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ISSN 1396-7002 **June 2003**

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A Formal Calculus for Categories

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PhD Dissertation

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A Formal Calculus for Categories

A Dissertation
Presented to the Faculty of Science
of the University of Aarhus
in Partial Fulfilment of the Requirements for the
PhD Degree

by
Mario José Cáccamo
6th July 2004

A toda mi familia, y en especial a Chola y Pora.

Abstract

This dissertation studies the logic underlying category theory. In particular we present a formal calculus for reasoning about universal properties. The aim is to systematise judgements about *functoriality* and *naturality* central to categorical reasoning. The calculus is based on a language which extends the typed lambda calculus with new binders to represent universal constructions. The types of the languages are interpreted as locally small categories and the expressions represent functors.

The logic supports a syntactic treatment of universality and duality. Contravariance requires a definition of universality generous enough to deal with functors of mixed variance. *Ends* generalise limits to cover these kinds of functors and moreover provide the basis for a very convenient algebraic manipulation of expressions.

The equational theory of the lambda calculus is extended with new rules for the definitions of universal properties. These judgements express the existence of natural isomorphisms between functors. The isomorphisms allow us to formalise in the calculus results like the Yoneda lemma and Fubini theorem for ends. The definitions of limits and ends are given as *representations* for special **Set**-valued functors where **Set** is the category of sets and functions. This establishes the basis for a more calculational presentation of category theory rather than the traditional diagrammatic one.

The presence of structural rules as primitive in the calculus together with the rule for duality give rise to issues concerning the *coherence* of the system. As for every well-typed expression-in-context there are several possible derivations it is sensible to ask whether they result in the same interpretation. For the functoriality judgements the system is coherent up to natural isomorphism. In the case of naturality judgements a simple example shows its incoherence. However in many situations to know there exists a natural isomorphism is enough. As one of the contributions in this dissertation, the calculus provides a useful tool to verify that a functor is continuous by just establishing the existence of a certain natural isomorphism.

Finally, we investigate how to generalise the calculus to enriched categories. Here we lose the ability to manipulate contexts through weakening and contraction and conical limits are not longer adequate.

Acknowledgements

In first place I would like to express my gratitude to my supervisor Glynn Winskel for his guidance. He has been a constant source of inspiration and I am indebted to him for so many discussions.

I would also like to thank the staff of BRICS for having created such an excellent environment, in particular to Mogens Nielsen for his enthusiasm and encouragement. Many thanks as well to Janne Christensen, Uffe Engberg and Karen Møller. I am also grateful to the Centre of the Danish National Research Foundation for financial support.

From my time in Århus I owe much to Daniel Fridlender; his help and support in the first two years of my studies have been fundamental to me. I would also like to thank my fellow PhD-students, in special to the members of the "casual category theory study group". Many thanks to Ulrich Kohlenbach and Paulo Oliva for clarifying many issues on proof theory.

I wish to thank the staff of the Computer Laboratory in Cambridge where I spent the second half of my studies. Special thanks to Marcelo Fiore for always providing positive feedback to my work, and Daniele Varacca for his friendship and enduring innumerable sessions of derivations and proofs. They have also read portions of this dissertation providing invaluable comments. I would also like to thank Federico Crazzolara, Giuseppe Milicia and Philip Matchett.

I would like to thank Edmund Robinson and Lars Birkedal for accepting to evaluate this dissertation.

Part of the work presented in this dissertation was inspired by a Cambridge Part III course on category theory taught by Martin Hyland, in particular its emphasis on end and coend manipulation.

Finally, I would like to thank Laura, Sofía and my family for their love and support in all these years. I dedicate this dissertation to them.

Mario José Cáccamo,
Cambridge, 6th July 2004.

Contents

Chapter 1

Introduction

At a first glance, a formalisation of category theory might appear as a formidable enterprise. Our goal here is less ambitious, we aim to model an expressive fragment: representable functors and their basic properties. This serves two purposes: firstly to develop a syntax where properties like functoriality are ensured by the form of the expressions; and secondly to provide a logic where we can formalise proofs about categorical statements. This logic is rich enough to capture the ingredients of the categorical reasoning like universality, duality and contravariance.

1.1 Motivation

Universal properties occur in mathematics and computer science under different forms: free constructions, join completions, least fixed points, most general unifiers, inductive definitions and so on. The presentation of these notions in an abstract setting is one of the main contributions of category theory. Much of the appealing nature of the theory rests on the fact that the properties of mathematical structures can be modelled in a unifying system whose language is based on diagrams of arrows and objects. Diagrams avoid the ambiguities of language and facilitates the communication and understanding of mathematical concepts.

Over the last three decades category theory has become a important tool in many areas of computer science. The use of categories in computer science does not appear as a mere organisational language, but as a suitable formal model to understand complex notions of computation. In a more foundational aspect computer science helped to bridge the gap between logic and categories resulting in a whole new are: categorical logic. Here the focus was traditionally set on the characterisation of logic constructors in terms of categorical ones. The best example is the categorical-theoretic model for intuitionistic logic provided by cartesian closed categories via the simply typed lambda-calculus [LS86, Jac99]. Other examples are the categorical models for linear logic [Amb92, Bie95].

In a pioneer work Szabo [Sza78] studied the logics obtained by considering different instances of monoidal categories. In this dissertation we study the logic underlying the manipulation of universal properties. As we talk about logic we need to define a suitable syntax for the new notions and a proof method which supports a sound manipulation of the language. This requires a more algebraic presentation of category theory rather than the usual one based on diagrams of objects and arrows. The approach relies on the theory of *representables*, special kinds of **Set**-valued functors, able to capture in a very compact way the notion of universal property.

The advantages of a language in which to carry out definitions are well-known in domain theory and denotational semantics. There the functions arising in giving semantics to programming languages need to be continuous, with a view to taking least fix points, a fact which holds automatically once the functions are expressed in the language. In the same spirit we aim to systematise judgements of the kind "an expression is functorial in its free parameters" or "two functors are natural isomorphic".

1.2 Category Theory

In this section we revise the basic concepts in category theory and fix the notation used in the next chapters. For more detailed introductions to category theory refer to [Mac98, Bor94a, BW90].

1.2.1 Categories, Functors and Natural Transformations

As observed by Eilenberg and Mac Lane: *"Category has been defined in order to define functor, and functor has been defined in order to define natural transformation"*.

A *category* $\mathcal{C}$ consists of a collection of objects and for every pair a, b of objects there is a collection of arrows or morphisms, written as $\mathcal{C}(a, b)$, where a is the domain and b the codomain. In a category the morphisms compose, the composition is associative and has identities.

We use calligraphic capital letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... to denote categories. Small letters from the beginning and the end of the alphabet $a, b, \ldots, x, y, z$ denote objects while small letters from the middle of the alphabet $f, g, h, \ldots$ are reserved for morphisms. We write $c \in \mathcal{C}$ to mean that c is an object in the category $\mathcal{C}$. To make domains and codomains explicit a morphism f in the collection $\mathcal{C}(c, d)$ is written as $f : c \to d$ or $c \xrightarrow{f} d$.

A category $\mathcal{C}$ is *locally small* if for any pair $c, d \in \mathcal{C}$ the collection $\mathcal{C}(c, d)$ is a set, the so-called *hom-set*. In particular if the collection of objects is also a set we say that the category is *small*. We use $\mathbb{C}, \mathbb{I}, \mathbb{J}, \ldots$ to denote small categories. Henceforth we use the word category with the assumption that we are referring to a locally small category the only exception being chapter 7 where the notion of $\mathcal{V}$-category is introduced.

There are many example of categories in mathematics and computer science. In this dissertation the category **Set** of sets and functions plays an important role. As an exception objects in **Set** will be denoted by capital letters $A, B, \ldots$ as it is customary with sets.

Functors are structure-preserving transformations between categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$, written as $F : \mathcal{C} \to \mathcal{D}$, is described by

- a mapping F from objects in $\mathcal{C}$ into the objects in $\mathcal{D}$, and
- for every pair $c, d \in \mathcal{C}$ a function

$$\mathcal{C}(c, d) \xrightarrow{F_{c,d}} \mathcal{D}(F(c), F(d))$$

such that identities and compositions are preserved. We use capital letters from the middle of the alphabet $F, G, H, \ldots$ to denote functors. In particular when for each $c, d \in \mathcal{C}$ the mapping $F_{c,d}$ is surjective we say that F is *full*, and if $F_{c,d}$ is injective the functors is said to be *faithful*. As it is the usual practice we will omit the subscripts in the mappings defined by a functor.

As functors map categories into categories it is sensible to ask which kind of properties are transferred, or *preserved*, in that transformation. An direct consequence from the definition of functors, for instance, is that functors preserve isomorphisms or more general section-retraction pairs. In the opposite direction, one speaks of the properties in the codomain of a functor that are *reflected* into the domain. In the case of the isomorphisms we have that full and faithful functors reflect them.

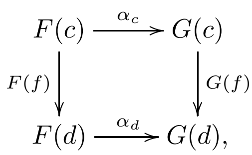
Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* α from F to G, written as $\alpha : F \Rightarrow G$, is defined by a $\mathcal{C}$-indexed family of arrows

$$\langle \alpha_c : F(c) \to G(c) \rangle_{c \in \mathcal{C}}$$

such that for any arrow $f : c \to d$ in $\mathcal{C}$ we have that

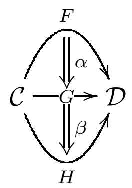
$$\alpha_d \circ F(f) = G(f) \circ \alpha_c.$$

This is graphically represented as the *commutativity* of the diagram



the so-called *naturality square*.

Compatible natural transformation compose componentwise. For functors and natural transformations



it is routine to verify that the family defined by the componentwise composition

$$\langle \beta_c \circ \alpha_c \rangle_{c \in \mathcal{C}}$$

is a natural transformation as well. This new family is written as $\beta \circ \alpha$. This composition is associative, and for a functor F it has an identity defined by the family $\langle \mathsf{id}_{F(c)} \rangle_{c \in \mathcal{C}}$.

Locally small categories and functors between them form the category **CAT**, though a bigger one which is not locally small. Indeed because of the presence of natural transformations, this category has extra structure and classifies as a 2-category. In few words in a 2-category there are arrows between arrows satisfying certain axioms (refer to [KS74] for an introduction of 2-category theory).

1.2.2 Constructions on Categories

Duality

A category $\mathcal{C}$ gives rise to another category by just reversing all the arrows. This new category called the dual or opposite is denoted by $\mathcal{C}^{\mathsf{op}}$. The composition of the arrows in $\mathcal{C}^{\mathsf{op}}$ can be expressed in terms of the arrows in $\mathcal{C}$ by reading the composition in the reverse order: $g \circ f$ in $\mathcal{C}^{\mathsf{op}}$ uniquely corresponds to $f \circ g$ in $\mathcal{C}$. In the same spirit, a functor $F : \mathcal{C} \to \mathcal{D}$ can be *dualised* to obtain a functor $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$. More generally a statement S involving a category $\mathcal{C}$ automatically gives a dual statement S^{op} obtained by reversing all the arrows. This is known as the *duality principle*.

Product of Categories

Given categories $\mathcal{C}$ and $\mathcal{D}$ there is a product category $\mathcal{C} \times \mathcal{D}$. The collection of objects is given by the cartesian product of the collection of objects in $\mathcal{C}$ and $\mathcal{D}$. The set of arrows from (c, d) to (c', d') is defined to be

$$\mathcal{C} \times \mathcal{D}((c, d)), (c', d')) = \mathcal{C}(c, c') \times \mathcal{D}(d, d').$$

Composition and identities are defined pairwise. By projecting on one object of a pair we obtain the *projections* functors

$$\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \quad \text{and} \quad \pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}.$$

Functor Categories

For functors $F, G : \mathcal{C} \to \mathcal{D}$ the collection of the natural transformations from F to G is denoted by the expression $\mathbf{Nat}(F, G)$. The natural transformations in $\mathbf{Nat}(F, G)$ compose, this composition is associative and has identities. Thus there is a "category" where the functors from $\mathcal{C}$ to $\mathcal{D}$ are objects and natural transformations are arrows: the so-called *functor category* written as $[\mathcal{C}, \mathcal{D}]$.

For locally small categories $\mathcal{C}$ and $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ is not necessary locally small and then may fall out of the universe considered here (see [Mac98, Cro93] for examples). If the domain is small, say $\mathbb{C}$, then the functor category $[\mathbb{C}, \mathcal{D}]$ is indeed locally small and we can write $[\mathbb{C}, \mathcal{D}](F, G)$, its hom-set, for $\mathbf{Nat}(F, G)$.

Bifunctors

Products of categories allows us to define functors of multiple arguments. It follows from the usual inductive argument and the associativity of products that it is enough to consider functors with two arguments, the so-called *bifunctors*. In this section we mention some basic yet important results related to bifunctors.

With the introduction of functor categories for many of the interesting cases we could just work with one-argument functors. This has a precise categorical meaning that we postpone until adjunctions. By now it is enough to observe that for a bifunctor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and for each $a \in \mathcal{A}$ there is a functor

$$F^a : \mathcal{B} \to \mathcal{C}$$

obtained by fixing a, what Kelly [Kel82] calls the *partial functor*. Some authors use the notation $F(a, \_)$ for F^a which may create confusion with the bifunctor F. The action of F^a is defined as

$$F^a(b) = F(a, b) \quad \text{and} \quad F^a(g) = F(\mathsf{id}_a, g)$$

for $g : b \to b'$ in $\mathcal{B}$. Similarly for every $b \in \mathcal{B}$ there is a partial functor

$$F_b : \mathcal{A} \to \mathcal{C}.$$

The collection of partial functors alone is not enough to characterise the bifunctor F. There is an extra requirement, namely that for arrows $f : a \to a'$ in $\mathcal{A}$ and $g : b \to b'$ in $\mathcal{B}$

$$F^{a'}(g) \circ F_b(f) = F_{b'}(f) \circ F^a(g).$$

This justifies the use of a more compact notation $F(a, g)$ and $F(f, b)$ instead of $F(\mathsf{id}_a, g)$ and $F(f, \mathsf{id}_b)$ respectively when referring to the bifunctor F.

This basically says that bifunctoriality can not be checked at each component. This contrasts with naturality, given a natural transformation $\alpha : F \Rightarrow G$ where $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, by fixing an object $a \in \mathcal{A}$ there is a *partial natural transformation*

$$\alpha_{a,\_} : F^a \Rightarrow G^a$$

and similarly for an object $b \in \mathcal{B}$. The collection of all these partial natural transformations uniquely determine α. Then naturality as opposed to functoriality can be checked at each component separately.

1.3 Functoriality and Naturality

The internal structure of **CAT** suggest a language where the expressions are given by the constructors on categories above. Indeed most of these constructions give functors, or in other words they are *functorial* in their variables. These functors can be combined via composition to generate complex expressions still functorial in their variables.

More formally, let $x_1 \in \mathcal{C}_1, \cdots, x_n \in \mathcal{C}_n$ be objects and an expression

$$E(x_1, \cdots, x_n)$$

denoting an object in $\mathcal{C}$. We say E is functorial in $x_1, \cdots, x_n$ if it defines an action consistent with the definition of functor when $x_1, x_2, \ldots, x_n$ stand for arrows instead of objects. Thus a functorial expression has two possible readings: as an object in $\mathcal{C}$ when the free variables $x_1, x_2, \ldots, x_n$ are interpreted as objects, and as an arrow in $\mathcal{C}$ when they are interpreted as arrows.

Given two expressions $E_1(x_1, \cdots, x_n)$ and $E_2(x_1, \cdots, x_n)$ functorial in the variables $x_1, \cdots, x_n$ it is sensible to ask whether there is a *natural* family between them. In general we shall be interested in natural isomorphisms, in this case one says that there is an isomorphism

$$E_1(x_1, \cdots, x_n) \cong E_2(x_1, \cdots, x_n)$$

natural in $x_1, \cdots, x_n$.

It is a usual practice in category theory to use diverse place-holders like $\_$ and $=$ to indicate variables in a functorial expression. We use place-holders in the introductory chapters but in the more complex examples where confusion can arise we use lambda-notation to indicate parameters and their scope.

Hom-Functors

For a category $\mathcal{C}$, the expression $\mathcal{C}(c, d)$ defines the hom-functor

$$\mathcal{C}(=, \_) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$$

where the action on arrows is

$$\begin{array}{ccc} c & d & \mathcal{C}(c,d) \\ f \uparrow & g \downarrow & \downarrow \mathcal{C}(f,g) = g \circ \_ \circ f \\ c' & d' & \mathcal{C}(c', d'). \end{array}$$

The same example serves to see that not all possible expressions give rise to a functor. The expression $\mathcal{C}(c, c)$ for a nontrivial category $\mathcal{C}$ is non-functorial since there is no action over arrows matching the action over objects.

Constant Functors and Diagonals

For an object $c \in \mathcal{C}$ there is a constant functor $\Delta c\colon \mathcal{A} \to \mathcal{C}$. This functor just maps any object in the category $\mathcal{A}$ into c and any arrow into the identity id_c.

For a small category $\mathbb{I}$, the constant functors define the so-called *"diagonal"* functor $\Delta\_ : \mathcal{C} \to [\mathbb{I}, \mathcal{C}]$. This functor maps an object $c \in \mathcal{C}$ into the functor Δc. Given an arrow $f\colon c \to c'$, the natural transformation Δf is defined to be the arrow f at each component.

Evaluation Functor

Given a functor $F : \mathbb{I} \to \mathcal{C}$ and an object $i \in \mathbb{I}$ we can evaluate F at i to obtain $F(i) \in \mathcal{C}$. This extends to a functor

$$\mathit{eval}\colon [\mathbb{I}, \mathcal{C}] \times \mathbb{I} \to \mathcal{C},$$

the *evaluation functor*. The action of this functor on arrows is defined as

$$\begin{array}{ccccc} F & & i & & F(i) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)\circ\alpha_i = \alpha_j \circ F(f)} \\ G & & j & & G(j). \end{array}$$

Functoriality as Naturality

A functor $F\colon\mathcal{C} \to \mathcal{D}$ defines for a pair $c, d \in \mathcal{C}$ a map

$$\mathcal{C}(c, d) \xrightarrow{F_{c,d}} \mathcal{D}\big(F(c), F(d)\big)$$

sending an arrow $f\colon c \to d$ into $F(f)\colon F(c) \to F(d)$. This mapping is indeed natural in c and d, in the sense that the family $\langle F_{c,d}\rangle_{c,d\in\mathcal{C}^{\mathsf{op}}\times\mathcal{C}}$ is a natural transformation from $\mathcal{C}(=,-)\colon\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ to $\mathcal{D}(F(=), F(-))\colon\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$. As naturality can be verified at each component: for an arrow $h\colon d \to d'$ the diagram

$$\begin{array}{ccc} \mathcal{C}(c, d) & \xrightarrow{F_{c,d}} & \mathcal{D}(F(c), F(d)) \\ {\scriptstyle h\circ-}\downarrow & & \downarrow{\scriptstyle F(h)\circ-} \\ \mathcal{C}(c, d') & \xrightarrow{F_{c,d'}} & \mathcal{C}(F(c), F(d')) \end{array}$$

commutes since for $f\colon c \to d$ we have

$$F(h) \circ F(f) = F(h \circ f)$$

by functoriality of F. Naturality in d follows by a similar argument. Thus functoriality is expressed here as a naturality condition.

Partial Functors

Let $F\colon\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor. For objects $a, a' \in \mathcal{A}$ there are partial functors $F^a, F^{a'}\colon\mathcal{B} \to \mathcal{C}$. An arrow $f\colon a \to a'$ gives rise to a family $F^f = \langle F(f, b)\colon F(a, b) \to F(a', b)\rangle_{b\in\mathcal{B}}$. Now, for an arrow $g\colon b \to b'$ the square

$$\begin{array}{ccc} F^a(b) = F(a, b) & \xrightarrow{F^f(b)} & F^{a'}(b) = F(a', b) \\ {\scriptstyle F^a(g)}\downarrow & & \downarrow{\scriptstyle F^{a'}(g)} \\ F^a(b') = F(a, b') & \xrightarrow[F^f(b')]{} & F^{a'}(b') = F(a', b') \end{array}$$

commutes since F is a bifunctor. Thus the family F^f is a natural transformation from F^a to $F^{a'}$.

Curried Functors and the Yoneda Embedding

For a functor $G\colon\mathcal{A}\times\mathbb{I}\to\mathcal{C}$ where $\mathbb{I}$ is a small category, the family of partial functors defines a higher-order functor:

$$\boldsymbol{\lambda}(G)=\lambda x.G^x\colon\mathcal{A}\to[\mathbb{I},\mathcal{C}]$$

the *curried* version of G.

In particular for a small category $\mathbb{C}$, the hom-functor $\mathbb{C}(=,-)\colon\mathbb{C}^{\mathsf{op}}\times\mathbb{C}\to\mathbf{Set}$ can be curried into a functor

$$\boldsymbol{\lambda}(\mathbb{C}(=,-))\colon\mathbb{C}^{\mathsf{op}}\to[\mathbb{C},\mathbf{Set}].$$

This is called the contravariant *Yoneda* functor and denoted by $\mathcal{Y}^{\mathsf{op}}$. From the composition of the hom-functor with the functor swapping arguments we obtain the covariant *Yoneda* functor

$$\mathcal{Y}\colon\mathbb{C}\to[\mathbb{C}^{\mathsf{op}},\mathbf{Set}].$$

This functor acts by mapping an object $c\in\mathbb{C}$ into $\mathbb{C}(-,c)$ and over arrows

$$\begin{array}{ccc} c & & \mathbb{C}(-,c) \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle \mathcal{Y}(f)=\langle f\circ-\rangle_{c\in\mathbb{C}}} \\ d & & \mathbb{C}(-,d) \end{array}$$

By definition $\mathcal{Y}$ is injective over objects. As a direct consequence of the Yoneda lemma, this functor is full and faithful and then an embedding.

