is a cover of $\bigwedge r$ when $\bigwedge r \notin r$.

- Assume (U_1) and (U_2), and let $x \leq x'$ and r a single lobe cover of x with $x' \notin r$. Let $r' = r \cap \mathsf{up}(x')$. If $x' \notin \mathsf{down}(r)$ then $r' = \emptyset$, and r' is certainly a cover of x'. Assume now that $x' \in \mathsf{down}(r)$.

  If there exists y such that $x \prec y$ and $r = \mathsf{up}(y)$ then, by (U_1), $x' \prec x' \vee y$, and thus $r' \cap \mathsf{up}(x') = \mathsf{up}(x' \vee y)$ is a cover of x'. Note that, since $x' \notin r$, $y \not\leq x'$, and since $x' \in \mathsf{down}(r)$, $x' \Uparrow y$.

  If no such y exists then $x = \bigwedge r$. By (U_2), it is either the case that

$$x' = x \vee x' = (\bigwedge r) \vee x' = \bigwedge(r \vee x') = \bigwedge r',$$

or else there exists z such that

$$x' = x \vee x' = (\bigwedge r) \vee x' \prec z$$

and $r' = r \vee x' = \mathsf{up}(z)$. In both cases, r' is a cover of x'. ∎

Corollary 6.31 *A U^+-domain is a U-domain.*

It is easy to see that flat domains are U^+-domains, and hence U-domains as well. As we have already indicated, property (U_2) is not preserved by product. This is easy to see, since, if $(\bigwedge X_i) \vee y_i \prec \bigwedge(X_i \vee y_i)$ for $i = 1, 2$, then

$$(\bigwedge X_1 \times X_2) \vee (y_1, y_2) \not\prec \bigwedge(X_1 \times X_2 \vee (y_1, y_2)).$$

That is, a covering step may take place in more than one component, so that the end result in the product will consist of several covering steps, rather than a single one. The same argument also demonstrates that the (continuous) function space does not preserve property (U_2). Since property (U_2^+) requires equality, it does not suffer from this problem.

Proposition 6.32 *The product of domains preserves each of the properties (U_1) and (U_2^+).*

Proof: If $(x_1, x_2) \prec (y_1, y_2)$ then either $x_1 \prec y_1$ and $x_2 = y_2$, or vice versa. If $(x, z) \prec (y, z)$, $(x, z) \leq (x', z')$, $(y, z) \Uparrow (x', z')$, and $(y, z) \not\leq (x', z')$ then $x \prec y$, $x \leq x'$, $y \Uparrow x'$, and $y \not\leq z'$, so that assumption of (U_1) for the underlying domains leads to $x' \prec y \vee x'$, so that $(x', z') \prec_(y \vee x', z') = (y, z) \vee (x', z')$. Symmetrically for the right-handed case, so that the product has property (U_1).

If X is a down-directed up-closed set and $(y_1, y_2) \in \mathsf{down}\, X$ then, for $i = 1, 2$, $X_i = \pi_i(X)$ is down-directed and up-closed (and is a self-lobe if so is X), with $y_i \in \mathsf{down}(X_i)$, so that $(\bigwedge X_i) \vee y_i = \bigwedge(X_i \vee y_i)$ by assumption of (U_2^+) for the component domains. Therefore

$$\begin{aligned}
(\bigwedge X) \vee (y_1, y_2) &= (\bigwedge X_1, \bigwedge X_2) \vee (y_1, y_2) \\
&= ((\bigwedge X_1) \vee y_1, (\bigwedge X_2) \vee y_2) \\
&= (\bigwedge(X_1 \vee y_1), \bigwedge(X_2 \vee y_2)) \\
&= \bigwedge(X \vee (y_1, y_2)),
\end{aligned}$$

and thus the product has property (U_2^+). ∎

6.4 Continuous Functions on U^+-domains

Using the results obtained for U-domains, we know that each of the sequential, stable and continuous function spaces between U^+-domains is an FM-domain. We proceed now to show that U^+-domains are closed under the continuous function space. As a matter of fact, we show that the continuous function space preserves property (U_2^+), and that it always has property (U_1), regardless of the underlying domains.

Recall that $[x, y] \leq^{\mathrm{P}} f$ iff $y \leq f(x)$. If $y_1 \neq \bot$ then $[x_1, y_1] \leq^{\mathrm{P}} [x_2, y_2]$ iff $x_2 \leq x_1$ and $y_1 \leq y_2$, and $[x_1, y_1] \Uparrow^{\mathrm{P}} [x_2, y_2]$ iff $x_1 \Uparrow x_2$ implies that $y_1 \Uparrow y_2$.

Definition 6.33 We say that a threshold function $[x_0, y_0]$ is an *atomic increment* for f iff

(1) $f \Uparrow^{\mathrm{p}} [x_0, y_0]$, that is, $f(x_0) \Uparrow y_0$;

(2) $f(x_0) \prec\!\!\!- f(x_0) \vee y_0$; and

(3) and $y_0 \leq f(x)$ for every $x > x_0$.

It is easy to see that for every atomic increment $[x_0, y_0]$ of f and for every $x \neq x_0$, $f(x) = (f \vee^{\mathrm{p}} [x_0, y_0])(x)$. •

Proposition 6.34 *In the continuous function space, $f \prec\!\!\!- g$ iff $g = f \vee^{\mathrm{p}} [x_0, y_0]$ for some atomic increment $[x_0, y_0]$ of f, with x_0 uniquely determined by f and g.*

Proof: Let $f \prec\!\!\!- g$. Clearly there exists a threshold function $[x_0, y_0]$ dominated by g but not below f. By definition of covering, $f < f \vee^{\mathrm{p}} [x_0, y_0] = g$.

If there exists z' such that $f(x_0) < z' < g(x_0)$ then there exists some isolated z such that $f(x_0) < f(x_0) \vee z < g(x_0)$, and hence $f < f \vee^{\mathrm{p}} [x_0, z] < g$, a contradiction. Therefore, $f(x_0) \prec\!\!\!-_g (x_0) = f(x_0) \vee y_0$.

If $x_0 < x$ then $g(x) = f(x) \vee y_0$. If x is isolated then $f \leq^{\mathrm{p}} f \vee^{\mathrm{p}} [x, y_0] < g$, and, by definition of covering, $f = f \vee^{\mathrm{p}} [x, y_0]$, so that $[x, y_0] \leq^{\mathrm{p}} f$, and $y_0 \leq f(x)$. If x is not isolated then there must exist an isolated approximation x' to x that does not approximate x_0, and the argument may be repeated for the isolated element $x' \vee x_0$, to show that $y_0 \leq f(x' \vee x_0) \leq f(x)$.

We have shown that $[x_0, y_0]$ is an atomic increment for f. To see that x_0 is uniquely determined by f and g, assume that there exists some other $[x, y]$ dominated by g, but not below f. Then, again, by definition of covering, $f < f \vee^{\mathrm{p}} [x, y] = g = f \vee^{\mathrm{p}} [x_0, y$