Duality Revisited

The operation taking a category $\mathcal{C}$ into $\mathcal{C}^{\mathsf{op}}$ can be extended to an endofunctor over **CAT**. Indeed $-^{\mathsf{op}}$ defines actions for objects, functors and natural transformations given in reality a 2-functor. It acts over categories as follows:

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C}^{\mathsf{op}} \\ F\ \overset{\alpha}{\Longrightarrow}\ G & \longmapsto & F^{\mathsf{op}}\ \overset{\alpha^{\mathsf{op}}}{\Longleftarrow}\ G^{\mathsf{op}} \\ \mathcal{D} & & \mathcal{D}^{\mathsf{op}} \end{array}$$

where $\alpha_c^{\mathsf{op}}\colon G^{\mathsf{op}}(c)\to F^{\mathsf{op}}(c)$ in $\mathcal{D}^{\mathsf{op}}$ corresponds to the arrow $\alpha_c\colon F(c)\to G(c)$ in $\mathcal{C}$. Then this functor is contravariant with respect to natural transformations only.

By definition of opposite category for objects $c,d\in\mathcal{C}$ there is an isomorphism of sets

$$\mathcal{D}(F(c),G(d))\cong\mathcal{D}^{\mathsf{op}}(G^{\mathsf{op}}(d),F^{\mathsf{op}}(c)). \tag{1.1}$$

These expressions are functorial in all variables and furthermore the isomorphism is natural in those variables as well. For natural transformations $\alpha : H\Rightarrow F$ and $\beta : G\Rightarrow K$ and morphisms $f\colon c'\to c$ and $g\colon d\to d'$ the diagram

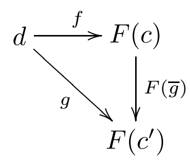
$$\begin{array}{ccc} \mathcal{D}\big(F(c),G(d)\big) & \cong & \mathcal{D}^{\mathsf{op}}\big(G^{\mathsf{op}}(d),F^{\mathsf{op}}(c)\big) \\ {\scriptstyle K(g)\circ\beta_d\circ-\circ F(f)\circ\alpha_{c'}}\downarrow & & \downarrow{\scriptstyle H^{\mathsf{op}}(f)\circ\alpha_c^{\mathsf{op}}\circ-\circ G^{\mathsf{op}}(g)\circ\beta_{d'}^{\mathsf{op}}} \\ \mathcal{D}\big(H(c'),K(d')\big) & \cong & \mathcal{D}^{\mathsf{op}}\big(K^{\mathsf{op}}(c'),H^{\mathsf{op}}(d')\big) \end{array}$$

commutes.

1.4 Universal Properties and Representables

Universal constructions appear in category theory under different flavours: limits, adjoints, Kan extensions and so on. There are several ways in which we can expressed the properties defining these constructions. These definitions, however, are special instances of the most abstract notion of universal arrow.

For a functor $F:\mathcal{C} \to \mathcal{D}$ and an object $d \in \mathcal{D}$, a *universal arrow* from d to F consists of a pair (f, c) where $f:d \to F(c)$ is an arrow in $\mathcal{D}$ such that for any other arrow $g:d \to F(c')$ there exists a *unique* arrow $\overline{g}:c \to c'$ in $\mathcal{C}$ making the diagram



commute. In the setting of locally small categories the definition of universal arrow is equivalent to universal element (see chapter 2).

A functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ is said to be *representable* if it is isomorphic to a hom-functor

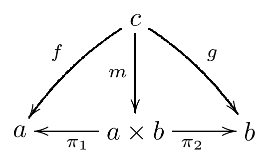
$$\mathcal{C}(-, c) \overset{\theta}{\cong} F$$

or in other words there is an isomorphism

$$\mathcal{C}(x, c) \overset{\theta_x}{\cong} F(x)$$

natural in x. The pair (c, θ) is called the representation for F. A fundamental result in category theory is that a universal construction determines and is determined by a representation for a **Set**-valued functor.

Consider as an example the categorical definition of product. Given two objects a, b in a category $\mathcal{C}$, their product is defined to be an object $a \times b$ equipped with morphisms $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$. The universal condition is that for any other object $c \in \mathcal{C}$ with arrows $f : c \to a$ and $g : c \to b$ there exists a unique arrow $m : c \to a \times b$ such that the triangles in the diagram



commute. In other words for each $c \in \mathcal{C}$ there is an isomorphism

$$\mathcal{C}(c, a \times b) \cong \mathcal{C}(c, a) \times \mathcal{C}(c, b).$$

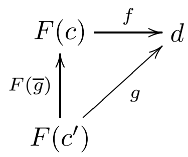
The commutativity of the diagram above implies that this isomorphism is natural in c, *i.e.* that the product defines a representation for the functor

$$\mathcal{C}(-, a) \times \mathcal{C}(-, b) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}.$$

The same result generalises for the case of limits. Among a collection of objects satisfying certain property in a category a limit selects the universal one. A property is described as a *diagram* in $\mathcal{C}$, *i.e.* a functor with a small category as domain. Formally an object satisfies the property defined by $D : \mathbb{I} \to \mathcal{C}$ if it is the vertex of cone for D. A *limit* for a diagram $D : \mathbb{I} \to \mathcal{C}$ is a representation

$$\mathcal{C}(-, \varprojlim_{\mathbb{I}} D) \cong [\mathbb{I}, \mathcal{C}](\Delta -, D). \tag{1.2}$$

By duality we can define a couniversal arrow from a functor $F:\mathcal{C} \to \mathcal{D}$ to an object $d \in \mathcal{D}$. In this case the couniversal arrow is given by a pair (f, c) where $f:F(c) \to d$ in $\mathcal{D}$ such that for any other arrow $g:F(c') \to d$ there exists a unique arrow $\overline{g}:c' \to c$ for which the diagram



commutes.

Couniversal arrows from F to d are in one-to-one correspondence with universal arrows from d to F^{op}. This follows by applying the 2-functor $(\_)^{\mathsf{op}}$ to the diagram above.

Limits as such do not account for contravariance, not at least in the categorical language present above. There are cases of functors, the typical example being the evaluation functor [Pow95, Str02], which demand a unified treatment of multiple variance. *Ends* are special kind of limits for functor of the form $G:\mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{C}$, the end of G is written as $\int_x G(x,x)$. Although ends and limits coincide, the former supports a convenient algebraic manipulation of functors over which we based our calculus for categories. In particular results like the Yoneda lemma can be expressed by an end expression. The hom-sets of functor categories correspond to end formulæ as well. Let $F, G:\mathbb{C} \to \mathcal{D}$ be functors. Then there is an isomorphism

$$[\mathbb{C}, \mathcal{D}](F(x), G(x)) \cong \int_x \mathcal{D}(F(x), G(x)).$$

Chapter 3 is devoted to the presentation of universal properties from the perspective of ends and its dual form coends.

1.5 The Calculus

1.5.1 Syntax

The categorical constructions above induce the language of the calculus. It basically extends the simply typed lambda calculus with expressions for hom-sets and new binders for ends and its dual coends. The types of the language denote locally small categories and a typing judgement of the form

$$\Gamma \vdash E:\mathcal{C}$$

is read as the expression E being functorial in its free variables occurring in the context Γ. Of course not all well-formed sequents result in functorial expressions. The typing rules of the calculus determine which are the legal expressions of the language.

The rules for lambda abstraction and application are basically the ones inherited from the lambda calculus with types, but already the rule for typing hom-sets poses an interesting example:

$$\frac{\Gamma_1 \vdash E_1:\mathcal{C} \quad \Gamma_2 \vdash E_2:\mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2):\mathbf{Set}}$$

it defines a bifunctor which is contravariant in the first argument. The modality Γ_1^{op} indicates that the types in Γ_1 are replaced by its dual form. This rule captures the usual practice in the manipulation of hom-expressions in category theory.

One of the important results of representables functors is that they can carry extra parameters in a functorial manner, the so-called parametrised representability. One consequence is that the introduction of an end binder over some parameters does not disturb the functoriality over the ones remaining free:

$$\frac{\Gamma, y : \mathbb{I}^{\mathsf{op}}, z : \mathbb{I} \vdash E(y, z) : \mathcal{C}}{\Gamma \vdash \int_x E(x, x) : \mathcal{C}.}$$

One of the novelties is the presence of duality explicitly as a rule

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}.}$$

The operation on expressions $(\_)^*$ turns limits and ends into their dual forms colimits and coends. The interaction of contravariant functors (as the hom-expression above) and duality forces a non-standard notion of substitution. In chapter 5 we show that despite these new ingredients the proof system is still closed under substitution.

The definition of limits and ends as representations supports an equational reasoning which extends the one provided by the lambda calculus. The syntactic judgement

$$\Gamma \vdash E_1 \cong E_2 : \mathcal{C}$$

is read as "the expressions E_1 and E_2 are naturally isomorphic in the free variables in Γ". For instance the definition of limit would give the rule

$$\frac{\Gamma, x : \mathbb{I} \vdash E(x) : \mathcal{C}}{\Gamma, z : \mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(z, \varprojlim_{\mathbb{I}} E(x)) \cong [\mathbb{I}, \mathcal{C}](\Delta z, E(x)) : \mathbf{Set}.}$$

1.5.2 Semantics

Types in the language are interpreted as locally small categories, and expressions correspond to functors. In this setting contexts are understood as the categorical product of types occurring in them. This interpretation is consistent with the structural rules for the manipulation of variables. The meaning of an expression in the language is defined by a derivation. Thus a derivation

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E : \mathcal{C} \end{array}$$

denotes a functor from the category denoted by Γ into $\mathcal{C}$.

The presence of structural rules and special rules for duality results in that the same final sequent can be obtained through different derivations. This calls for a coherence result, *i.e.* we need to verify that different derivations lead to the same interpretation. In chapter 5 we show that the syntax is coherent with respect to the semantics up to natural isomorphism.

A derivation of an isomorphism judgement is interpreted as a natural isomorphism between the functors determined by the expressions. In this case the interpretation is not coherent with respect to the syntax. Thus an isomorphisms judgement only expresses the existence of one or more natural isomorphisms.

The choice for structural rules and restrictions on their use play an important role in the different aspects of the calculus. In the last chapter we study how to extend the calculus from the locally small categories to an enriched setting. Indeed, the 2-category obtained by putting together $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural isomorphisms gives rise to an interpretation of a substructural version of the calculus.

1.6 Related Work and Applications

In categorical logic we encounter the classical references to the work initiated by Lambek about the relation between λ-calculus and cartesian-closed categories [LS86]. Asperti and Longo [AL91] give a detailed introduction to categorical semantics, although they prefer to elaborate on the sequent-based calculi instead of natural deduction. Crole in the book *Types for Categories* [Cro93] and later Pitts [Pit00] and Jacobs [Jac99] extended the work by Lambek and Scott to study the categorical models behind higher-order logics.

The study of the logics behind categorical constructions is more recent. An example is the work by Cockett and Seely [CS01] on a deductive system for categories with finite products and coproducts. In a related work although from a more mathematical perspective Freyd [Fre02] presents a cartesian logic where the associated language comes from the logic of unique existential assertions.

Rydeheard and Burstall [RB88] formalised various universal constructions in the functional language ML. This inspired many of the later attempts to find a mechanisation of the categorical reasoning although the seem to follow a somewhat different tack that the approach taken in this dissertation. In particular Takeyama [Tak95] presents a computer checked language with the goal of supplying a categorical framework for type theory. In a similar development Huet and Saïbi [HS00] implement a formalisation of category theory where the main ingredient is the representation of hom-sets as setoids.

Category theory supplies an adequate abstract language for denotational semantics within the scope of different paradigms in computer science. Domain theory [AJ94, AC98], for example, has been traditionally studied under a categorical perspective. Recently this approach has been extended to enriched categories of domains, an example is the work by Fiore [Fio96]. One of our motivations is the increasing use of category theory in denotational semantics, often as categories as domains, where universal properties like limits and ends, and functors preserving those properties have a useful consequence [Cat99]. This approach begun with the work by Winskel and Nielsen [WN95] on the formalisation of operations over seemingly different models for concurrency like transition systems, Petri nets and event structures as instance of the same categorical constructions. These models embody a categorical definition of bisimulation in terms of open maps. Joyal *et al.* [JNW96] showed that spans of open maps in the category of transition systems corresponds to Milner-Park bisimulation. Cattani and Winskel [Cat99, CW] extended this work by proposing a systematic treatment of bisimulation for concurrent process languages like higher-order CCS and π-calculus. In a related work categories have been used to bridge the gap between the operational and denotational world [Tur96, TP97, FPT99]. It is interesting that in these applications presheaf categories play a central role in the theory. Under a different perspective an algebraic approach to categories would help the task of mechanising the theory in a theorem prover. The last decade has seen an impressive growth in this area where several formal theories have been mechanised, regrettably there is no satisfactory implementation of category theory yet.

1.7 Synopsis

Chapters 2 and 3 We present the basic categorical background from the viewpoint of representables. In particular we study the implications of parametrised representability in different scenarios like limits and adjoints. The goal of these two chapters is not to rewrite what can be found in a good introductory book; rather to present the theory by emphasising the results which are relevant to the calculus as well as giving examples of the kind of categorical reasoning supported.

Chapter 4 In this chapter we present a characterisation of continuous functors which can be formalised in the calculus. This presentation gives a fresh approach to the verification of continuity, and we consider is one of the original contributions in this dissertation.

Chapter 5 This chapter and the following present the syntax, semantics and equational theory of the calculus. Chapter 5 is devoted to functoriality. The main results in this chapter are the admissibility of substitution and the "coherence" of the interpretation for functorial judgements.

Chapter 6 This chapter focuses on the rules for natural isomorphisms and their interpretations. We show how the definitions of colimits and coends can be derived from their dual limits or ends plus a simple rule for opposite categories. At the end of this chapter we propose a syntax for weighted limits.

Chapter 7 We study how to generalise the calculus to cover the 2-category of $\mathcal{V}$-enriched categories for a suitable locally small category $\mathcal{V}$. The calculus is reduced to a substructural version of the original one where the rules for weakening and contraction of contexts are not longer present. This reflects the gap open in the model between ends and weighed limits, where the latter gives a more general definition.

Chapter 8 In the last chapter we give the conclusions and elaborate on possible extensions of this work.

Chapter 2

Representability

We review the basic concepts of category theory such as limits and adjunctions from the perspective of representable functors. In particular we study different instances of parametrised representability which will be later used to justify the semantics of the calculus. Despite the fact that most of the results in this chapter and the next one are part of the folklore in category theory in some occasions we sketch their proofs to show the kind of reasoning we aim to formalise in the calculus.

2.1 Universal Properties

The notion of universal arrows reduces to that of universal element when considering locally small categories.

Definition 2.1.1 (Universal Element) Given a functor $G:\mathcal{C} \to \mathbf{Set}$, a *universal element* of G is a pair (c, x) where $c \in \mathcal{C}$ and $x \in G(c)$, such that for any other pair (c', y) where $y \in G(c')$ there exists a *unique* arrow $f:c \to c'$ in $\mathcal{C}$ with $y = G(f)(x)$.

Given a functor $F:\mathcal{C} \to \mathcal{D}$ and $d \in \mathcal{D}$ we can define a functor $\mathcal{D}(d, F(-)):\mathcal{C} \to \mathbf{Set}$. The condition that $h:d \to F(c)$ is universal is equivalent to saying that (c, h) is a universal element of $\mathcal{D}(d, F(-))$. Conversely, a universal element $x \in G(c)$ for a functor $G:\mathcal{C} \to \mathbf{Set}$ can be regarded as an arrow $x:1 \to G(c)$. Thus the condition that the element x is universal is equivalent to saying that $x:1 \to G(c)$ is universal. A couniversal arrow corresponds to a universal element for a contravariant **Set**-valued functor.

The notion of universal arrow can be defined as the initial object of the comma category $x \downarrow F$. This expresses the general notion of universality in terms of a particular one. Similarly, a universal element for $G:\mathcal{C} \to \mathbf{Set}$ may be defined as the initial element of the categories of elements of G. Of course we replace *initial object* by *final object* when talking about couniversality.

Definition 2.1.2 (Representable Functor) A functor $F:\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ is *representable* if for some $c \in \mathcal{C}$ there exists a natural isomorphism

$$\mathcal{C}(-, c) \overset{\theta}{\cong} F.$$

The pair (c, θ) is a *representation* for F and $\varepsilon_c = \theta_c(\mathsf{id}_c) \in F(c)$ is called its *counit*. The dual definition: a functor $G:\mathcal{C} \to \mathbf{Set}$ is representable if there exists an object $d \in \mathcal{C}$ and a natural isomorphism

$$\mathcal{C}(d, -) \overset{\beta}{\cong} G.$$

In this case $\eta_d = \beta_d(\mathsf{id}_d)$ is called the *unit* of the representation.

By abuse of the language we refer to a representation by mentioning only the first component of the pair assuming the natural isomorphism is understood from the context. In some occasions to simplify the notation we omit the subindices of counits and units.

The Yoneda lemma provides a close connection between representable functors and universal elements.

Theorem 2.1.3 (Yoneda Lemma) Given a functor $F\colon\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ the mapping

$$\alpha \mapsto \alpha_c(\mathsf{id}_c)$$

defines an isomorphism

$$[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]\big(\mathcal{C}(-, c), F\big) \cong F(c)$$

natural in c and F.

The inverse of the mapping above sends an element $x \in F(c)$ for some $c \in \mathcal{C}$ to the natural transformation $\widetilde{x} : \mathcal{C}(-, c) \Rightarrow F$ whose component at d is

$$\widetilde{x}_d(f) = F(f)(x).$$

In particular if a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ has a representation (c, θ) then the counit $\varepsilon_c \in F(c)$ is a universal element of F since $\widetilde{\varepsilon_c}$ is an isomorphism for each c. Conversely, if a functor $F\colon\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ has a universal element (c, x) then F is representable since $\widetilde{x}$ is by definition a natural isomorphism. In this case for an element $y \in F(d)$ we say that the unique f such that

$$y = F(f)(x)$$

is the *mediating* arrow defined by y.

Theorem 2.1.4 Let $F\colon\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ be a functor. F is representable if and only if it has a universal element.

Proof. It follows from the discussion above. The most relevant part of the proof for the rest of this work is that the counit of a representation is a universal element. □

We shall say that an object satisfies a *universal property* when it is a representation for a **Set**-valued functor. The **Set**-valued functor defines the kind of elements the property describes.

An immediate corollary of the Yoneda lemma is that the Yoneda functor is full and faithful, this gives the following fundamental result:

Proposition 2.1.5 Representations are unique up to isomorphism: if (a, α) and (b, β) are both representations for a functor $F\colon\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ then $a \cong b$ in $\mathcal{C}$.

Terminal objects give an example of universal construction. A terminal object $\top$ in a category $\mathcal{C}$ is such that for any object $a \in \mathcal{C}$ there is only one arrow from a to $\top$:

$$\mathcal{C}(a, \top) \cong 1.$$

In other words, the constant contravariant functor $\Delta 1\colon\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ is representable. Similarly as exemplified in the introduction the categorical product can be expressed as a representation where the counit is the pair of projections (π_1, π_2).

By considering **Set**-valued bifunctors we can generalise the notion of representable functors to carry extra arguments.

Theorem 2.1.6 (Parametrised Representability) Let $F : \mathcal{A} \times \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$ be a bifunctor such that for every $a \in \mathcal{A}$ there exists a representation $(G[a], \theta^a)$ for the partial functor $F^a : \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$. Then there is a *unique* extension of the mapping $a \mapsto G[a]$ to a functor $G : \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{B}\big(b, G(a)\big) \overset{\theta^a_b}{\cong} F(a, b) \tag{2.1}$$

is natural in $a \in \mathcal{A}$ and $b \in \mathcal{B}^{\mathsf{op}}$.

Proof. In this proof as in other examples we omit the subindices to indicate the components of the natural transformations to simplify the notation. Given an arrow $f : a \to a'$ there is a diagram

$$\begin{array}{ccc} \mathcal{B}(-, G[a]) & \overset{\theta^a}{\cong} & F^a \\ \Big\downarrow{\scriptstyle (\theta^{a'})^{-1} \circ F^f \circ \theta^a} & & \Big\downarrow{\scriptstyle F^f} \\ \mathcal{B}(-, G[a']) & \overset{\theta^{a'}}{\cong} & F^{a'}. \end{array} \tag{2.2}$$

Since the Yoneda functor is full and faithful there must exist a unique arrow, say $G(f)$, such that

$$\mathcal{B}(-, G(f)) = (\theta^{a'})^{-1} \circ F^f \circ \theta^a.$$

It is now routine to verify that this definition gives a functor. □

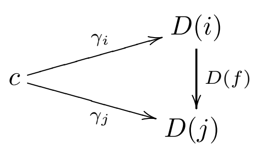
A concrete definition of $G(f)$ above can be obtain by chasing $\mathsf{id}_{G[a]}$ around the commuting diagram (2.2):

$$G(f) = \Big((\theta^{a'})^{-1} \circ F(f, G[a]) \circ \theta^a\Big)\,(\mathsf{id}_{G[a]}) = \Big((\theta^{a'})^{-1} \circ F(f, G[a])\Big)\,(\varepsilon_{G[a]}). \tag{2.3}$$

We now review the notions of limits and adjoints at the light of this result.

2.2 Limits

A limit for a digram $D : \mathbb{I} \to \mathcal{C}$ is given by a universal cone. A *cone* from an object $c \in \mathcal{C}$ to a diagram D is a family of arrows $\langle \gamma_i : c \to D(i) \rangle_{i \in \mathbb{I}}$ such that for every $f : i \to j$ in $\mathbb{I}$ the corresponding triangle

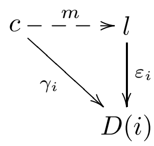


commutes. This is equivalent to saying that the family γ is a natural transformation from the constant functor Δc to D.

A cone $\varepsilon : \Delta l \Rightarrow D$ is universal or a *limiting* if given any other cone $\gamma : \Delta c \Rightarrow D$ there exists a *unique* arrow $m : c \to l$ such that the diagram of natural transformations

$$\begin{array}{ccc} \Delta c & \overset{\Delta m}{- - \to} & \Delta l \\ {\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\ D & \overset{\mathsf{id}}{\cong} & D \end{array} \tag{2.4}$$

commutes, *i.e.* for every $i \in \mathbb{I}$ the triangle



commutes. We say that ε is a *limit* for D. This definition can be given as a representation for a special **Set**-valued functor.

Definition 2.2.1 (Limit) A *limit* for a diagram $D : \mathbb{I} \to \mathcal{C}$ is a representation (l, θ) for the functor $[\mathbb{I}, \mathcal{C}](\Delta -, D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$. The counit of this representation

$$\varepsilon = \theta_l(\mathsf{id}_l) \in [\mathbb{I}, \mathcal{C}](\Delta l, D)$$

is a limiting cone.

From the discussion above, the elements of $[\mathbb{I}, \mathcal{C}](\Delta c, D)$ are the cones from c to D. From Theorem 2.1.4 the counit ε of the representation (l, θ) is a universal element. Then for $\gamma \in [\mathbb{I}, \mathcal{C}](\Delta c, D)$ there exists a unique arrow $m : c \to l$ in $\mathcal{C}$ such that

$$\gamma = [\mathbb{I}, \mathcal{C}](\Delta m, D)(\varepsilon) = \varepsilon \circ \Delta m.$$

By abuse of the language we call limit the object part of the representation. For a given diagram D there might be several representations for the functor above. By Proposition 2.1.5 all those limits give representations where the objects are isomorphic. This allows us to talk about *"the limit"* of D meaning a special choice for a limiting cone whose vertex is written as $\varprojlim_{\mathbb{I}} D$.

2.2.1 Completeness and Parameters

Terminal objects are limits for the empty diagram and products are limits for diagrams whose domain is a discrete two-object category. More complicated universal constructions are equalisers and pullbacks. Not all categories have limits for every diagram. Categories with all limits, called *complete categories*, are of special interest.

The category **Set** is complete: given a diagram $D : \mathbb{I} \to \mathbf{Set}$ our canonical choice for the limit of D shall be

$$\varprojlim_{\mathbb{I}} D = \{\langle x_i \rangle_{i \in \mathbb{I}} \mid x_i \in D(i) \text{ and for all } f : i \to j \quad x_j = D(f) x_i\},$$

where $\langle x_i \rangle_{i \in \mathbb{I}}$ is a tuple indexed by the elements in $\mathbb{I}$. The limiting cone is given by the projections.

Proposition 2.2.2 (Limits with Parameters) Let $G : \mathcal{B} \times \mathbb{I} \to \mathcal{C}$ be a bifunctor such that for each $b \in \mathcal{B}$ the induced diagram $G^b : \mathbb{I} \to \mathcal{C}$ has a limit in $\mathcal{C}$. Then the mapping $b \mapsto \varprojlim_{\mathbb{I}} G^b$ uniquely extends to a functor

$$\lambda x. \varprojlim_{\mathbb{I}} G^x : \mathcal{B} \to \mathcal{C}.$$

Proof. For every $b \in \mathcal{B}$ the limit of G^b determines a representation

$$\mathcal{C}\big(-, \varprojlim_{\mathbb{I}} G^b\big) \overset{\theta^b}{\cong} [\mathbb{I}, \mathcal{C}]\big(\Delta -, G^b\big).$$

The result follows by applying Theorem 2.1.6 to the bifunctor

$$\lambda x, y. [\mathbb{I}, \mathcal{C}]\big(\Delta y, G^x\big) : \mathcal{B} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}.$$

□

From (2.3) we have a concrete understanding of the action of $\lambda x.\varprojlim_{\mathbb{I}} G^x : \mathcal{B} \to \mathcal{C}$ over arrows: given $f : b \to b'$ in $\mathcal{B}$

$$\varprojlim_{\mathbb{I}} G^f = \Big((\theta^{b'})^{-1} \circ [\mathbb{I}, \mathcal{C}]\big(\Delta \varprojlim_{\mathbb{I}} G^b, G^f\big)\Big)\,(\varepsilon) = (\theta^{b'})^{-1}\big(G^f \circ \varepsilon\big)$$

where ε is the limiting cone (or counit) associated to $\varprojlim_{\mathbb{I}} G^b$. Thus $\varprojlim_{\mathbb{I}} G^f$ is the unique arrow defined by the universal property of $\varprojlim_{\mathbb{I}} G^{b'}$ with respect to the cone $G^f \circ \varepsilon : \Delta \varprojlim_{\mathbb{I}} G^b \Rightarrow G^{b'}$. In other words $\varprojlim_{\mathbb{I}} G^f$ is the unique mediating arrow such that the diagram of natural transformations

$$\begin{array}{ccc}
\Delta \varprojlim_{\mathbb{I}} G^b & \xrightarrow{\Delta \varprojlim_{\mathbb{I}} G^f} & \Delta \varprojlim_{\mathbb{I}} G^{b'} \\
{\scriptstyle \varepsilon}\downarrow & & \\
G^b & & \downarrow{\scriptstyle \varepsilon'} \\
{\scriptstyle G^f}\downarrow & & \\
G^{b'} & \overset{\mathsf{id}}{\cong} & G^{b'}
\end{array}$$

commutes where $\varepsilon' : \Delta \varprojlim_{\mathbb{I}} G^{b'} \Rightarrow G^{b'}$ is limiting.

An important special case arises by taking $\mathcal{B} = [\mathbb{I}, \mathcal{C}]$ and G to be the evaluation functor $\mathit{eval} : [\mathbb{I}, \mathcal{C}] \times \mathbb{I} \to \mathcal{C}$ where $\mathcal{C}$ has all limits for $\mathbb{I}$-indexed diagrams. For every diagram $D \in [\mathbb{I}, \mathcal{C}]$ there is a representation

$$\mathcal{C}\big(-, \varprojlim_{\mathbb{I}} \mathit{eval}^D\big) \cong [\mathbb{I}, \mathcal{C}]\big(\Delta -, \mathit{eval}^D\big).$$

By the result above and since $\mathit{eval}^D = D$ there is a functor

$$\lambda x.\varprojlim_{\mathbb{I}} \mathit{eval}^x = \varprojlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C},$$

the *limit functor*, such that

$$\mathcal{C}(y, \varprojlim_{\mathbb{I}} D) \cong [\mathbb{I}, \mathcal{C}](\Delta y, D)$$

natural in y and $D \in [\mathbb{I}, \mathcal{C}]$. Given a natural transformation $\alpha : D \Rightarrow D'$ the arrow $\varprojlim_{\mathbb{I}} \alpha$ is just the mediating arrow associated with the cone resulting from the composition of the limiting cone $\varepsilon : \Delta \varprojlim_{\mathbb{I}} D \Rightarrow D$ with α. Thus $\varprojlim_{\mathbb{I}} \alpha$ is the unique arrow making the diagram of natural transformations

$$\begin{array}{ccc}
\Delta \varprojlim_{\mathbb{I}} D & \xrightarrow{\Delta \varprojlim_{\mathbb{I}} \alpha} & \Delta \varprojlim_{\mathbb{I}} D' \\
{\scriptstyle \varepsilon}\downarrow & & \\
D & & \downarrow{\scriptstyle \varepsilon'} \\
{\scriptstyle \alpha}\downarrow & & \\
D' & \overset{\mathsf{id}}{\cong} & D'
\end{array} \tag{2.5}$$

commute, where $\varepsilon' : \Delta \varprojlim_{\mathbb{I}} D' \Rightarrow D'$ is limiting.

As functors preserve isomorphisms a direct corollary of this result is that for diagrams $D, D' : \mathbb{I} \to \mathcal{C}$

$$D \cong D' \Rightarrow \varprojlim_{\mathbb{I}} D \cong \varprojlim_{\mathbb{I}} D'.$$

Remark 2.2.3 The definition of the limit functor gives in reality a family of functors. A specific limit functor depends on the choice for the limit of each $D \in [\mathbb{I}, \mathcal{C}]$. The limit functors, however, are isomorphic and we can talk about "the limit functor" with respect to some particular choice for the representations.

Sometimes it is of interest to consider only a subcategory $\mathcal{K} \subseteq [\mathbb{I}, \mathcal{C}]$ of diagrams. If all diagrams in $\mathcal{K}$ have limits in $\mathcal{C}$, *i.e.* $\mathcal{C}$ is $\mathcal{K}$-complete, we can define a limit functor

$$\varprojlim_{\mathbb{I}} - : \mathcal{K} \to \mathcal{C}.$$

In this case a limit for a diagram $D \in \mathcal{K}$ is a representation for

$$\mathcal{K}(\Delta\_\, , D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

and parametrised representability works as before. The full subcategory induced by all the diagrams in $[\mathbb{I}, \mathcal{C}]$ with limit in $\mathcal{C}$ is denoted by $\mathit{diag}[\mathbb{I}, \mathcal{C}]$.

2.2.2 Colimits

A cone from a diagram $D : \mathbb{I} \to \mathcal{C}$ to an object $c \in \mathcal{C}$, sometimes call *cocone*, is a natural transformation from D to Δc. A colimit for D is a universal cone $\eta : D \Rightarrow \Delta c$ in the sense that for any other cone $\gamma : D \Rightarrow \Delta a$ there exists a unique mediating arrow $m : c \to a$ such that the diagram of natural transformations

$$\begin{array}{ccc} D & \overset{\mathsf{id}}{\cong} & D \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \Delta c & \xrightarrow[\Delta m]{} & \Delta a \end{array} \tag{2.6}$$

commutes. This condition can be expressed as a representation for a covariant functor.

Definition 2.2.4 (Colimit) A *colimit* for a diagram $D : \mathbb{I} \to \mathcal{C}$ is a representation (c, θ) for the functor $[\mathbb{I}, \mathcal{D}](D, \Delta\_) : \mathcal{D} \to \mathbf{Set}$. The counit of this representation

$$\theta_c(\mathsf{id}_c) = \eta \in [\mathbb{I}, \mathcal{D}](D, \Delta)$$

is a colimiting cone.

By abuse of the language we call the object c the colimit of D and write it as $\varinjlim_{\mathbb{I}} D$ or $\mathrm{colim}_{\mathbb{I}} D$.

By duality we can define initial objects, coproducts and more generally colimits in a category $\mathcal{C}$ as simply being terminal objects, products and limits in $\mathcal{C}^{\mathsf{op}}$. In 2-categorical terms the property illustrated by (2.6) can be translated to the universal property of limits by applying the functor $(-)^{\mathsf{op}}$:

$$\begin{array}{ccc} D^{\mathsf{op}} & \overset{\mathsf{id}}{\cong} & D^{\mathsf{op}} \\ \uparrow{\scriptstyle\eta^{\mathsf{op}}} & & \uparrow{\scriptstyle\gamma^{\mathsf{op}}} \\ (\Delta c)^{\mathsf{op}} & \xleftarrow[(\Delta m)^{\mathsf{op}}]{} & (\Delta a)^{\mathsf{op}} \end{array}$$

Remember that $(-)^{\mathsf{op}}$ reverses the direction of natural transformations only! This makes η^{op} a universal cone, *i.e.* a limit for $D^{\mathsf{op}} : \mathbb{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$.

Categories with colimits for all diagrams are said to be *cocomplete*. The category **Set** is cocomplete: given a diagram $D : \mathbb{I} \to \mathbf{Set}$ our choice for the colimit of D is

$$\mathrm{colim}_{\mathbb{I}} D = \biguplus_{i \in \mathbb{I}} D(i) / \sim$$

where $\sim$ is the least equivalence relation on $\biguplus_{i\in\mathbb{I}} D(i) \times \biguplus_{i\in\mathbb{I}} D(i)$ such that

$$(i,x) \sim (j,y) \Leftrightarrow \exists\, f\colon i \to j \text{ in } \mathbb{I} \quad (D(f))x = y\ .$$

For $(i,x) \in \biguplus_{i\in\mathbb{I}} D(i)$ we let $[(i,x)]_\sim$ denote the equivalence class of $\sim$ containing (i,x). The colimiting cone is given by the collection of injection functions

$$x \mapsto [(i,x)]_\sim.$$

By the dual version of Theorem 2.1.6 for a $\mathcal{K}$-cocomplete category $\mathcal{C}$ there is a colimit functor

$$\mathrm{colim}_{\mathbb{I}}\textbf{--}\colon \mathcal{K} \to \mathcal{C}$$

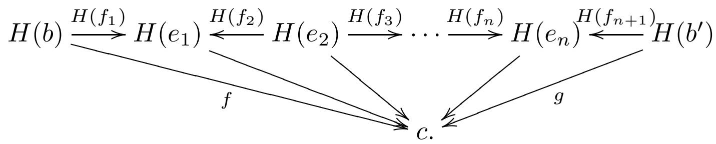
for a subcategory $\mathcal{K} \subseteq [\mathbb{I}, \mathcal{C}]$.

2.2.3 Initial Functors

A functor $H : \mathcal{B} \to \mathcal{C}$ is called initial if for each $c \in \mathcal{C}$ the corresponding comma category $H\!\downarrow\! c$ is non-empty and connected. In other words for each c there exists at least one object $b \in \mathcal{B}$ and an arrow $H(b) \to c$; and for any two such arrows $f\colon H(b) \to c$, $g\colon H(b') \to c$ there are arrows

$$b \xrightarrow{f_1} e_1 \xleftarrow{f_2} e_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} e_n \xleftarrow{f_{n+1}} b'$$

in $\mathcal{B}$ and arrows in $\mathcal{C}$ to form a commutative diagram



Proposition 2.2.5 Let $D\colon \mathbb{I} \to \mathbb{J}$ and $F\colon \mathbb{J} \to \mathcal{C}$ be diagrams. If D is initial then we have

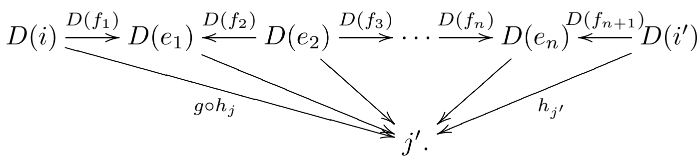
$$[\mathbb{I}, \mathcal{C}](\Delta c, F \circ D) \cong [\mathbb{J}, \mathcal{C}](\Delta c, F)$$

natural in c.

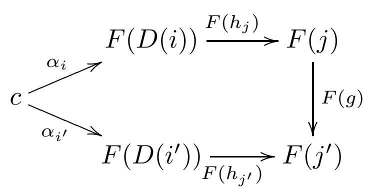
Proof. For every $j \in \mathbb{J}$ pick one arrow $h_j\colon D(i) \to j$; this is possible by initiality of D. For a cone $\alpha : \Delta c \Rightarrow F \circ D$ and for every $j \in \mathbb{J}$ there is an arrow

$$c \xrightarrow{\alpha_i} F(D(i)) \xrightarrow{F(h_j)} F(j).$$

Now, for an arrow $g : j \to j'$ by initiality of D there is a sequence of arrows in $\mathbb{J}$ connecting $h_{j'}\colon D(i') \to j'$ with $g \circ h_j$ such that they form a commutative diagram



By applying F to this diagram and since α is a cone we have that the diagram



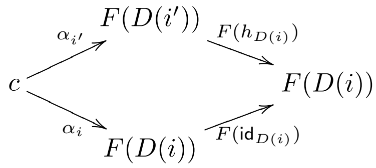
commutes, and then the family $\widehat{\alpha} = \langle F(h_j) \circ \alpha_i : c \to F(j) \rangle_{j \in \mathbb{J}}$ is a cone.

Conversely, for a cone $\beta : \Delta c \Rightarrow F$ we just pre-compose with D to obtain a cone $\beta D : \Delta c \Rightarrow F \circ D$. We now check this is an isomorphism of sets:

- For $\alpha : \Delta c \Rightarrow F \circ D$ we have

$$(\widehat{\alpha} D)_i = \widehat{\alpha}_{D(i)} = F(h_{D(i)}) \circ \alpha_{i'}.$$

Notice that i' might be different to i, and then $F(h_{D(i)})$ is not necessarily the identity but by initiality of D and since α is a cone the diagram



commutes, and then $F(h_{D(i)}) \circ \alpha_{i'} = \alpha_i$.

- For $\beta : \Delta c \Rightarrow F$ we have

$$\begin{aligned} (\widehat{\beta D})_j &= F(h_j) \circ \beta_{D(i)} && \text{by definition of } (\widehat{-}), \\ &= \beta_j && \text{since } \beta \text{ is a cone.} \end{aligned}$$

Now we can conclude $\widehat{\beta D} = \beta$.

Naturality in c follows by simply verifying that for an arrow $f : c \to c'$ we have that

$$(\widehat{\alpha \circ \Delta f})_j = F(h_j) \circ \alpha_i \circ f = (\widehat{\alpha} \circ \Delta f)_j$$

for any $j \in \mathbb{J}$. □

Thus we can conclude the important result of this section:

Proposition 2.2.6 Let $D : \mathbb{I} \to \mathbb{J}$ and $F : \mathbb{J} \to \mathcal{C}$ be diagrams. If D is initial and the limit for $F \circ D$ exists then the limit of F exists and there is an isomorphism

$$\varprojlim_{\mathbb{J}} F \cong \varprojlim_{\mathbb{I}} (F \circ D).$$

Proof. By the definition of limit and the proposition above

$$\mathcal{C}(c, \varprojlim_{\mathbb{I}} F \circ D) \cong [\mathbb{I}, \mathcal{C}](\Delta c, F \circ D) \cong [\mathbb{J}, \mathcal{C}](\Delta c, F)$$

natural in c. Then by uniqueness of universal properties

$$\varprojlim_{\mathbb{J}} F \cong \varprojlim_{\mathbb{I}} (F \circ D).$$

□

We shall use this result when showing the relation between the different flavours of universality in the next chapter. There we see an example of initial functors given by the forgetful functor defined by a special category of elements. In general we say that a subcategory $\mathcal{C}$ of $\mathcal{D}$ is initial when the inclusion functor is initial. For a simple non-initial category consider a discrete subcategory of a non-discrete category, clearly in this case the inclusion functor is not initial since the corresponding comma category is empty.

2.2.4 Preservation of Limits

Informally, a functor $F:\mathcal{C} \to \mathcal{D}$ is said to preserve products if for $c, d \in \mathcal{C}$ we have that

$$F(c \times d) \cong F(c) \times F(d).$$

Despite this isomorphism ensures the object $F(c \times d)$ indeed gives rise to a product it says nothing about the projections. If F does not map the projections of $c \times d$ into the projections for $F(c \times d)$ the isomorphism above would not be natural. Preservation of product in the categorical sense requires that the projections are preserved as well.

Definition 2.2.7 (Preservation of Limits) Let $D : \mathbb{I} \to \mathcal{C}$ be a diagram. A functor $G:\mathcal{C} \to \mathcal{D}$ preserves a limiting cone $\varepsilon : \Delta c \Rightarrow D$ if the composition

$$G\varepsilon : \Delta G(c) \Rightarrow G \circ D$$

is a limiting cone.

A functor $G:\mathcal{C} \to \mathcal{D}$ preserves any cone in $\mathcal{C}$ but the definition above requires more: limiting cones should be mapped into limiting cones. We say that a functor is *continuous* when it preserves all limiting cones.

Proposition 2.2.8 For a category $\mathcal{C}$ the hom-functor $\mathcal{C}(a, -):\mathcal{C} \to \mathbf{Set}$ is continuous.

Proof. Let $D:\mathbb{I} \to \mathcal{C}$ be a diagram and $\varepsilon : \Delta c \Rightarrow D$ a limiting cone. By applying $\mathcal{C}(a, -)$ we obtain the cone

$$\langle \varepsilon_i \circ - \rangle_{i \in \mathbb{I}} : \Delta\mathcal{C}(a, c) \Rightarrow \mathcal{C}(a, D(-)).$$

It is routine to verify that since ε is limiting and from our choice of limits in **Set** this is indeed a limiting cone. $\square$

This proposition allows us to express a limit in an arbitrary category $\mathcal{C}$ as a limit in **Set** since for a diagram $D:\mathbb{I} \to C$ we have

$$\mathcal{C}(c, \varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} \mathcal{C}(c, D(-)). \tag{2.7}$$

A colimit in $\mathcal{C}$, however, corresponds to a limit in **Set**:

$$\begin{aligned}
\mathcal{C}(\mathrm{colim}_{\mathbb{I}} F, c) &\cong \mathcal{C}^{\mathsf{op}}(c, \varprojlim_{\mathbb{I}^{\mathsf{op}}} F^{\mathsf{op}}) && \text{by duality,}\\
&\cong \varprojlim_{\mathbb{I}^{\mathsf{op}}} \mathcal{C}^{\mathsf{op}}(c, F^{\mathsf{op}}(-)) && \text{since hom-functors preserve limits,}\\
&\cong \varprojlim_{\mathbb{I}^{\mathsf{op}}} \mathcal{C}(F(-), c) && \text{by duality (1.1).}
\end{aligned}$$

Proving preservation of limits by closely following the definition above requires a concrete reasoning in terms of diagrams and limiting cones. This can be rather tedious and involve a fair amount of bookkeeping, something that in category theory is traditionally left implicit. In chapter 4 we study in which situations a more equational approach can be taken when checking whether a functor preserves universal properties.

2.3 Adjunctions

The symmetry between universal and couniversal properties is captured by the notion of adjunction.

Definition 2.3.1 (Adjunction) Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be functors. An *adjunction* in which F is the *left adjoint* and G is the *right adjoint* consists of an isomorphism

$$\mathcal{D}(F(c), d) \overset{\varphi_{c,d}}{\cong} \mathcal{C}(c, G(d)) \tag{2.8}$$

natural in c, d. When such a natural isomorphism exists we write $\varphi : F \dashv G$.

The bijection of arrows defined by an adjunction is sometimes represented as

$$\frac{F(c) \xrightarrow{f} d}{c \xrightarrow{\overline{f}} G(d)} \quad \text{and} \quad \frac{c \xrightarrow{g} G(d)}{F(c) \xrightarrow{\overline{g}} d}$$

and then $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. Notice that by naturality of the isomorphism (2.8) for $h : c' \to c$ in $\mathcal{C}$ and $k : d \to d'$ in $\mathcal{D}$ we have the following identities

$$\frac{F(c') \xrightarrow{F(h)} F(c) \xrightarrow{f} d \xrightarrow{k} d'}{c' \xrightarrow{h} c \xrightarrow{\overline{f}} G(d) \xrightarrow{G(k)} G(d')} \quad \text{and} \quad \frac{c' \xrightarrow{h} c \xrightarrow{g} G(d) \xrightarrow{G(k)} G(d')}{F(c') \xrightarrow{F(h)} F(c) \xrightarrow{\overline{g}} d \xrightarrow{k} d'.}$$

The isomorphism (2.8) admits two readings: as a covariant representation by fixing $c \in \mathcal{C}^{\mathsf{op}}$ or as a contravariant representation by fixing $d \in \mathcal{D}$.

From Theorem 2.1.6, to give a right adjoint to $F : \mathcal{C} \to \mathcal{D}$ is equivalent to giving for each $d \in \mathcal{D}$ a representation

$$\mathcal{C}(\_, G[d]) \overset{\theta^d}{\cong} \mathcal{D}(F(-), d). \tag{2.9}$$

The mapping $d \mapsto G[d]$ induces a unique (up to isomorphism) functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$\mathcal{C}(c, G(d)) \overset{\theta^d_c}{\cong} \mathcal{D}(F(c), d)$$

natural in c, d.

For every $d \in \mathcal{D}$ the counit $\varepsilon_{G[d]}$ for the representation (2.9) is an arrow from $F(G(d))$ to d – as G is understood from the context we henceforth write ε_d instead of $\varepsilon_{G[d]}$ when referring to the counit of an adjunction. From the definition of counit we have

$$\frac{F(G(d)) \xrightarrow{\varepsilon_d} d}{G(d) \xrightarrow{\mathsf{id}_{G(d)}} G(d).}$$

By (2.3) from the proof of Theorem 2.1.6 we have that the action of the induced right adjoint G on $g : d \to d'$ is defined to be

$$G(g) = \big((\theta^{d'})^{-1} \circ \mathcal{D}(F(G[d]), g)\big)(\varepsilon_d) = (\theta^{d'})^{-1}(g \circ \varepsilon_d)$$

that can be expressed as

$$\frac{G(d) \xrightarrow{G(g)} G(d')}{F(G(d)) \xrightarrow{\varepsilon_d} d \xrightarrow{g} d'.}$$

By universality of $\varepsilon_{d'}$ we have

$$g \circ \varepsilon_d = \mathcal{D}\big(F(G(g)), d'\big)(\varepsilon_{d'}) = \varepsilon_{d'} \circ F(G(g)).$$

Then the diagram

$$\begin{array}{ccc} F(G(d)) & \xrightarrow{\varepsilon_d} & d \\ {\scriptstyle F(G(g))}\downarrow & & \downarrow{\scriptstyle g} \\ F(G(d')) & \xrightarrow[\varepsilon_{d'}]{} & d' \end{array}$$

commutes. Therefore the family of counits $\langle \varepsilon_d : F(G(d)) \to d \rangle_{d \in \mathcal{D}}$ is a natural transformation from $F \circ G$ to the identity functor.

Counits are universal elements, but from the discussion at the beginning of section 2.1 in this special case they can be regarded as universal arrows as well. Thus $G(g)$ is the unique arrow from $G(d)$ to $G(d')$ which corresponds to

$$g \circ \varepsilon_d : F(G(d)) \to d'$$

under the isomorphism defined by the adjunction.

Dually, to give a left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ is equivalent to giving for each $c \in \mathcal{C}$ a representation

$$\mathcal{D}(F[c], -) \overset{\rho^c}{\cong} \mathcal{C}(c, G(-)).$$

In this case the unit $\eta_{F[c]}$ of the covariant representation is an arrow from c to $G(F(c))$ – we drop F and write η_c instead of $\eta_{F[c]}$. By definition of unit we have

$$\frac{c \xrightarrow{\eta_c} G(F(c))}{F(c) \xrightarrow{\mathsf{id}_{F(c)}} F(c)}$$

and for an arrow $f : c \to c'$

$$\frac{F(c) \xrightarrow{F(f)} F(c')}{c \xrightarrow{f} c' \xrightarrow{\eta_{c'}} G(F(c')).}$$

The family $\langle \eta_c : c \to G(F(c)) \rangle_{c \in \mathcal{C}}$ is a natural transformation from the identity functor to $G \circ F$. By Proposition 2.1.5 we can conclude that left and right adjoints are unique up to natural isomorphism.

Left and right adjoint can carry extra parameters. It follows below the result for the left adjoint; there is of course a dual result for right adjoints.

Proposition 2.3.2 (Adjunction with Parameters) Let $F : \mathcal{C} \times \mathcal{A} \to \mathcal{D}$ be a bifunctor such that for each $a \in \mathcal{A}$ the induced partial functor $F_a : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G^a : \mathcal{D} \to \mathcal{C}$ via the isomorphism

$$\mathcal{D}(F_a(c), d) \overset{\varphi^a_{c,d}}{\cong} \mathcal{C}(c, G^a(d))$$

natural in c and d. Then there is a unique extension of the mapping $(a, d) \mapsto G_a(d)$ to a bifunctor $G : \mathcal{A}^{\mathsf{op}} \times \mathcal{D} \to \mathcal{C}$ such that

$$\mathcal{D}(F(c, a), d) \overset{\varphi^a_{c,d}}{\cong} \mathcal{C}(c, G(a, d)) \tag{2.10}$$

natural in c, d and a.

Proof. For fixed $d \in \mathcal{D}$ and $a \in \mathcal{A}$ there is a representation

$$\mathcal{C}(-, G^a(d)) \overset{(\varphi^a)^{-1}}{\cong} \mathcal{D}(F_a(-), d).$$

Then from Theorem 2.1.6 applied to the **Set**-valued functor

$$\lambda x, y, z.\mathcal{D}\big(F_x(z), y\big) : \mathcal{A}^{\mathsf{op}} \times \mathcal{D} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

the mapping $(a, d) \mapsto G^a(d)$ uniquely extends to a bifunctor $G : \mathcal{A}^{\mathsf{op}} \times \mathcal{D} \to \mathcal{C}$ where $G(a, y) = G^a(y)$. □

The action of the functor $G : \mathcal{A}^{\mathsf{op}} \times \mathcal{D} \to \mathcal{C}$ over arrows $f : a' \to a$ in $\mathcal{A}$ and $g : d \to d'$ in $\mathcal{D}$ is concretely given by

$$G(f, g) = \big(\varphi^{a'} \circ \mathcal{D}\big(F(G(a, d), f), g\big)\big)(\varepsilon^a_d) = \varphi^{a'}\big(g \circ \varepsilon^a_d \circ F(G(a, d), f)\big)$$

where ε^a and $\varepsilon^{a'}$ are the counits defined by $F_a \dashv G^a$ and $F_{a'} \dashv G^{a'}$ respectively. This can be expressed as

$$\frac{G(a, d) \xrightarrow{G(f,g)} G(a', d')}{F(G(a, d), a') \xrightarrow{F(G(a,d),f)} F(G(a, d), a) \xrightarrow{\varepsilon^a_d} d \xrightarrow{g} d'.} \tag{2.11}$$

2.3.1 Examples of Adjunctions

Quoting Mac Lane: "Adjunctions arise everywhere". It is not our intention here to enumerate examples of adjunctions. Instead we review some of the notions seen already and introduce exponentials as special case of adjunctions.

Let $\mathcal{C}$ be a complete category. The limit functor $\varprojlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$ for a small category $\mathbb{I}$ is such that

$$\mathcal{C}(c, \varprojlim_{\mathbb{I}} D) \cong [\mathbb{I}, \mathcal{C}](\Delta c, D)$$

natural in c and $D \in [\mathbb{I}, \mathcal{C}]$. Then the limit functor is a right adjoint to the diagonal functor where the counit is given by the collection of limiting cones and the unit at component c is the unique arrow from c to $\varprojlim_{\mathbb{I}} \Delta c$. Similarly in a cocomplete category a colimit functor is a left adjoint to the diagonal functor.

As a special case, a category $\mathcal{A}$ has all binary products if the diagonal functor $\Delta - : \mathbf{2} \to \mathcal{A}$ has right adjoint (where $\mathbf{2}$ is the discrete category with two objects), *i.e.*

$$\mathcal{A}(a, \varprojlim_{\mathbb{I}} D) \cong [\mathbf{2}, \mathcal{C}](\Delta a, D)$$

natural in a and D. The diagram D is a tuple selecting two objects in $\mathcal{A}$. The limit functor is in this case the product functor $- \times - : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. In a similar way a terminal object can be given as a right adjoint of the diagonal constant in $\mathbf{1}$. A category with all finite products, *i.e.* binary products and terminal object, is called a *cartesian category*. Some authors add to this definition the existence of equalisers, *i.e.* all finite limits [FS90, Joh02].

Given a cartesian category $\mathcal{C}$ and an object $b \in \mathcal{C}$, we say b is *exponentiable* if there is a right adjoint to the partial functor $- \times b : \mathcal{C} \to \mathcal{C}$, *i.e.* an isomorphism

$$\mathcal{C}(a \times b, c) \overset{\lambda_{a,b}}{\cong} \mathcal{C}(a, [b, c]) \tag{2.12}$$

natural in a and c where $[b, -] : \mathcal{C} \to \mathcal{C}$ is a functor. The object $[b, c]$ is called the exponential. The counit of this adjunction is the evaluation morphism

$$eval^b_c = \varepsilon_c : [b, c] \times b \to c$$

and the unit is given by

$$\eta_a : a \to [b, (a \times b)].$$

By Proposition 2.3.2, if for every $b \in \mathcal{C}$ we have the adjunction above, then the mapping $b \mapsto [b, -]$ uniquely extends to a bifunctor

$$[=, -] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}.$$

The action of $[-, c]$ over $g : b' \to b$ is defined as (see (2.11))

$$\frac{[b,c] \xrightarrow{[g,c]} [b',c]}{[b,c] \times b' \xrightarrow{[b,c]\times g} [b,c] \times b \xrightarrow{eval_c^b} c.} \tag{2.13}$$

In this case the category $\mathcal{C}$ is said to be *cartesian closed*.

Notation 2.3.3 In the special case of **Set** we have

$$\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, \mathbf{Set}(Y, Z))$$

natural in X, Y, Z. We henceforth write $[Y, Z]$ for $\mathbf{Set}(Y, Z)$.

The category of locally small categories **CAT** is a cartesian category since products are defined for every pair of categories where the empty product (or terminal object) is given by the singleton category. **CAT** is not cartesian closed, though, as functor categories are not always defined for arbitrary locally small categories. Exponentials exists, however, if the domain is a small category.

Let $\mathbb{I}$, $\mathbb{J}$ be a small categories. A bifunctor $F : \mathbb{I} \times \mathbb{J} \to \mathcal{C}$ can be *"curried"* into a functor $\boldsymbol{\lambda}(F) : \mathbb{I} \to [\mathbb{J}, \mathcal{C}]$. This functor maps $i \in \mathbb{I}$ into the partial functor $F^i : \mathbb{J} \to \mathcal{C}$, and an arrow $f : i \to i'$ in $\mathbb{I}$ into the family $F(f, -) = F^f$ which from the discussion on partial functors in § 1.3 is a natural transformation.

This construction suggest an adjunction, one which falls outside the universe of locally small categories. The evaluation functor $eval : [\mathbb{I}, \mathcal{C}] \times \mathbb{I} \to \mathcal{C}$ is the counit of the "adjunction"

$$\mathbf{CAT}(\mathbb{I} \times \mathbb{J}, \mathcal{C}) \overset{\boldsymbol{\lambda}}{\cong} \mathbf{CAT}(\mathbb{I}, [\mathbb{J}, \mathcal{C}]). \tag{2.14}$$

From the result on counits and adjunctions above we have, as expected, that for $F : \mathbb{I} \times \mathbb{J} \to \mathcal{C}$

$$F = eval \circ (\boldsymbol{\lambda}(F) \times \mathbb{J}).$$

The exponentials which exists in **CAT** are important to our development. Exponentials in a cartesian category essentially correspond to the arrow type in the lambda calculus [LS86, Tay99, Joh02]. Although in this case as **CAT** is not cartesian closed we shall need to restrict the introduction of the arrow type in the logic to the cases where the domain is small.

Chapter 3

Ends and Coends

As pointed by Street [Str02] category theory officially began with the definition of naturality. There are situations, however, where a more general notion of "naturality" is required as for example in the evaluation morphism for cartesian closed categories. Kelly [Kel65] first identified the so-called extraordinary natural transformations in the development of enriched category theory. This notion was later generalised by Dubuc and Street [DS70] to define the dinatural transformations. The notion of end arises as universal dinatural transformations of a special form. In the setting of locally small categories ends and limits coincide. Ends, however, support an algebraic manipulation of universal properties which lays the foundations for a formal language and logic for categories. In the first part of this chapter we introduce ends and its properties. In the second part we concentrate on presheaf categories and weighted limits giving a different approach to universality where the relevance of representable functors is made explicit through the notion of weight.

3.1 Dinatural Transformations

A family of evaluation morphisms in a cartesian closed category $\mathcal{C}$

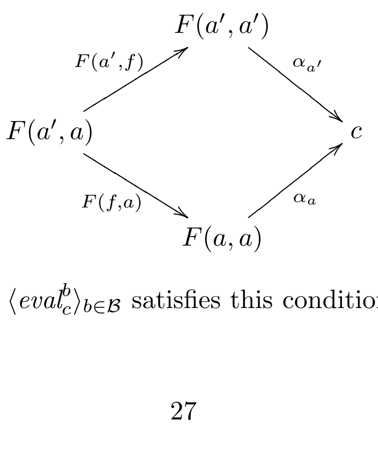
$$eval^b_c : [b, c] \times b \to c \tag{3.1}$$

is natural in c since it is the counit of $\_ \times b \dashv [b, \_]$. The expression $[b, c] \times b$ is functorial in b as well. In fact we have a multivariance bifunctor:

$$[=, c] \times - : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}.$$

Kelly [Kel65] identified the sense in which the family (3.1) can be considered natural in b calling the new notion *"extraordinary naturality"*.

For a multivariance functor $F : \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{C}$ a family $\langle \alpha_a : F(a, a) \to c \rangle_{a \in \mathcal{A}}$ for some $c \in \mathcal{C}$ is natural in this sense if for every arrow $f : a \to a'$ the *"wedge"*



commutes. We can check $\langle eval^b_c \rangle_{b \in \mathcal{B}}$ satisfies this condition: for an arrow $g : b \to b'$ in $\mathcal{C}$

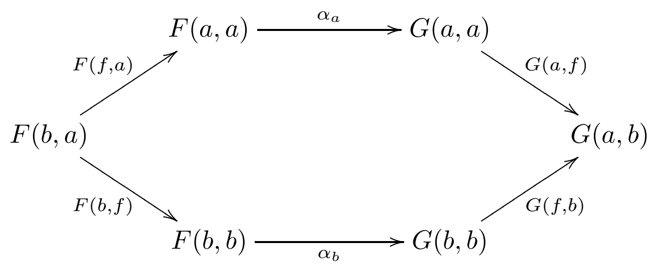
$$\dfrac{\dfrac{[b',c]\times b \xrightarrow{[g,c]\times b} [b,c]\times b \xrightarrow{eval_c^b} c}{[b',c] \xrightarrow{[g,c]} [b,c]}}{[b',c]\times b \xrightarrow{[b',c]\times g} [b',c]\times b' \xrightarrow{eval_c^{b'}} c,}$$

where the first step follows from universality of $eval_c^b$ and the second by definition of $[g,c]$. Hence

$$eval_c^b \circ ([g,c]\times b) = eval_c^{b'} \circ ([b',c]\times g).$$

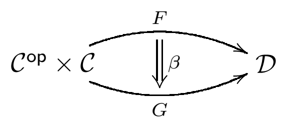
Dubuc and Street [DS70] generalised the extraordinary naturality to define a special notion of transformation between functors with multivariance:

Definition 3.1.1 Let $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A *dinatural transformation* $\alpha : F \ddot{\to} G$ consists of a family $\langle \alpha_a : F(a,a) \to G(a,a)\rangle_{a\in\mathcal{C}}$ of arrows in $\mathcal{D}$ such that for every arrow $f : a \to b$ in $\mathcal{C}$ the diagram

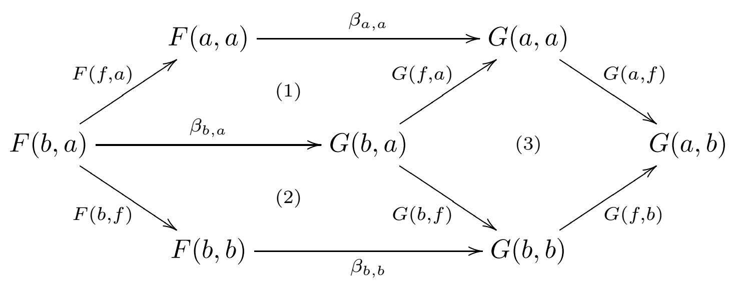


commutes.

Every ordinary natural transformation gives rise to a dinatural transformation. Given the natural transformation



then the family of arrows $\langle \beta_{a,a} : F(a,a) \to G(a,a)\rangle_{a\in\mathcal{C}}$ is a dinatural transformation since for an arrow $f : a \to b$ there is a diagram



where

(1) commutes since it is the naturality square associated with (f, id_a),

(2) commutes since it is the naturality square associated with (id_b, f), and

(3) commutes since G is a bifunctor,

and then the outermost hexagon commutes as well.

As with natural transformations, dinaturality can be verified at each component independently. Let $H, K : \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \times \mathcal{B}^{\mathsf{op}} \times \mathcal{B} \to \mathcal{C}$ be functors. A family

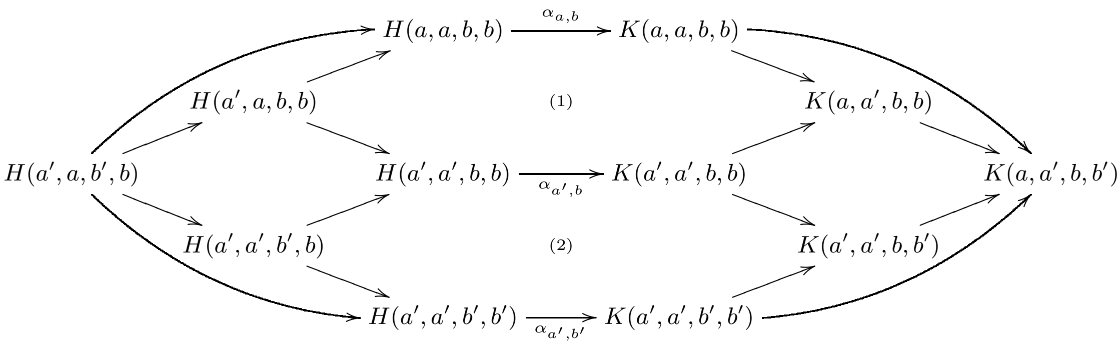
$$\langle \alpha_{a,b} : H(a,a,b,b) \to K(a,a,b,b) \rangle_{a \in \mathcal{A}, b \in \mathcal{B}}$$

is dinatural if and only if the induced families

$\alpha_{a,-} = \langle \alpha_{a,b} \rangle_{b \in \mathcal{B}}$ from $H^{a,a}$ to $K^{a,a}$ for each $a \in \mathcal{A}$, and

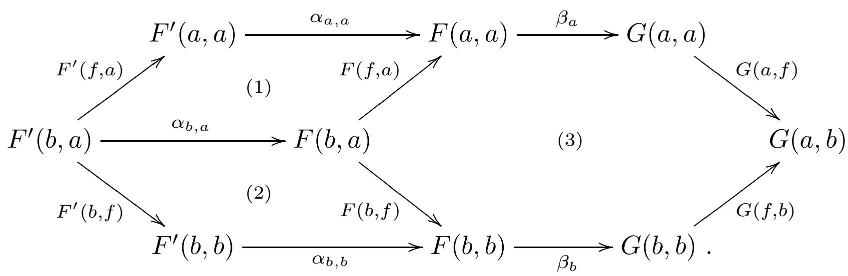
$\alpha_{-,b} = \langle \alpha_{a,b} \rangle_{a \in \mathcal{A}}$ from $H_{b,b}$ to $K_{b,b}$ for each $b \in \mathcal{B}$

are dinatural. This follows by considering the diagram



for arrows $a \longrightarrow a'$ in $\mathcal{A}$ and $b \longrightarrow b'$ in $\mathcal{B}$. To simplify the diagram we just write $H^{a,a}(b,b)$ as $H(a,a,b,b)$ and so on. The hexagon (1) commutes since $\langle \alpha_{a,b} \rangle_a$ is a dinatural family, similarly the hexagon (2) commutes since $\langle \alpha_{a',b} \rangle_b$ is dinatural. The rest of the internal diagrams commute since H, K are functors of multiple variables.

The most remarkable aspect of dinatural transformations is the failure of composition. In general the attempt of merging two hexagons componentwise is fruitless (see § 3.1.3). Therefore, we cannot regard dinatural transformation as arrows in a category. The situation, however, is not so negative since dinatural transformations do compose with natural transformations. Given $\alpha : F' \Rightarrow F$ and $\beta : F \ddot{\to} G$ (assume functors of the right type), the outermost hexagon in



commutes for $f : a \to b$ since

(1) and (2) commute from naturality of α, and

(3) commutes from dinaturality of β.

Similarly we can compose a natural transformation on the right. This composition gives rise to the bifunctor:

$$\mathbf{Dinat}(=,-) : [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}] \to \mathbf{Set}$$

acting on functors and natural transformations as

$$\begin{array}{ccc} F & G & \mathbf{Dinat}\big(F,G\big) \\ \eta \Big\Uparrow & \Big\Downarrow \varphi & \Big\downarrow \mathbf{Dinat}\big(\eta,\varphi\big)=\varphi\circ-\circ\eta \\ H & K & \mathbf{Dinat}\big(H,K\big) \end{array}.$$

3.1.1 Dinaturality generalises naturality

A functor $F:\mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{D}$ is *dummy* in its first argument if there exists a functor $F_0:\mathcal{C}\to\mathcal{D}$ such that F is the composition

$$\mathcal{C}^{\mathsf{op}}\times\mathcal{C}\xrightarrow{\pi_2}\mathcal{C}\xrightarrow{F_0}\mathcal{D}$$

where $\pi_2:\mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{C}$ is the projection to the second argument. Analogously, we can define "dummy" in the second argument.

Let $F,G:\mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{D}$ be functors dummy in their first argument and $\alpha:F\ddot{\to}G$ a dinatural transformation. Then for any $f:a\to b$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccccc} & F(b,b)=F_0(b) & \xrightarrow{\alpha_b} & G(b,b)=G_0(\mathsf{id}_b) & \\ {\scriptstyle F(b,f)=F_0(f)}\nearrow & & & & \searrow{\scriptstyle G(f,b)=G_0(b)} \\ F(b,a)=F_0(a) & & & & G(a,b)=G_0(b) \\ {\scriptstyle F(f,a)=F_0(a)}\searrow & & & & \nearrow{\scriptstyle G(a,f)=G_0(f)} \\ & F(a,a)=F_0(\mathsf{id}_a) & \xrightarrow[\alpha_a]{} & G(a,a)=G_0(a) & \end{array}$$

commutes. Hence the diagram

$$\begin{array}{ccc} F_0(b) & \xrightarrow{\alpha_b} & G_0(b) \\ {\scriptstyle F_0(f)}\Big\uparrow & & \Big\uparrow{\scriptstyle G_0(f)} \\ F_0(a) & \xrightarrow{\alpha_a} & G_0(a) \end{array}$$

commutes and α is a natural transformation from F_0 to G_0.

3.1.2 Wedges

A special case of a dinatural transformation arises when one of the functors is constant. For instance a dinatural transformation $\theta:\Delta c\ddot{\to}G$ is given by a family $\langle\theta_a:c\to G(a,a)\rangle_{a\in\mathcal{A}}$ such that given a morphism $f:a\to b$ the diagram

$$\begin{array}{ccccc} & & G(a,a) & & \\ & {\scriptstyle\theta_a}\nearrow & & \searrow{\scriptstyle G(a,f)} & \\ c & & & & G(a,b) \\ & {\scriptstyle\theta_b}\searrow & & \nearrow{\scriptstyle G(b,f)} & \\ & & G(b,b) & & \end{array}$$

commutes. This corresponds to the definition of extraordinary naturality above.

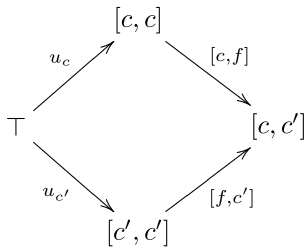
3.1.3 Examples

Identities

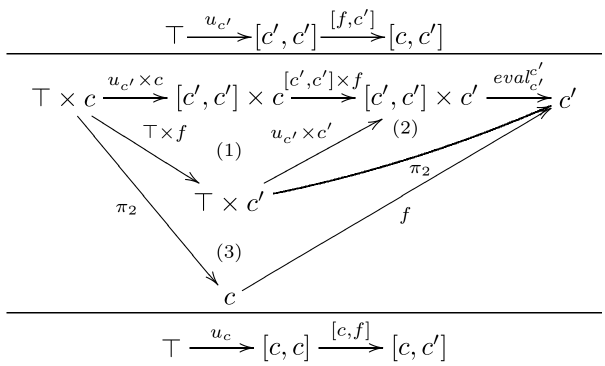
Cartesian close categories give another example of dinatural family. Let $\mathcal{C}$ be a cartesian closed category. Take the constant functor $\Delta\top$ where $\top$ is the terminal object of $\mathcal{C}$. The family $\langle u_c : \top \to [c, c]\rangle_c$ where u_c is defined under the adjunction (2.12) as

$$\frac{\top \xrightarrow{u_c} [c, c]}{\top \times c \xrightarrow{\pi_2} c}$$

is dinatural in c. For $f : c \to c'$ we need to verify the "wedge" condition, *i.e.* that the diagram



commutes. This follows from



where first bar corresponds to (2.13) plus naturality, and

- (1) commutes since the product gives a bifunctor,
- (2) by definition of $u_{c'}$ and universality of *eval*, and
- (3) by the universal property of the product.

The last bar follows from the definition of u_c and naturality. Hence $[f, c'] \circ u_{c'} = [c, f] \circ u_c$. The arrow u_c is sometimes referred as "the name" of id_c, in the special case of **Set** the arrow u_X selects the identity from $[X, X] = \mathbf{Set}(X, X)$.

Dinaturals for Fixed-Point Operators [BFSS90, Mul90, Sim93]

Consider a cartesian closed category $\mathcal{C}$ where every set $\mathcal{C}(x, y)$ comes equipped with a partial-order relation and for every $c, d, e \in \mathcal{C}$ the composition

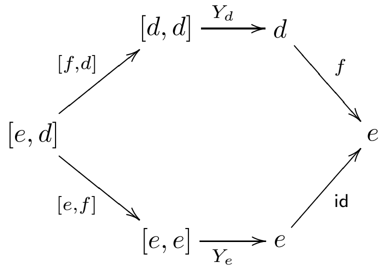
$$\mathcal{C}(d, e) \times \mathcal{C}(c, d) \xrightarrow{- \circ -} \mathcal{C}(c, e)$$

is monotonic. The elements $\mathcal{C}(\top, c)$ are called the "global elements" of c and are interpreted as being the element of a partially-ordered set. We also require the functor $\mathcal{C}(\top, \_)$ to be faithful. In this way the category $\mathcal{C}$ can be concretely regarded as a full cartesian closed subcategory of the category **Poset** of partial-order sets and monotone functions.

A *fix-dinatural* in $\mathcal{C}$ is a family of arrows

$$Y_d : [d, d] \to d$$

dinatural in d. The right-hand side functor is dummy on the contravariant argument. Thus for an arrow $f : d \to e$ the hexagon



(3.2)

commutes. For a global element

$$\frac{\top \xrightarrow{g} [e, d]}{\top \times e \cong e \xrightarrow{g} d}$$

the commutativity of (3.2) can be rephrased by using a set-theoretic notation as

$$f\,(Y_d(f \circ g)) = Y_e(g \circ f).$$

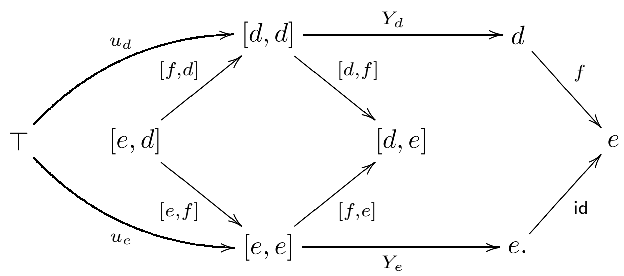
By taking $e = d$ and $g = \mathsf{id}_d$ we have

$$f(Y_d(f)) = Y_d(f)$$

and thus $Y_d(f)$ (or more abstractly the global element $Y_d \circ f$) is a fixed point for f.

An Example of Non-composition of Dinaturals

Identities and fix-dinaturals give an example of dinatural transformations whose composition does not result in a dinatural transformation. Indeed, if the family $\langle Y_d \circ u_d \rangle_d$ in a category $\mathcal{C}$ is dinatural then $\mathcal{C}$ is trivial in the sense the set $\mathcal{C}(c, c')$ is a singleton for any $c, c' \in \mathcal{C}$. To explain this consider the diagram defined by the arrow $f : d \to e$ in $\mathcal{C}$



The commutativity of the outermost hexagon can be expressed as

$$f\,(Y_d(\mathsf{id}_d)) = Y_e(\mathsf{id}_e).$$

If we take $d = \top$ we have that any $f : \top \to e$ is equal to $Y_e(\mathsf{id}_e)$, hence there is a unique global element for every $e \in \mathcal{C}$. In particular for any pair c, c' there is a unique global element for $[c, c']$ which in a cartesian closed category is equivalent to having a unique mapping from c to c'.

Some Applications

There are many examples in the literature where multivariant functorial expressions give rise to dinatural families. One example of dinatural family is given by *traced monoidal categories* introduced by Joyal, Street and Verity [JSV96]. In this setting a *trace* is a family of functions

$$\mathcal{C}(x \otimes u, y \otimes u) \xrightarrow{t^u_{x,y}} \mathcal{C}(x, y)$$

natural in x, y and dinatural in u plus some axioms for a suitable tensor bifunctor $\_ \otimes =$. From a computer science viewpoint the arrow $t^u_{x,y}(f)$ should be thought as a feedback along u giving a categorical interpretation to cyclic data structures. Indeed, Hyland and Hasegawa [Has97, Has02] independently observed a bijective correspondence between fixpoint operators and traces with finite products.

An interesting example of the application of dinatural transformations in computer science is the work by Bainbridge *et al.* on models for parametric polymorphism [BFSS90]. The key observation is that naturality fails to capture polymorphisms in the presence of function spaces due to the occurrence of arguments in contravariant position. Then, dinaturality comes to the rescue. Blute [Blu91] in his PhD dissertation refines this work establishing connections with linear logic and deductive systems in general.

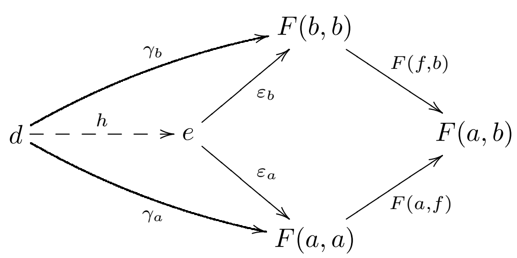
Mulry [Mul90] first showed that the least-fixed-point operator is a dinatural transformation. Then Simpson [Sim93] established the necessary conditions on a subcategory of **CPO**, complete partial orders and continuous functions, for which the property of dinaturality completely characterises the least-fixed-point operator. This is of interest since it equationally determines the fixed-point operator in a purely categorical setting. For more examples of dinatural transformations arising in computer science refer to [Sco00].

3.2 Ends

An end for a multivariant diagram $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is given by a universal wedge for F, dinatural transformations allow us to present this notions as a representation.

Definition 3.2.1 (End) An *end* for $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation (e, θ) for the functor $\mathbf{Dinat}(\Delta\_, F) : \mathcal{D}^{\mathsf{op}} \to \mathbf{Set}$. The counit of this representation is a universal wedge $\varepsilon : \Delta e \ddot{\to} F$.

More concretely for any wedge $\gamma : \Delta d \ddot{\to} F$ there exists a unique morphism $h : d \to e$ such that for $f : a \to b$ in $\mathbb{I}$ the triangles in



commute. In other words, h is the unique arrow such that

$$\gamma = \varepsilon \circ \Delta h$$

where the right-hand side refers to the composition of the natural transformation Δh with the dinatural transformation ε.[1] As usual by abuse of the language we call the object e the "end of F" and is written as $\int_x F(x,x)$. The reasons for this notation are made explicit by exploring the properties of ends like point-wise computation and Fubini theorem.

As limits or any other universal constructions there are special cases of the parametricity results for ends. Thus given a category $\mathcal{C}$ with enough ends we can define the *"end functor"* mapping a diagram $F \in [\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathcal{D}]$ into an end object $\int_x F(x,x)$. For a natural transformation $\alpha : F \Rightarrow G$ the expression $\int_x \alpha_{x,x}$ denotes the unique arrow defined by the wedge $\alpha \circ \varepsilon$ where ε is the universal wedge associated to $\int_x F(x,x)$. By the definition of end as a representation Remark 2.2.3 applies to end functors as well.

3.2.1 Limits are Ends

A diagram $D_0 : \mathbb{I} \to \mathcal{C}$ gives rise to a multivariant diagram $D : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{C}$ where the first argument is dummy. For $c \in \mathcal{C}$ a natural transformations from Δx to D_0 uniquely corresponds to a dinatural transformation from Δx to D and then

$$[\mathbb{I}, \mathcal{D}](\Delta x, D_0) = \mathbf{Dinat}\big(\Delta x, D\big).$$

Hence,

$$\mathcal{D}(\Delta x, \varprojlim_{\mathbb{I}} D_0) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{D}](\Delta x, D_0) = \mathbf{Dinat}\big(\Delta x, D\big)$$

natural in x, *i.e.* $(\varprojlim_{\mathbb{I}} F_0, \theta)$ is a representation for $\mathbf{Dinat}\big(\Delta -, F\big)$ and consequently $\varprojlim_{\mathbb{I}} F_0 \cong \int_x F(x,x)$. This justifies writing $\int_x D_0(x)$ for $\varprojlim_{\mathbb{I}} D_0$, the notation we shall use later in the formal language.

3.2.2 Ends are Limits

Perhaps less intuitive at a first glance is that ends are limits. Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ be a diagram whose end is given by the representation

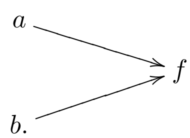
$$\mathcal{D}(-, \textstyle\int_x F(x,x)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big).$$

We construct a category $\mathbb{I}^{\S}$ and a functor $d^{\S} : \mathbb{I}^{\S} \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ such that

$$\textstyle\int_x F(x,x) \cong \varprojlim_{\mathbb{I}^{\S}} (F \circ d^{\S}).$$

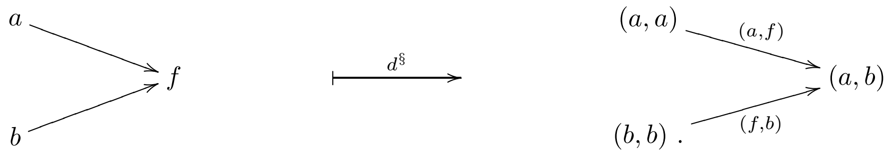
The category $\mathbb{I}^{\S}$ is built from $\mathbb{I}$ as follows:

- the objects of $\mathbb{I}^{\S}$ are the objects and arrows of $\mathbb{I}$, and
- the arrows of $\mathbb{I}^{\S}$ are the identities and for every $f : a \to b$ in $\mathbb{I}$ the arrows



[1] Notice that this equation cannot be expressed as a commutative diagram of functors and natural transformations as in 2.4 for limits.

Observe that the only possible composition in $\mathbb{I}^\S$ is with identities. Now, we define the functor $d^\S : \mathbb{I}^\S \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ acting on objects and arrows as



The cones in $[\mathbb{I}^\S, \mathcal{D}](\Delta x, F \circ d^\S)$ are exactly the wedges in $\mathbf{Dinat}\big(\Delta x, F\big)$ and hence

$$\mathcal{D}(-, \textstyle\int_x F(x,x)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big) = [\mathbb{I}^\S, \mathcal{D}](\Delta -, F \circ d^\S)\ .$$

Therefore $\int_x F(x,x)$ is a limit for $F \circ d^\S$. As a consequence of these two results we can deduce that if $\mathcal{D}$ has all limits then $\mathcal{D}$ has all ends and vice versa. We can also write the equation (2.7) for ends as

$$\mathcal{D}(d, \textstyle\int_x F(x,x)) \cong \int_x \mathcal{D}(d, F(x,x)). \tag{3.3}$$

3.2.3 Ends in Set

As **Set** is complete all multivariant **Set**-valued diagrams have ends. From the choice for limits in **Set** and the result linking the two notions we can deduce our choice for ends in **Set**.

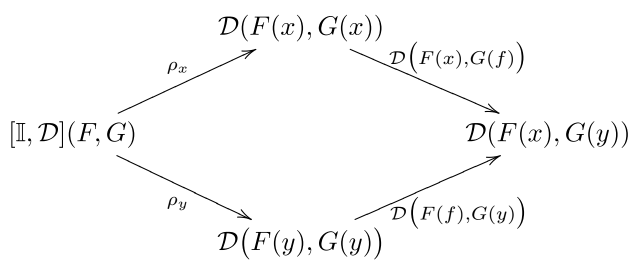
For a multivariant functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$ there is a functor $F \circ d^\S : \mathbb{I}^\S \to \mathbf{Set}$. The limit of $F \circ d^\S$ is given by the set

$$\begin{aligned} \varprojlim\nolimits_{\mathbb{I}}(F \circ d^\S) &= \{\langle x_a\rangle_{a\in\mathbb{I}^\S} \mid x_a \in (F\circ d^\S)(a) \text{ and for all } u : a \to b \text{ in } \mathbb{I}^\S\ \ x_b = (F \circ d^\S)(u)(x_a)\} \\ &\cong \{\langle x_i\rangle_{i\in\mathbb{I}} \mid x_i \in F(i,i) \text{ and for all } f : i \to j \text{ in } \mathbb{I}\ \ F(i,f)(x_i) = F(f,j)(x_j)\} \end{aligned}$$

where the projections are the universal cone.

3.2.4 (Di)Naturality Formula

Let $\mathbb{I}$ be a small category and $\mathcal{D}$ a locally small category. For diagrams $F, G : \mathbb{I} \to \mathcal{D}$ consider the family $\langle \rho_x : [\mathbb{I}, \mathcal{D}](F,G) \to \mathcal{D}(F(x), G(x))\rangle_{x\in\mathcal{D}}$ where ρ_x maps a natural transformation $\alpha : F \Rightarrow G$ into the component $\alpha_x : F(x) \to G(x)$. For an arrow $f : x \to y$ the diagram



commutes since for $\alpha : F \Rightarrow G$

$$G(f) \circ \alpha_x = \alpha_y \circ F(f).$$

Thus the family $\langle \rho_x \rangle_x$ defines a wedge for

$$\mathcal{D}\big(F(=), G(-)\big) : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}.$$

Moreover from our choice of ends in **Set** this is an end for this functor(where every natural transformations is interpreted as a tuple of components). Then there is an isomorphism

$$[\mathbb{I},\mathcal{D}](F,G) \cong \int_x \mathcal{D}\big(F(x),G(x)\big). \tag{3.4}$$

Furthermore as a consequence of parametrised representability if $\mathcal{D}$ is complete enough the expression above gives rise to a functor

$$[\mathbb{I},\mathcal{D}](=,-) : [\mathbb{I},\mathcal{D}]^{\mathsf{op}} \times [\mathbb{I},\mathcal{D}] \to \mathbf{Set}$$

and the isomorphism above is natural in F, G. Henceforth the isomorphism (3.4) is referred as the *"naturality formula"*.

In a similar way dinaturalities can be regarded as ends. For multivariant functors $H, K : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ the set $\mathbf{Dinat}\big(H,K\big)$ is an end for

$$\mathcal{D}\big(H(-,=),K(=,-)\big) : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$$

and thus

$$\mathbf{Dinat}\big(H,K\big) \cong \int_x \mathcal{D}\big(H(x,x),K(x,x)\big) \tag{3.5}$$

natural in H, K. This isomorphism is called the *"dinaturality formula"*.

3.2.5 Iterated Ends: Fubini

Dinaturality over functors with multiple variables can be verified componentwise. This property allows us to deduce a general result for the manipulation of "integrals" in expressions: the so-called Fubini theorem for ends.

Proposition 3.2.2 Let $G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \times \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathcal{B}$ be a functor. If for every pair $d_1, d_2 \in \mathbb{D}$ the end $\int_x G_{(d_1,d_2)}(x,x)$ exists in $\mathcal{B}$, where $G_{(d_1,d_2)} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathcal{B}$ is induced by G, then there is an isomorphism

$$\int_{(x,y)} G(x,x,y,y) \cong \int_y \int_x G_{(y,y)}(x,x).$$

Here we also mean that if one side of the isomorphism exists then so does the other.

Proof. Given a wedge $\alpha : \Delta b \ddot{\to} \lambda d_1, d_2. \int_x G_{(d_1,d_2)}(x,x)$ we can define for a pair $c, d \in \mathbb{C} \times \mathbb{D}$ an arrow $\widetilde{\alpha}_{c,d}$ as the composition

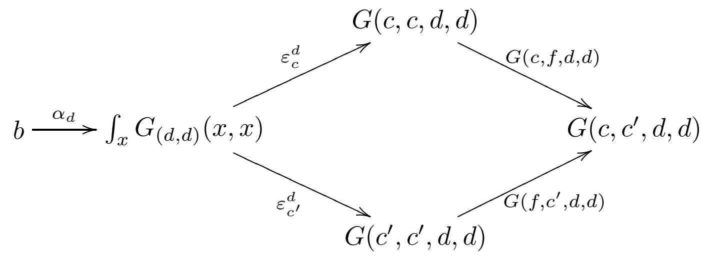
$$b \xrightarrow{\alpha_d} \int_x G_{(d,d)}(x,x) \xrightarrow{\varepsilon^d_c} G(c,c,d,d)$$

where ε^d is the universal wedge defined by $\int_x G_{(d,d)}(x,x)$. The family $\langle \widetilde{\alpha}_{c,d} \rangle_{c,d}$ is a wedge from Δb into G, this can be verified at each variable apart:

- By fixing $d \in \mathbb{D}$ it is clear that the family

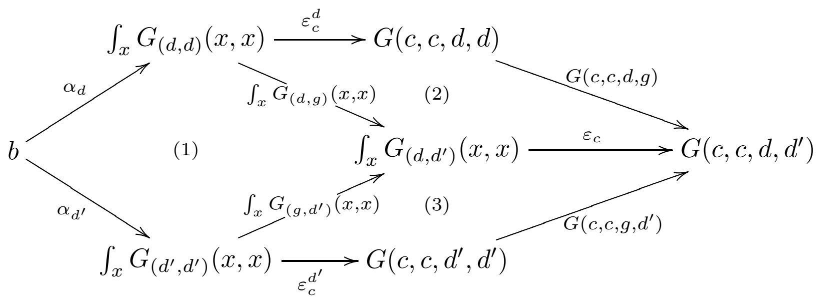
  $$\langle \varepsilon^d_c \circ \alpha_d \rangle_{c \in \mathbb{C}}$$

  is a wedge since ε^d is itself a wedge and then for $f : c \to c'$ the corresponding diagram

  commutes.

- By fixing $c \in \mathbb{C}$ and for $g : d \to d'$ in $\mathbb{D}$ the outer hexagon



commutes since

(1) commutes because α is a wedge;

(2) commutes since by definition $\int_x G_{(d,g)}(x,x)$ is the mediating arrow defined by the wedge $G_{d,g} \circ \varepsilon^d$ (consequence of parametrised representability (2.3)); and

(3) commutes for the same reason as (2) but applied to the arrow $\int_x G_{(g,d)}(x,x)$.

Conversely, given a wedge $\gamma : \Delta b \ddot{\to} G$ the family $\langle \gamma_{c,d} \rangle_c$ defines for a fixed d a wedge from b to $G_{d,d}$, which in turn by universality of ε^d gives an arrow γ_d from b to $\int_x G_{d,d}(x,x)$. By using the universality property of the ends it is routine to verify that the collection $\langle \gamma_d \rangle_d$ forms a wedge from b to $\lambda d_1, d_2. \int_x G_{d_1,d_2}(x,x)$.

Thus the mapping $\alpha \mapsto \widetilde{\alpha}$ is a bijection and then there is an isomorphism

$$\mathbf{Dinat}\big(\Delta b, \lambda d_1, d_2. \textstyle\int_x G_{(d_1,d_2)}(x,x)\big) \cong \mathbf{Dinat}\big(\Delta b, G\big)$$

which is easily seen to be natural in $b \in \mathcal{B}$. Hence

$$\begin{aligned} \mathcal{B}(b, \textstyle\int_y \int_x G_{(y,y)}(x,x)) &\cong \mathbf{Dinat}\big(\Delta b, \lambda d_1, d_2. \textstyle\int_x G_{(d_1,d_2)}(x,x)\big) \\ &\cong \mathbf{Dinat}\big(\Delta b, G\big) \\ &\cong \mathcal{B}(b, \textstyle\int_{(x,y)} G(x,x,y,y)) \end{aligned}$$

all natural in b. □

Analogously, if for every pair $c_1, c_2 \in \mathbb{C}$ the end $\int_y G^{(c_1,c_2)}(y,y)$ is defined in $\mathcal{B}$, then there is a canonical isomorphism

$$\textstyle\int_{(x,y)} G(x,x,y,y) \cong \int_x \int_y G^{(x,x)}(y,y).$$

Corollary 3.2.3 (Fubini) Let $G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \times \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathcal{B}$ be a functor such that the ends $\int_x G_{(d_1,d_2)}(x,x)$ and $\int_y G^{(c_1,c_2)}(y,y)$ exist for every pair $d_1, d_2 \in \mathbb{D}$ and $c_1, c_2 \in \mathbb{C}$ then there is a canonical isomorphism

$$\textstyle\int_x \int_y G^{(x,x)}(y,y) \cong \int_y \int_x G_{(y,y)}(x,x)$$

meaning that one side of the isomorphism exists if and only if so does the other. By parametrised representability the isomorphism above is natural in G.

As limits are ends over functors with "dummy" extra arguments Fubini works for limits as well. Thus, for a diagram $D : \mathbb{I} \times \mathbb{J} \to \mathcal{C}$ we have

$$\varprojlim_{\mathbb{J}} (\lambda y. \varprojlim_{\mathbb{I}} D_y) \cong \varprojlim_{\mathbb{I} \times \mathbb{J}} D.$$

3.2.6 Ends in Functor Categories

Ends in functor categories are computed pointwise. A functor $F:\mathbb{I}^{\mathsf{op}}\times\mathbb{I}\to[\mathbb{J},\mathcal{C}]$ corresponds under (2.14) to a functor

$$G=\boldsymbol{\lambda}^{-1}(F):\mathbb{I}^{\mathsf{op}}\times\mathbb{I}\times\mathbb{J}\to\mathcal{C}.$$

For every $j\in\mathbb{J}$ the functor G induces a partial functor $G_j:\mathbb{I}^{\mathsf{op}}\times\mathbb{I}\to\mathcal{C}$. In fact by definition

$$G_j(i_1,i_2)=F(i_1,i_2)(j).$$

Proposition 3.2.4 Let $F:\mathbb{I}^{\mathsf{op}}\times\mathbb{I}\to[\mathbb{J},\mathcal{C}]$ be a functor and let $G=\boldsymbol{\lambda}^{-1}(F)$. If for each $j\in\mathbb{J}$ the end $\int_x G_j(x,x)$ exists then the end $\int_x F(x,x)$ exists in $[\mathbb{J},\mathcal{C}]$ and there is an isomorphism

$$\textstyle\int_x F(x,x)\cong\lambda y.\int_x G_y(x,x).$$

Proof.

$$\begin{aligned}
[\mathbb{J},\mathcal{C}]\big(H,\lambda y.\textstyle\int_x G_y(x,x)\big) &\cong \textstyle\int_y \mathcal{C}(H(y),\int_x G_y(x,x)) && \text{by naturality formula,}\\
&\cong \textstyle\int_y \mathbf{Dinat}\big(\Delta H(y),G_y\big) && \text{by definition of end,}\\
&\cong \textstyle\int_y\int_x \mathcal{C}(H(y),G_y(x,x)) && \text{by dinaturality formula,}\\
&\cong \textstyle\int_x\int_y \mathcal{C}(H(y),G_y(x,x)) && \text{by Fubini for ends,}\\
&\cong \textstyle\int_x [\mathbb{J},\mathcal{C}](H,\lambda y.G_y(x,x)) && \text{by naturality formula,}\\
&\cong \mathbf{Dinat}\big(\Delta H,\lambda x_1,x_2.\lambda y.G_y(x_1,x_2)\big) && \text{by dinaturality formula,}\\
&\cong \mathbf{Dinat}\big(\Delta H,F\big).
\end{aligned}$$

□

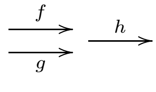
Again as limits are ends the same result is valid for limits. For a diagram $D:\mathbb{I}\to[\mathbb{J},\mathcal{C}]$ there is a canonical isomorphism

$$\varprojlim_{\mathbb{I}} D\cong\lambda x.\varprojlim_{\mathbb{I}} K_x$$

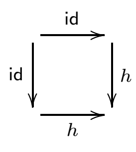
where $K:\mathbb{I}\times\mathbb{J}\to\mathcal{C}$ is the bifunctor defined by D under (2.14).

A consequence of pointwise computation of limits is that for a complete category $\mathcal{C}$ the functor category $[\mathbb{I},\mathcal{C}]$ is complete as well. For instance, the category **Set** is complete and so is a functor category $[\mathbb{I},\mathbf{Set}]$.

If $\mathcal{C}$ is not complete, then non-pointwise limits may exist. The following example is extracted from [Kel82]. The category of arrows of $\mathcal{C}$ is the functor category $[\{\to\},\mathcal{C}]$ where $\{\to\}$ represents the category with two objects and one non-identity arrow. An object in this category is just an arrow in $\mathcal{C}$, and a morphisms corresponds to a commuting square in $\mathcal{C}$. Now consider the category of arrows of the category induced by the graph



where $h\circ f=h\circ g$. The square



is not a pullback since $f \neq g$. On the other hand in the category of arrows the diagram

$$\begin{array}{ccc} g & \xrightarrow{\mathsf{id}} & g \\ {\scriptstyle \mathsf{id}}\downarrow & & \downarrow{\scriptstyle (f,h)} \\ g & \xrightarrow[(f,h)]{} & h \end{array}$$

is trivially a pullback since there is no other arrow but the identity with g as codomain.

3.3 Coends

A coend for a diagram $G\colon \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is given by a universal wedge $\eta : G \stackrel{..}{\to} \Delta c$. This condition can be expressed as a representation for a covariant **Set**-valued functor.

Definition 3.3.1 (Coend) A *coend* for a functor $G\colon \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation (c, θ) for the functor $\mathbf{Dinat}\big(G, \Delta_{-}\big)\colon \mathcal{D} \to \mathbf{Set}$. The unit of this representation is a universal wedge $\eta : G \stackrel{..}{\to} \Delta c$.

As usual we call the object c the "coend of G" and write it as $\int^x G(x,x)$. By duality we can translate the results for ends into the dual ones for coends. Thus colimits are instances of coends and coends can be expressed as colimits.

This gives us a choice for coends in **Set**, for a functor $G : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$ the coend $\int^x G(x,x)$ is the set

$$\mathrm{colim}_{\mathbb{I}^{\S}}(G \circ d^{\S}) = \biguplus_{a \in \mathbb{I}^{\S}} (G \circ d^{\S})(a) / \sim$$

where $\sim$ is the least equivalence relation where

$$(a,x) \sim (b,y) \Leftrightarrow \exists u\colon a \to b \text{ in } \mathbb{I}^{\S} \text{ such that } (G \circ d^{\S})(u)(x) = y.$$

By definition of $\mathbb{I}^{\S}$ and $d^{\S}$ this can be rewritten as

$$\mathrm{colim}_{\mathbb{I}^{\S}}(G \circ d^{\S}) = \biguplus_{i \in \mathbb{I}} G(i,i) / \stackrel{\diamond}{\sim}$$

where $\stackrel{\diamond}{\sim}$ is given as the least equivalence relation where

$$(i,x) \stackrel{\diamond}{\sim} (i,y) \Leftrightarrow \exists f\colon i \to j \text{ in } \mathbb{I} \text{ such that } G(i,f)(x) = G(f,j)(y).$$

As with colimits, coends in a category $\mathcal{D}$ are translated into ends in **Set**. Thus, for a functor $G\colon \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ we have an isomorphism

$$\mathcal{D}(\textstyle\int^x G(x,x), d) \cong \int_x \mathcal{D}(G(x,x), d). \tag{3.6}$$

3.4 Powers and Copowers

When the factors of a product are all equal, say $\Pi_{x \in X} b$ where $b \in \mathcal{C}$ and $X \in \mathbf{Set}$ the product is called a *power* and is written $[X, b]$. So there is an isomorphism

$$\mathcal{C}(a, [X,b]) = \mathcal{C}(a, \prod_{x \in X} b) \cong \prod_{x \in X} \mathcal{C}(a,b) = [X, \mathcal{C}(a,b)]$$

natural in $a \in \mathcal{C}$. Thus a power $[X, b]$ just gives a representation for the functor

$$[X, \mathcal{C}(-, b)] : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}.$$

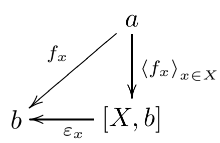
The counit of this representation is a function

$$\varepsilon : X \to \mathcal{C}([X, b], b)$$

equivalent to a X-indexed family of projections. For an X-indexed family

$$\langle f_x : a \to b \rangle_{x \in X}$$

there is a unique mediating morphism $\langle f_x \rangle_{x \in X} : a \to [X, b]$ such that the triangle



commutes for each $x \in X$.

Powers are fully alive in the case $\mathcal{C} = \mathbf{Set}$ where

$$\prod_{x \in X} B = \mathbf{Set}(X, B) = [X, B]$$

and $[X, B]$ is just the exponential justifying the notation.

If a category $\mathcal{C}$ has all powers the mapping $X \mapsto [X, b]$ extends to a functor

$$[-, b] : \mathbf{Set}^{\mathsf{op}} \to \mathcal{C}.$$

This follows by applying parametrised representability (Theorem 2.1.6) to the bifunctor

$$[=, \mathcal{C}(-, b)] : \mathbf{Set}^{\mathsf{op}} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}.$$

Copowers give the dual notion. Given a category $\mathcal{C}$, a set X and an object $b \in \mathcal{C}$ a coproduct $\sum_{x \in X} b$ called a copower and is written $X \otimes b$. More formally a copower is a representation for the functor

$$[X, \mathcal{C}(b, -)] : \mathcal{C} \to \mathbf{Set}.$$

The unit of this representation is a function

$$\eta : X \to \mathcal{C}(b, X \otimes b),$$

i.e. a X-indexed family of injections. In the special case of $\mathcal{C} = \mathbf{Set}$ a copower is just a cartesian product

$$\sum_{x \in X} B = X \times B.$$

If a category $\mathcal{C}$ has all copowers for every $b \in \mathcal{C}$ then the mapping $X \mapsto X \otimes b$ uniquely extends to a functor

$$- \otimes b : \mathbf{Set} \to \mathcal{C}.$$

This follows by applying the dual of Theorem 2.1.6 to the bifunctor

$$[=, \mathcal{C}(b, -)] : \mathbf{Set}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}.$$

In general as powers and copowers are special kind of limits they are computed pointwise in functor categories. Thus if $\mathcal{D}$ is a complete category and $F : \mathbb{C} \to \mathcal{D}$ is a functor then

$$[A, F](x) \cong \lambda x.[A, F(x)].$$

The dual result holds for copowers.

3.5 Presheaf Categories

For a small category $\mathbb{C}$, the category $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ is called the category of *presheaves* over $\mathbb{C}$. The dual construction $[\mathbb{C}, \mathbf{Set}]$ is the category of covariant presheaves. Representables are objects of presheaf categories, therefore the properties of those categories are of the special interest. Presheaf categories are complete; this follows from the fact that **Set** is complete and limits are computed pointwise.

In the case of presheaf categories the Yoneda lemma expresses as an end for a **Set**-valued functor. For a presheaf $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ we have

$$\begin{aligned} F(c) &\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\mathcal{C}(-, c), F\big) && \text{by the Yoneda lemma,} \\ &\cong \textstyle\int_x [\mathcal{C}(x, c), F(x)] && \text{by naturality formula.} \end{aligned}$$

Indeed the family

$$\langle \lambda x. F(-)(x) : F(c) \to [\mathbb{C}(c', c), F(c')] \rangle_{c' \in \mathbb{C}}$$

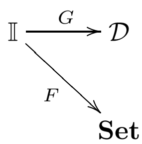
defines a wedge from $F(c)$ to

$$[\mathbb{C}(-, c), F(=)] : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}.$$

3.5.1 Weighted Limits

From the theory of representables and presheaf categories arises another approach to universality given by the notion of weighted limit. In the case of locally small categories weighted limits are showed to coincide with ends (and then with limits). Weighted limits, however, are particularly relevant to the manipulation of presheaves.

Definition 3.5.1 (Weighted Limit) Given the diagrams



the limit of G weighted by F is a representation for the functor

$$[\mathbb{I}, \mathbf{Set}]\big(F, \lambda x. \mathcal{D}(-, G(x))\big) : \mathcal{D}^{\mathsf{op}} \to \mathbf{Set},$$

i.e. an isomorphism

$$\mathcal{D}(y, d) \overset{\theta_y}{\cong} [\mathbb{I}, \mathbf{Set}]\big(F, \lambda x. \mathcal{D}(y, G(x))\big) \tag{3.7}$$

natural in y for some object $d \in \mathcal{D}$.

As usual by abuse of the language we called the object d the weighted limit and write it as $\varprojlim_F G$ or $\{F, G\}$, in this dissertation we prefer the second. This concept was first introduced by Borceux and Kelly [BK75] as a proposal for a notion of "limit" in the most general setting of enriched categories. Then Kelly in [Kel82] developed the theory in detail, we reproduce some of the results here. There the concept is referred as indexed limits, with the time the term "weighted limit" prevailed.

As for limits and ends there is a more concrete understanding of the notion of weighted limits in terms of universal *weighted cones*. Given a natural transformation

$$\alpha \in [\mathbb{I}, \mathcal{D}]\big(F, \mathcal{D}(y, G(-))\big),$$

a component α_i is a function mapping elements in $F(i)$ to elements in $\mathcal{D}(y, G(i))$ such that for an arrow $f : i \to j$ in $\mathbb{I}$ the square

$$\begin{array}{ccc} F(i) & \xrightarrow{\alpha_i} & \mathcal{D}(y, G(i)) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)\circ -} \\ F(j) & \xrightarrow{\alpha_j} & \mathcal{D}(y, G(j)) \end{array}$$

commutes. Thus for every element $x \in F(i)$ the triangle

$$\begin{array}{ccc} & {\scriptstyle \alpha_i(x)}\nearrow & G(i) \\ y & & \big\downarrow{\scriptstyle G(f)} \\ & {\scriptstyle \alpha_j(F(f)(x))}\searrow & G(j) \end{array} \tag{3.8}$$

commutes. Then we say that α is a cone from y to G weighted by F. In the terminology used by Kelly [Kel82] this is a (F, y)-cylinder over G.

The counit ε of the representation (3.7) is a universal weighted cone in the sense that for another cone τ from b to G weighted by F there is a unique mediating arrow $h : b \to \{F, G\}$ such that for every $i \in \mathbb{I}$ and $x \in F(i)$ the triangle

$$\begin{array}{ccc} b & & \\ {\scriptstyle h}\,\big\downarrow & \searrow{\scriptstyle \tau_i(x)} & \\ \{F, G\} & \xrightarrow[\varepsilon_i(x)]{} & G(i) \end{array}$$

commutes. In other words, h is the unique morphism making the diagram of natural transformations

$$\begin{array}{ccc} F & \overset{\mathsf{id}}{\cong} & F \\ {\scriptstyle \varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \mathcal{D}(\{F, G\}, G(-)) & \xrightarrow[\langle - \circ h\rangle]{} & \mathcal{D}(b, G(-)) \end{array}$$

commute.

"Conical" Limits and Weighted Limits

Let $G : \mathbb{I} \to \mathcal{D}$ be a functor and $\Delta 1 : \mathbb{I} \to \mathbf{Set}$ be the constant functor over a singleton set. There is a bijection

$$[\mathbb{I}, \mathcal{D}]\big(\Delta y, G\big) \cong [\mathbb{I}, \mathbf{Set}]\big(\Delta 1, \mathcal{D}(y, G(-))\big) \tag{3.9}$$

natural in y. Given a cone $\beta \in [\mathbb{I}, \mathcal{D}]\big(\Delta y, G\big)$ and a morphism $f : i \to j$ the diagram

$$\begin{array}{ccc} & {\scriptstyle \beta_i}\nearrow & G(i) \\ y & & \big\downarrow{\scriptstyle G(f)} \\ & {\scriptstyle \beta_j}\searrow & G(j) \end{array}$$

commutes. Take the family $\langle \alpha_i : 1 \to \mathcal{D}(y, G(i))\rangle_i$ defined by $\alpha_i(*) = \beta_i$ where $*$ is the unique element in 1. Then from the triangle above we have that the diagram

$$\begin{array}{ccc} & {\scriptstyle \alpha_i}\nearrow & \mathcal{D}(y, G(i)) \\ 1 & & \big\downarrow{\scriptstyle G(f)\circ -} \\ & {\scriptstyle \alpha_j}\searrow & \mathcal{D}(y, G(j)) \end{array}$$

commutes and then α is a weighted cone. Conversely given a weighted cone

$$\gamma \in [\mathbb{I}, \mathbf{Set}]\big(\Delta 1, \lambda x.\mathcal{D}(y, G(x))\big)$$

the family $\langle \gamma_i(*) \rangle_i$ is a cone from y to G. This is clearly a bijection. As the construction is preserved through pre-composition in the first argument it follows the naturality in y.

Hence usual limits, called in this context *conical* limits, can be expressed as weighted limits:

$$\mathcal{D}(-, \varprojlim_{\mathbb{I}} G) \cong [\mathbb{I}, \mathcal{D}](\Delta -, G) \cong [\mathbb{I}, \mathcal{D}]\big(\Delta 1, \lambda x.\mathcal{D}(-, G(x))\big).$$

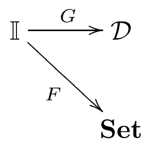
To show that weighted limits can be expressed as conical limits we need to define the category of elements associated to a presheaf or weight F.

Definition 3.5.2 Given a functor $F : \mathbb{I} \to \mathbf{Set}$, the *category of elements* of F, written as $\mathsf{els}(F)$, has:

- the pairs (i, x) where $x \in F(i)$ as objects , and
- $f : (i, x) \to (j, y)$ for every $f : i \to j$ in $\mathbb{I}$ where $y = F(f)(x)$ as arrows.

It can be easily checked that this definition indeed gives a category where composition and identities are inherited from $\mathbb{I}$. There is a forgetful functor $U_F : \mathsf{els}(F) \to \mathbb{I}$ projecting on the first component of an object and acting on arrows accordingly.

Given functors



there is a bijection

$$[\mathbb{I}, \mathbf{Set}]\big(F, \mathcal{D}(y, G(-))\big) \cong [\mathsf{els}(F), \mathcal{D}]\big(\Delta y, G \circ U_F\big)$$

natural in y. For a weighted cone $\alpha \in [\mathbb{I}, \mathbf{Set}]\big(F, \mathcal{D}(y, G(-))\big)$ by definition of the category $\mathsf{els}(F)$ the family $\langle \alpha_i(x) \rangle_{(i,x) \in \mathsf{els}(F)}$ is a cone from y to $G \circ U_F$ (see the diagram (3.8)). For the inverse, given a cone $\beta : \Delta y \Rightarrow G \circ U_F$ the family $\langle \lambda x.\beta_{(i,x)} \rangle_{i \in \mathbb{I}}$ is a natural transformation from F to $\mathcal{D}(y, G(-))$. Again, as the construction is preserved through pre-composition the isomorphism is natural in y.

Hence

$$\mathcal{D}(y, \{F, G\}) \cong [\mathbb{I}, \mathbf{Set}]\big(F, \mathcal{D}(y, G(-))\big) \cong [\mathsf{els}(F), \mathcal{D}](\Delta y, G \circ U_F)$$

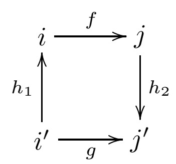
natural in y and $\{F, G\} \cong \varprojlim_{\mathsf{els}(F)} G \circ U_F$.

Ends and Weighted Limits

A special case of categories of elements arises by considering the hom-functor

$$\mathbb{I}(=, -) : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}.$$

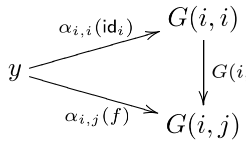
The objects of $\mathsf{els}(\mathbb{I}(=, -))$ can be regarded as just the arrows of $\mathbb{I}$. An arrow from $f : i \to j$ to $g : i' \to j'$ is a pair of arrows $(h_1 : i' \to i, h_2 : j \to j')$ in $\mathbb{I}$ such that the diagram



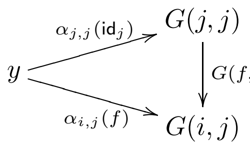
commutes. Mac Lane [Mac98] calls $\mathsf{els}(\mathbb{I}(=,-))$ the *twisted* category of $\mathbb{I}$. Given a multivariant functor $G:\mathbb{I}^{\mathsf{op}}\times\mathbb{I}\to\mathcal{D}$ the weighted limit $\{\mathbb{I}(=,-),G\}$ is given by a representation

$$\mathcal{D}\big(y,\{\mathbb{I}(=,-),G\}\big)\cong[\mathbb{I}^{\mathsf{op}}\times\mathbb{I},\mathbf{Set}]\big(\mathbb{I}(=,-),\mathcal{D}\big(y,G(=,-)\big)\big)$$

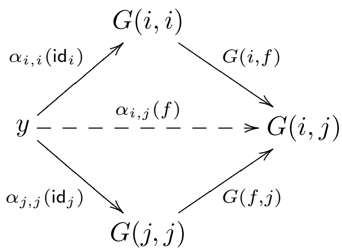
natural in y. A cone α from y to G weighted by $\mathbb{I}(=,-)$ is such that for the pair $(\mathsf{id}_i:i\to i,f:i\to j)$ the diagram



commutes, and for the pair $(f:i\to j,\mathsf{id}_j:j\to j)$ the diagram



commutes. Then by pasting these diagrams together we have that the outer diagram in



commutes for $f:i\to j$ and the family $\langle\alpha_{i,i}(\mathsf{id}_i):y\to G(i,i)\rangle_{i\in\mathbb{I}}$ is a wedge.

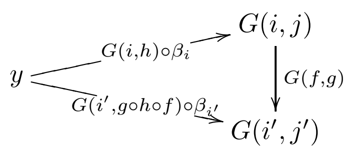
Conversely, for a wedge β from y to G and for a morphism $h:i\to j$ we can define the arrow

$$G(i,h)\circ\beta_i=G(h,j)\circ\beta_j.$$

For arrows $f:i'\to i$, $g:j\to j'$ and $h:i\to j$ in $\mathbb{I}$ we have that

$$\begin{aligned}
G(f,g)\circ G(i,h)\circ\beta_i &= G(f,j')\circ G(i,g\circ h)\circ\beta_i\\
&= G(f,j')\circ G(g\circ h,j')\circ\beta_{j'} && \beta\text{ is a wedge (applied to } g\circ h),\\
&= G(g\circ h\circ f,j')\circ\beta_{j'}\\
&= G(i',g\circ h\circ f)\circ\beta_{i'} && \beta\text{ is a wedge (applied to } g\circ h\circ f).
\end{aligned}$$

Then the diagram



commutes and the family

$$\langle\mathbb{I}(i,j)\xrightarrow[G(-,j)\circ\beta_j]{G(i,-)\circ\beta_i}\mathcal{D}(y,G(i,j))\rangle_{i,j\in\mathbb{I}^{\mathsf{op}}\times\mathbb{I}}$$

is a cone from y to G weighted by $\mathbb{I}(=,-)$.

It is straightforward to check that these two mappings between weighted cones and wedges give rise to a bijection of sets. Hence

$$[\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathbf{Set}]\big(\mathbb{I}(=,-), \mathcal{D}(y, G(=,-))\big) \cong \mathbf{Dinat}\big(\Delta y, G\big)$$

natural in y and then

$$\{\mathbb{I}(=,-), G\} \cong \textstyle\int_x G(x,x).$$

Naturality in the parameter y follows by observing that these constructions are preserved through pre-composition.

At this point one might wonder if the *twisted* category gives a different way of linking the notions of limits and ends. The answer comes from expressing the weighted limit $\{\mathbb{I}(=,-), G\}$ as a limit:

$$\textstyle\int_x G(x,x) \;\cong\; \{\mathbb{I}(=,-), G\} \;\cong\; \varprojlim_{\mathsf{els}(\mathbb{I}(=,-))} G \circ U$$

where $U : \mathsf{els}(\mathbb{I}(=,-)) \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ is the corresponding forgetful functor.

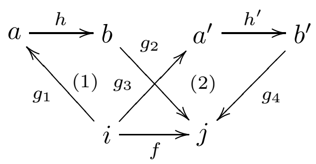
The category $\mathbb{I}^{\S}$ (defined in § 3.2.2) can be seen as a subcategory of the *twisted* category of $\mathbb{I}$. Remember that objects in $\mathbb{I}^{\S}$ are the objects and arrows in $\mathbb{I}$. The "inclusion"

$$\mathbb{I}^{\S} \overset{\iota}{\hookrightarrow} \mathsf{els}(\mathbb{I}(=,-))$$

maps $i \in \mathbb{I}$ to the identity id_i and an arrow to itself. The action of this inclusion on the arrow $i \longrightarrow f$ (for $f : i \to j$ in $\mathbb{I}$) results in the pair (id_i, f), dually the arrow $j \longrightarrow f$ is sent to the pair (f, id_j).

In fact $\mathbb{I}^{\S}$ is an *"initial subcategory"* of $\mathsf{els}(\mathbb{I}(=,-))$ (see §2.2.3) since for each arrow $f : i \to j$ in $\mathbb{I}$ (*i.e.* an object in the *twisted* category):

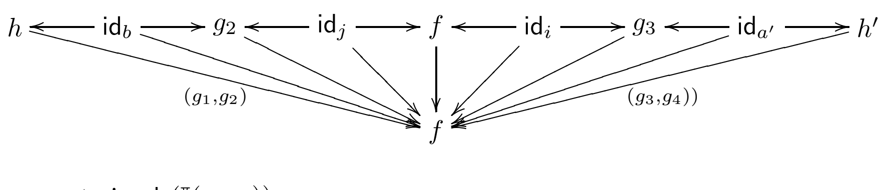
- there is at least the morphisms $(f, \mathsf{id}_j) : \mathsf{id}_j \to f$ and $(\mathsf{id}_i, f) : \mathsf{id}_i \to f$ in $\mathsf{els}(\mathbb{I}(=,-))$, and
- for any pair of arrows $h \to f, h' \to f$ in $\mathsf{els}(\mathbb{I}(=,-))$ there are arrows



in $\mathbb{I}$ where (1) and (2) commute by definition of $\mathsf{els}(\mathbb{I}(=,-))$; then there is a connection

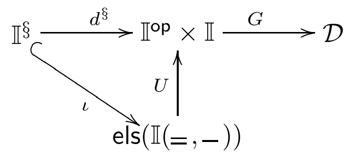
$$h \longleftarrow b \longrightarrow g_2 \longleftarrow j \longrightarrow f \longleftarrow i \longrightarrow g_3 \longleftarrow a' \longrightarrow h'$$

in $\mathbb{I}^{\S}$ such that the diagrams



commute in $\mathsf{els}(\mathbb{I}(=,-))$.

The general situation is illustrated by the diagram



where the triangle commutes. By putting all this information together it follows that

$$\int_x G(x,x) \cong \varprojlim_{\mathbb{I}^\S} G \circ d^\S \cong \varprojlim_{\mathbb{I}^\S} G \circ U \circ \iota \overset{(1)}{\cong} \varprojlim_{\mathsf{els}(\mathbb{I}(=,-))} G \circ U \cong \{\mathbb{I}(=,-), G\}.$$

where the isomorphism (1) is the one defined by Proposition 2.2.5.

Clearly as weighted limits are conical limits they are ends as well. If for all $i \in \mathbb{I}$ the power $[F(i), G(i)]$ exists in $\mathcal{D}$ there is an alternative way of expressing weighted limits as ends.

Proposition 3.5.3 Let $G : \mathbb{I} \to \mathcal{C}$ be a diagram and $F : \mathbb{I} \to \mathbf{Set}$ a weight such that for each $i \in \mathbb{I}$ the power $[F(i), G(i)]$ exists. Then, there is an isomorphism

$$\{F, G\} \cong \int_x [F(x), G(x)]$$

meaning that one side of the isomorphism exists if and only if so does the other.

Proof.

$$\begin{aligned} \mathcal{C}(c, \textstyle\int_x [F(x), G(x)]) &\cong \textstyle\int_x \mathcal{C}(c, [F(x), G(x)]) && \text{since hom-functors preserve ends,} \\ &\cong \textstyle\int_x \big[F(x), \mathcal{C}(c, G(x))\big] && \text{by definition of powers,} \\ &\cong [\mathbb{I}, \mathbf{Set}](F, \mathcal{C}(c, G(-))) && \text{by naturality formula.} \end{aligned}$$

□

Weighted Limits with Parameters

As with conical limits the expression $\{F, G\}$ is functorial in $G \in [\mathbb{I}, \mathcal{D}]$ provided that $\mathcal{D}$ has enough weighted limits. From parametrised representability $\{F, G\}$ is functorial in F as well, in this case as a contravariant argument. This follows by applying Theorem 2.1.6 to the functor

$$\lambda F, y.[\mathbb{I}, \mathcal{D}]\big(F, \mathcal{D}(y, G(-))\big) : [\mathbb{I}, \mathbf{Set}]^{\mathsf{op}} \times \mathcal{D}^{\mathsf{op}} \to \mathbf{Set}.$$

That is if for every $F \in [\mathbb{I}, \mathbf{Set}]^{\mathsf{op}}$ there is a representation

$$\mathcal{D}(-, \{F, G\}) \overset{\theta^F}{\cong} [\mathbb{I}, \mathcal{D}]\big(F, \lambda x.\mathcal{D}(-, G(x))\big).$$

Then there is a unique extension of the mapping $F \mapsto \{F, G\}$ to a functor

$$\{-, G\} : [\mathbb{I}, \mathbf{Set}]^{\mathsf{op}} \to \mathcal{D}.$$

Concretely given a natural transformation $\alpha : F \Rightarrow H$ where $F, H : \mathbb{I} \to \mathbf{Set}$ by the proof of Theorem 2.1.6 we have

$$\{\alpha, G\} = \left(\theta^F\right)^{-1} (\gamma \circ \alpha)$$

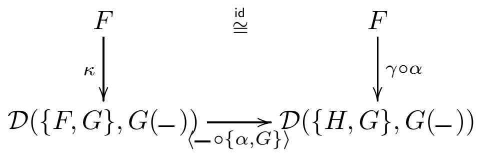
where γ is the universal weighted cone defined by $\{H, G\}$. The composition

$$\gamma \circ \alpha : F \Rightarrow \mathcal{D}(\{H, G\}, G(-))$$

gives a cone from $\{H, G\}$ to G weighted by F. Now by universality

$$\{\alpha, G\} : \{H, G\} \to \{F, G\}$$

is the unique morphism making the diagram



commutes where

$$\kappa = \theta^F(\mathsf{id}_{\{F,G\}})$$

is the universal weighted cone defined by $\{F, G\}$. Therefore, if $\mathcal{D}$ is a complete category there is a bifunctor

$$\lambda G, F.\{F, G\} : [\mathbb{I}, \mathcal{D}] \times [\mathbb{I}, \mathbf{Set}]^{\mathsf{op}} \to \mathcal{D}.$$

Iterated Weighted Limits

As expected weighted limits of diagrams over functor categories are computed pointwise. Let $H : \mathbb{I} \to [\mathbb{J}, \mathcal{C}]$ be a functor such that for a weight $F : \mathbb{I} \to \mathbf{Set}$ and for every $j \in \mathbb{J}$ the limit $\{F, G_j\}$ exists where $G_j : \mathbb{I} \to \mathcal{C}$ is the partial functor corresponding to $G = \boldsymbol{\lambda}^{-1}(H)$. Then there is an isomorphism

$$\{F, H\} \cong \lambda x.\{F, G_x\}.$$

This follows since for $K : \mathbb{J} \to \mathcal{C}$

$$\begin{aligned}
[\mathbb{J}, \mathcal{C}](K, \lambda x.\{F, G_x\}) &\cong \textstyle\int_x \mathcal{C}(K(x), \{F, G_x\}) && \text{by naturality formula,} \\
&\cong \textstyle\int_x [\mathbb{I}, \mathbf{Set}]\big(F, \mathcal{C}(K(x), G_x(-))\big) && \text{by definition of weighted limits,} \\
&\cong [\mathbb{I}, \mathbf{Set}](F, \textstyle\int_x \mathcal{C}(K(x), G_x(-))) && \text{since hom-functors preserve ends,} \\
&\cong [\mathbb{I}, \mathbf{Set}]\big(F, [\mathbb{J}, \mathcal{C}](K, H(-))\big) && \text{by naturality formula.}
\end{aligned}$$

As ends and limits, weighted limits can be iterated. First we need to state what an iterated weighted limit would mean. Let $H : \mathbb{I} \to [\mathbb{J}, \mathcal{C}]$ be a functor and $F_1 : \mathbb{I} \to \mathbf{Set}$ and $F_2 : \mathbb{J} \to \mathbf{Set}$ be weights. We define a *combined weight*

$$F_1 \otimes F_2 : \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$$

to be the functor which maps $(i, j) \in \mathbb{I} \times \mathbb{J}$ into the copower $F_1(i) \otimes F_2(j)$, which in this case is jut the product of sets $F_1(i) \times F_2(j)$. If the limit $\{F_1, H\}$ exists pointwise in $[\mathbb{J}, \mathcal{C}]$ then there is a canonical isomorphism

$$\{F_2, \{F_1, H\}\} \cong \{F_1 \otimes F_2, G\}$$

where $G = \boldsymbol{\lambda}^{-1}(H)$. This follows since for $c \in \mathcal{C}$

$$
\begin{aligned}
&\mathcal{C}\big(c,\{F_2,\{F_1,H\}\}\big) \\
&\quad\cong [\mathbb{J},\mathbf{Set}]\big(F_2,\mathcal{C}(c,\{F_1,H\}(-))\big) && \text{by definition of weighted limit,}\\
&\quad\cong \textstyle\int_x \big[F_2(x),\mathcal{C}\big(c,\{F_1,H\}(x)\big)\big] && \text{by naturality formula,}\\
&\quad\cong \textstyle\int_x \big[F_2(x),\mathcal{C}\big(c,\{F_1,G_x\}\big)\big] && \text{by pointwise computation,}\\
&\quad\cong \textstyle\int_x \big[F_2(x),[\mathbb{I},\mathbf{Set}]\big(F_1,\mathcal{C}(c,G_x(-))\big)\big] && \text{by definition of weighted limit,}\\
&\quad\cong \textstyle\int_x \big[F_2(x),\int_y[F_1(y),\mathcal{C}(c,G_x(y))]\big] && \text{by naturality formula,}\\
&\quad\cong \textstyle\int_x\int_y \big[F_2(x),[F_1(y),\mathcal{C}(c,G(x,y))]\big] && \text{since hom-functors preserve ends,}\\
&\quad\cong \textstyle\int_x\int_y [F_1(y)\otimes F_2(x),\mathcal{C}(c,G(x,y))] && \text{by definition of tensor and symmetry,}\\
&\quad\cong \textstyle\int_{(x,y)} [F_1(y)\otimes F_2(x),\mathcal{C}(c,G(x,y))] && \text{by Fubini,}\\
&\quad\cong [\mathbb{I}\times\mathbb{J},\mathbf{Set}]\big(F_1\otimes F_2,\mathcal{C}(c,G(=,-))\big) && \text{by naturality formula.}
\end{aligned}
$$

3.5.2 Weighted Colimits

As covariant presheaves are weights for limits, contravariant presheaves are weights for colimits. A weighted colimit for a diagram $D\colon\mathbb{I}\to\mathcal{D}$ is nothing but a weighted limit for $D^{\mathsf{op}}\colon\mathbb{I}^{\mathsf{op}}\to\mathcal{D}^{\mathsf{op}}$.

Definition 3.5.4 Let $D\colon\mathbb{I}\to\mathcal{D}$ and $F\colon\mathbb{I}^{\mathsf{op}}\to\mathbf{Set}$ be diagrams. The colimit of D weighted by F is a representation for the functor

$$[\mathbb{I}^{\mathsf{op}},\mathbf{Set}]\big(F,\lambda x.\mathcal{D}(D(x),-)\big)\colon\mathcal{D}\to\mathbf{Set},$$

i.e. an isomorphism

$$\mathcal{D}(c,y) \overset{\theta_y}{\cong} [\mathbb{I}^{\mathsf{op}},\mathbf{Set}]\big(F,\mathcal{D}(D(-),y)\big) \tag{3.10}$$

natural in y.

We call the object c the the weighted colimit and it is written as $\mathrm{Colim}_F D$ or $F\star D$; in this dissertation we use the second. The unit η of the representation (3.10) is a universal weighted cone in the sense that for another weighted cone τ from D to b weighted by F there is a unique mediating arrow $h\colon F\star D\to b$ such that for every $i\in\mathbb{I}$ and $x\in F(i)$ the triangle

$$
\begin{array}{ccc}
D(i) & \xrightarrow{\eta_i(x)} & F\star D\\
 & \searrow^{\tau_i(x)} & \downarrow h\\
 & & b
\end{array}
$$

commutes. That is, h is the unique morphism making the diagram of natural transformations

$$
\begin{array}{ccc}
F & \overset{\mathsf{id}}{\cong} & F\\
\eta\downarrow & & \downarrow\tau\\
\mathcal{D}(D(-),F\star D) & \xrightarrow[\langle h\circ -\rangle]{} & \mathcal{D}(D(-),b)
\end{array}
$$

commute. By means of the natural isomorphism (1.1) we obtain the diagram

$$
\begin{array}{ccc}
F & \overset{\mathsf{id}}{\cong} & F\\
\eta\downarrow & & \downarrow\tau\\
\mathcal{D}^{\mathsf{op}}((F\star D)^{\mathsf{op}},D^{\mathsf{op}}(-)) & \xrightarrow[\langle -\circ h^{\mathsf{op}}\rangle]{} & \mathcal{D}^{\mathsf{op}}(b,D^{\mathsf{op}}(-))
\end{array}
$$

which shows the expression $(F \star D)^{\mathsf{op}}$ as a weighted limit of D^{op} in $\mathcal{D}^{\mathsf{op}}$.

By duality the properties of weighted limits are exported to the case of weighted colimits. Thus weighted colimits correspond to "conical" colimits and coends. For a functor $G{:}\mathbb{I} \to \mathcal{D}$ and a weight $F{:}\mathbb{I}^{\mathsf{op}} \to \mathbf{Set}$ we have

$$F \star G \cong \mathrm{colim}_{\mathsf{els}(F)} G \circ U_F \tag{3.11}$$

where $\mathsf{els}(F)$ is the category of elements for the presheaf F and U_F is the corresponding forgetful functor. For a contravariant presheaf F the definition of $\mathsf{els}(F)$ needs a slight modification: the objects are pairs (i, x) where $x \in F(i)$ as before, but an arrow $f{:}(i, x) \to (j, y)$ in $\mathsf{els}(F)$ corresponds to an arrow $f{:}i \to j$ in $\mathbb{I}$ where $x = F(f)(y)$.

For a functor $G{:}\mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$, the coend of G can be expressed as a weighted colimit:

$$\textstyle\int^x G(x, x) \cong \mathbb{I}(=, -) \star G.$$

Now, the "inclusion" functor $\iota : \mathbb{I}^{\S} \to \mathsf{els}(\mathbb{I}(=, -))$ is final instead. Finality here means that for every arrow f in $\mathbb{I}$ the comma category $f \downarrow \iota$ is connected. Thus, from (3.11) and the dual form of Proposition 2.2.5 we have:

$$\textstyle\int^x G(x, x) \cong \mathrm{colim}_{\mathbb{I}^{\S}} G \circ d^{\S} \cong \mathrm{colim}_{\mathbb{I}^{\S}} G \circ U \circ \iota \cong \mathrm{colim}_{\mathsf{els}(\mathbb{I}(=,-))} G \circ U \cong \mathbb{I}(=, -) \star G.$$

Let $G : \mathbb{I} \to \mathcal{C}$ be a diagram and $F : \mathbb{I} \to \mathbf{Set}$ a weight. From the dual of Proposition 3.5.3 we have that if the copower $F(i) \otimes G(i)$ exists for each $i \in \mathbb{I}$ then there is an isomorphism

$$F \star G \cong \textstyle\int^x F(x) \otimes G(x).$$

It is interesting to verify how the parametrised representability works for weighted colimits. Given a cocomplete category $\mathcal{D}$ there is a bifunctor

$$\lambda G, F.F \star G{:}[\mathbb{I}, \mathcal{D}] \times [\mathbb{I}^{\mathsf{op}}, \mathbf{Set}] \to \mathcal{D}$$

where differently to the weighted-limit case the expression is covariant in the parameter for the weights.

The following result relates contravariant with covariant presheaves.

Proposition 3.5.5 Given the functors $F{:}\mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ and $G{:}\mathbb{C} \to \mathbf{Set}$ there is an isomorphism

$$F \star G \cong G \star F$$

Proof. For a set Z we have

$$\begin{aligned}
[F \star G, Z] &\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](F, [G(-), Z]) && \text{by definition of weighted colimit,}\\
&\cong \textstyle\int_x [F(x), [G(x), Z]] && \text{by naturality formula,}\\
&\cong \textstyle\int_x [F(x) \otimes G(x), Z] && \text{by definition of copower,}\\
&\cong \textstyle\int_x [G(x) \otimes F(x), Z] && \text{by symmetry,}\\
&\cong \textstyle\int_x [G(x), [F(x), Z]] && \text{by definition of copower,,}\\
&\cong [\mathbb{C}, \mathbf{Set}]\big(G, [F(-), Z]\big) && \text{by naturality formula.}
\end{aligned}$$

Thus by definition of weighted colimit we have $F \star G \cong G \star F$. □

Corollary 3.5.6 For a diagram $D{:}\mathbb{C} \to [\mathbb{D}, \mathbf{Set}]$ and a weight $F{:}\mathbb{C} \to \mathbf{Set}$ there is an isomorphism

$$F \star D \cong \lambda d.(G_d \star F)$$

where $G_d{:}\mathbb{D} \to \mathbf{Set}$ is the functor induced by the bifunctor $G = \boldsymbol{\lambda}^{-1}(D)$.

Proof.

$$\begin{aligned} F \star D &\cong \lambda d.(F \star G_d) && \text{pointwise computation,} \\ &\cong \lambda d.(G_d \star F) && \text{by the Proposition above.} \end{aligned}$$

□

3.5.3 Density

A subcategory $\mathbb{C} \hookrightarrow \mathcal{D}$ is said to be a *dense generator* if each object in $\mathcal{D}$ can be expressed as a colimit of objects in $\mathbb{C}$. More generally a functor $K : \mathbb{C} \to \mathcal{D}$ is *dense* if for each $d \in \mathcal{D}$ there is a canonical isomorphism

$$d \cong \mathrm{colim}(\, K \downarrow d \xrightarrow{P} \mathbb{C} \xrightarrow{K} \mathcal{D}\,)$$

where P is the projection functor associated to the comma category $K \downarrow d$. Here we show that for a category $\mathbb{C}$ the Yoneda embedding $\mathcal{Y} : \mathbb{C} \to [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ is a dense functor.

Proposition 3.5.7 For a presheaf $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ we have

$$F \star \mathcal{Y} \cong F.$$

Proof.

$$\begin{aligned} &[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(F \star \mathcal{Y}, H\big) \\ &\qquad \cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(F, \lambda x.[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}(x), H)\big) && \text{by definition of weighted colimit,} \\ &\qquad \cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(F, \lambda x.\textstyle\int_c [\mathbb{C}(c, x), H(c)]\big) && \text{by naturality formula,} \\ &\qquad \cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(F, H\big) && \text{by the Yoneda lemma.} \end{aligned}$$

□

This result is an important property which says that a presheaf $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ can always be expressed as a colimit of solely representables:

$$F \cong F \star \mathcal{Y} \cong \mathrm{colim}_{\mathsf{els}(F)} \mathcal{Y} \circ U_F.$$

From the Yoneda lemma we have that the the diagram of categories and functors

$$\begin{array}{ccc} \mathsf{els}(F) & \cong & \mathcal{Y} \downarrow F \\ {\scriptstyle U_F}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ \mathbb{C} & \overset{\mathsf{id}}{\cong} & \mathbb{C} \end{array}$$

commutes and then the Yoneda embedding is dense.

By Corollary 3.5.6 there is another way of expressing a presheaf $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$ as a weighted colimit (or "conical" colimit or coend)

$$F \cong F \star \mathcal{Y} \cong \lambda c.(\mathbb{C}(c, \_) \star F).$$

Since $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ is cocomplete and then has all copowers we have

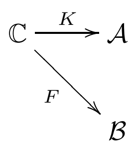
$$\begin{aligned} F \cong F \star \mathcal{Y} &\cong \textstyle\int^x F(x) \otimes \lambda y.\mathbb{C}(y, x) && \text{by dual of Proposition 3.5.3,} \\ &\cong \lambda y.\textstyle\int^x F(x) \otimes \mathbb{C}(y, x) && \text{pointwise computation.} \end{aligned}$$

This gives a third way of expressing a presheaf as a coend:

$$F \cong \lambda y. \int^x F(x) \otimes \mathbb{C}(y, x) \tag{3.12}$$

the so called *density formula*.

For functors



(3.13)

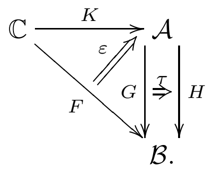
a *left Kan extension* of F along K is given by a "universal arrow"

$$(G : \mathcal{A} \to \mathcal{B}, \varepsilon : F \Rightarrow G \circ K)$$

from F to $\_ \circ K : [\mathcal{A}, \mathcal{B}] \to [\mathbb{C}, \mathcal{B}]$. In other words, for a pair $(H : \mathcal{A} \to \mathcal{B}, \beta : F \Rightarrow H \circ K)$ there is a unique natural transformation $\tau : G \Rightarrow H$ such that

$$\beta = K\tau \circ \varepsilon.$$

The following diagram illustrates the situation



Usually G is written as $\mathrm{Lan}_K F$. By definition the mapping sending β into τ establishes an isomorphism

$$\mathbf{Nat}\big(F, H \circ K\big) \cong \mathbf{Nat}\big(\mathrm{Lan}_K F, H\big)$$

"natural" in H. Kan extensions, however, cannot be expressed as a representation for a **Set**-valued functor simply because the category of functors $[\mathcal{A}, \mathcal{B}]$ is not necessarily locally small and we shall not develop their theory here.

The most relevant fact to our work is that left Kan extensions can be computed pointwise as coend formulæ. If the copower $\mathcal{A}(K(y), x) \otimes F(y')$ exists in $\mathcal{B}$ for every x, y, y' then

$$\mathrm{Lan}_K F = \lambda y. \int^x \mathcal{A}(K(x), y) \otimes F(x) \cong \lambda y. \mathcal{A}(K(\_), y) \star F. \tag{3.14}$$

For instance by the Yoneda lemma the density formula (3.12) corresponds to the left Kan extension of F composed with the Yoneda embedding.

There is of course a dual notion: for functors like in (3.13) the right Kan extension of F along K, sometimes written $\mathrm{Ran}_K F$, is given by an isomorphism

$$\mathbf{Nat}\big(H \circ K, F\big) \cong \mathbf{Nat}\big(H, \mathrm{Ran}_K F\big)$$

"natural" in H. If the power $[\mathcal{A}(x, K(y)), F(y')]$ exists in $\mathcal{B}$ for every x, y, y' then

$$\mathrm{Ran}_K F = \lambda y. \int_x [\mathcal{A}(y, K(x)), F(x)] \cong \lambda y. \{\mathcal{A}(K(\_), y), F\}.$$

For a proof of these facts refer to [Kel82, Mac98].

3.5.4 Exponentials

The last property of presheaves we study here is their cartesian closed structure. As the category $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ is complete we have all products, in fact they are computed pointwise

$$(F \times G)(c) \cong F(c) \times G(c).$$

The exponential $[F, G]$ should satisfy

$$\begin{aligned}
[F,G](c) &\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}(c), [F,G]) && \text{by the Yoneda lemma,}\\
&\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}(c) \times F, G) && \text{by the required adjunction,}\\
&\cong \textstyle\int_x [\mathbb{C}(x,c) \times F(x), G(x)] && \text{by naturality formula and pointwise computation.}
\end{aligned}$$

Let us verify that

$$\lambda y. \textstyle\int_x [\mathbb{C}(x,y) \times F(x), G(x)]$$

is indeed an exponential:

$$\begin{aligned}
[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](H, [F,G]) &\cong \textstyle\int_y [H(y), [F,G](y)] && \text{by naturality formula,}\\
&\cong \textstyle\int_y [H(y), \int_x [\mathbb{C}(x,y) \times F(x), G(x)]] && \text{by definition,}\\
&\cong \textstyle\int_y \int_x [H(y), [\mathbb{C}(x,y) \times F(x), G(x)]] && \text{since hom-functors preserve ends,}\\
&\cong \textstyle\int_x \int_y [H(y), [\mathbb{C}(x,y) \times F(x), G(x)]] && \text{by Fubini result,}\\
&\cong \textstyle\int_x \int_y [H(y), [\mathbb{C}(x,y), [F(x), G(x)]]] && \text{since } \mathbf{Set} \text{ is closed,}\\
&\cong \textstyle\int_x \int_y [H(y) \times \mathbb{C}(x,y), [F(x), G(x)]] && \text{since } \mathbf{Set} \text{ is closed,}\\
&\cong \textstyle\int_x [\int^y H(y) \times \mathbb{C}(x,y), [F(x), G(x)]] && \text{by (3.6),}\\
&\cong \textstyle\int_x [H(x), [F(x), G(x)]] && \text{by density formula,}\\
&\cong \textstyle\int_x [H(x) \times F(x), G(x)] && \text{since } \mathbf{Set} \text{ is closed,}\\
&\cong \textstyle\int_x [(H \times F)(x), G(x)] && \text{by pointwise computation,}\\
&\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](H \times F, G) && \text{by naturality formula.}
\end{aligned}$$

This gives another example of the power of the algebraic manipulation of ends and coends we exploit in this dissertation.

Chapter 4

Preservation of Limits

Functors preserving limits and/or colimits play an important role in the applications of category theory. A central result in [CW], for instance, is that cocontinuous functors between presheaf categories preserve open-map bisimulation. Another example arises in categorical logic [Jac99, Pit00] where a model for a simple type theory is defined to be a product-preserving functor with domain the corresponding classifying category. In general verifying that a functor preserves limits by following the definition can be tedious and rather involved. It seems to require a concrete reasoning in terms of cones. In this chapter we study how naturality may help to prove preservation of universal properties in a more "equational" manner. The main result of the chapter is that functors which preserve terminal objects and distribute over the limit functor up to natural isomorphism preserve limits. The presentation of this result constitutes one of the contributions of this dissertation and it is intimately related to the choice of rules for the calculus for categories presented in the subsequent chapters.

4.1 Limiting Cones

A cone from $c \in \mathcal{C}$ to a diagram $D : \mathbb{I} \to \mathcal{C}$ is a natural transformation from Δc to D. A limit for D is just a universal cone, *i.e.* a cone $\varepsilon : \Delta c \Rightarrow D$ such that for any other cone $\varepsilon' : \Delta c' \Rightarrow D$ there exists a unique mediating morphism $m : c' \to c$ for which the diagram

$$\begin{array}{ccc} \Delta c' & \xrightarrow{\Delta m} & \Delta c \\ {\scriptstyle \varepsilon'}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D & \overset{\mathsf{id}}{\cong} & D \end{array}$$

commutes. Alternatively a limit for D is a representation for the set-valued functor

$$[\mathbb{I}, \mathcal{C}](\Delta -, D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

where the counit is the limiting cone (see 2.2).

A pair of arrows $s : c \to d$, $r : d \to c$ form a section-retraction pair (or s is a split monic) if the composition $r \circ s$ yields the identity id_c and it is denoted by

$$c \underset{r}{\overset{s}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} d.$$

Proposition 4.1.1 Let $D, D' : \mathbb{I} \to \mathcal{C}$ be small diagrams such that there are cones $\varepsilon : \Delta c \Rightarrow D$ and $\varepsilon' : \Delta c' \Rightarrow D'$ together with section-retraction pairs (s_1, r_1) and

(s_2, r_2) such that the squares in the following diagram of natural transformations

$$\begin{array}{ccc} \Delta c' & \underset{\Delta r_1}{\overset{\Delta s_1}{\rightleftarrows}}\ \triangleleft & \Delta c \\ {\scriptstyle \varepsilon'}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D' & \underset{r_2}{\overset{s_2}{\rightleftarrows}}\ \triangleleft & D \end{array}$$

commute, *i.e.*

$$\varepsilon \circ \Delta s_1 = s_2 \circ \varepsilon' \quad \text{and} \quad \varepsilon' \circ \Delta r_1 = r_2 \circ \varepsilon. \tag{4.1}$$

If ε is a limiting cone then also ε' is a limiting cone.

Proof. We want to show ε' is universal, *i.e.* that for any cone $\kappa : \Delta x \Rightarrow D'$ there exists a unique factorisation of κ through ε'.

- Existence: The composition $s_2 \circ \kappa$ is a cone from x to D. Let $m : x \to c$ be the unique mediating arrow such that

$$\varepsilon \circ \Delta m = s_2 \circ \kappa. \tag{4.2}$$

  Hence,

$$\begin{aligned} \varepsilon' \circ \Delta r_1 \circ \Delta m &= r_2 \circ \varepsilon \circ \Delta m && \text{by (4.1)}, \\ &= r_2 \circ s_2 \circ \kappa && \text{by (4.2)}, \\ &= \kappa && (s_2, r_2) \text{ is a section-retraction pair.} \end{aligned}$$

- Uniqueness: Let $h : x \to c'$ be an arrow such that

$$\varepsilon' \circ \Delta h = \kappa.$$

  Hence,

$$\begin{aligned} \varepsilon \circ \Delta s_1 \circ \Delta h &= s_2 \circ \varepsilon' \circ \Delta h && \text{by (4.1)}, \\ &= s_2 \circ \kappa && \text{by definition of } \Delta h. \end{aligned}$$

  By uniqueness of m

$$m = s_1 \circ h,$$

  and since (s_1, r_1) is a section-retraction pair

$$r_1 \circ m = h.$$

□

Corollary 4.1.2 Let $D : \mathbb{I} \to \mathcal{C}$ be a diagram. Given a limiting cone $\varepsilon : \Delta c \Rightarrow D$ and an isomorphism $c' \overset{f}{\cong} c$ then $\varepsilon \circ \Delta f : \Delta c' \Rightarrow D$ is a limiting cone.

Proof. By direct application of the proposition above to

$$\begin{array}{ccc} \Delta c' & \overset{\Delta f}{\cong} & \Delta c \\ {\scriptstyle \varepsilon \circ \Delta f}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D & \overset{\mathsf{id}}{\cong} & D. \end{array}$$

□

4.2 Preservation of Limiting Cones

Let $G:\mathcal{C} \to \mathcal{D}$ be a functor. Given a cone $\gamma : \Delta a \Rightarrow D$ for a diagram $D:\mathbb{I} \to \mathcal{C}$ the natural transformation $G\gamma : \Delta G(a) \Rightarrow G \circ D$ is a cone as well. Thus, given a limiting cone $\varepsilon : \Delta d \Rightarrow G \circ D$ there is a unique mediating morphism $m:G(a) \to d$ such that the diagram

$$\begin{array}{ccc} \Delta G(a) & \xrightarrow{\Delta m} & \Delta d \\ {\scriptstyle G\gamma}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ G \circ D & \overset{\mathsf{id}}{\cong} & G \circ D \end{array}$$

commutes.

The functor G preserves a limiting cone $\kappa : \Delta c \Rightarrow D$ if the cone $G\kappa : \Delta G(c) \Rightarrow G \circ D$ is limiting (see Definition 2.2.7). This is equivalent to requiring the mediating arrow defined by $G\kappa$ to be an isomorphism. In fact many authors use this as the definition of preservation of limits (see [MM92]).

Proposition 4.2.1 Let $D, D':\mathbb{I} \to \mathcal{C}$ be small diagrams such that there are cones $\varepsilon : \Delta c \Rightarrow D$ and $\varepsilon' : \Delta c' \Rightarrow D'$ together with section-retraction pairs (s_1, r_1) and (s_2, r_2) making the squares in the diagram of natural transformations

$$\begin{array}{ccc} \Delta c' & \underset{\Delta r_1}{\overset{\Delta s_1}{\rightleftarrows}}\ \triangleleft & \Delta c \\ {\scriptstyle \varepsilon'}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ D' & \underset{r_2}{\overset{s_2}{\rightleftarrows}}\ \triangleleft & D \end{array}$$

commute. If ε is a limiting cone and $G:\mathcal{C} \to \mathcal{D}$ preservers it then G preserves the limiting cone ε'.

Proof. It follows by applying G to the diagram above and by Proposition 4.1.1. As $G\varepsilon$ is limiting then $G\varepsilon'$ is limiting and the corresponding mediating arrow must be an isomorphism. Notice that the mediating arrow is not necessary $\Delta G(s_1)$. □

As an immediate corollary we have that if a functor $G:\mathcal{C} \to \mathcal{D}$ preserves one limiting cone for a diagram $D:\mathbb{I} \to \mathcal{C}$ then it preserves any other limiting cone for D. We shall henceforth consider a chosen limiting cone when checking for preservation.

Proposition 4.2.2 Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors such that $F \cong G$. For every limiting cone $\kappa : \Delta c \Rightarrow D$ for some diagram $D:\mathbb{I} \to \mathcal{C}$

$$F \text{ preserves } \kappa \text{ if and only if } G \text{ preserves } \kappa.$$

Proof. Assume F preserves κ, then $F\kappa$ is a limiting cone and from the commutativity of the diagram

$$\begin{array}{ccc} \Delta G(c) & \cong & \Delta F(c) \\ {\scriptstyle G\kappa}\big\downarrow & & \big\downarrow{\scriptstyle F\kappa} \\ G \circ D & \cong & F \circ D \end{array}$$

we conclude by Proposition 4.1.1 that $G\kappa$ is a limiting cone as well. □

An immediate corollary from this and Proposition 2.2.8 is that representable functors are continuous.

In some cases we are interested in a subcategory of diagrams $\mathcal{K} \subseteq [\mathbb{I}, \mathcal{C}]$. Then we talk about a functor preserving $\mathcal{K}$-limits or being $\mathcal{K}$-continuous. Typically we take subcategories of diagrams $\mathcal{K}$ for which all limits exists in $\mathcal{C}$, *i.e.* $\mathcal{C}$ is $\mathcal{K}$-complete, in such a way that we can talk about a functor $\varprojlim_{\mathbb{I}}:\mathcal{K} \to \mathcal{C}$ as defined in § 2.2.1.

In order to prove that a functor $G:\mathcal{C} \to \mathcal{D}$ preserves the limit of a diagram $D:\mathbb{I} \to \mathcal{C}$ is not enough to exhibit an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} G \circ D.$$

Indeed, the action of G over arrows may result in a family that is not universal. As an example consider the category **Count** of countably infinite sets and functions. Clearly the objects of **Count** are all isomorphic. There is a functor $\_+1:\mathbf{Count} \to \mathbf{Count}$ that acts over sets by adding a new element: given $X \in \mathbf{Count}$ then $X+1 = X \cup \{X\}$ and given a function $f:X \to Y$, the function $f+1$ sends $a \in X$ to $f(a)$ and $\{X\}$ to $\{Y\}$. There is an isomorphism

$$(X \times Y) + 1 \cong (X+1) \times (Y+1),$$

but this functor does not preserve products; the arrow $\pi_X + 1:(X \times Y)+1 \to X+1$ is not a projection.

If the categories $\mathcal{C}$ and $\mathcal{D}$ have enough limits the expressions

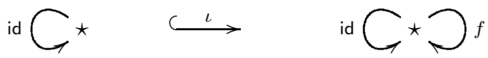
$$G(\varprojlim_{\mathbb{I}} D) \quad \text{and} \quad \varprojlim_{\mathbb{I}}(G \circ D)$$

are both functorial in D. For every D there is a mediating arrow m_D defined by $G\varepsilon$ where ε is the universal cone associated to $\varprojlim_{\mathbb{I}} D$. The family $\langle m_D \rangle_D$ is natural, this follows directly from the universality of the mediating arrows. Thus if G is $\mathcal{K}$-continuous there is a canonical isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}}(G \circ D)$$

natural in $D \in \mathcal{K}$.

This isomorphism is not always unique. Consider, for instance, the category $\mathbf{1}$ with one object, say $\star$, and the identity arrow. The functor category $[\mathbf{1},\mathbf{1}]$ has only one object: the "constant" functor $\Delta\star$. The limit for this functor is the object $\star$ itself where the limiting cone is the identity. We can extend $\mathbf{1}$ with an extra morphism



where f is also an isomorphism, *i.e.* $f \circ f = \mathsf{id}$. The inclusion functor ι clearly preserves the limit of the diagram $\Delta\star$. The mediating arrow is given by the identity on $\star$ which is an isomorphism and trivially natural. The arrow f, however, gives another isomorphism

$$\iota(\varprojlim_{\mathbf{1}} \Delta\star) \cong \varprojlim_{\mathbf{1}} \iota \circ (\Delta\star),$$

naturality here is trivial as well.

Often checking the isomorphism between the limiting objects follows from a more-or-less direct calculation. To prove that a functor preserves a limiting cone by closely following the definition can be rather tedious and involve a fair amount of bookkeeping. In this chapter we study under which conditions an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} G \circ D$$

is enough to ensure that G preserves the limit of D.

4.3 Connected Diagrams

A category $\mathcal{E}$ is *connected* if for any pair of objects $a, b \in \mathcal{E}$ there is a chain of arrows

$$a \to e_1 \leftarrow e_2 \to \ldots \to e_n \leftarrow b.$$

A small category $\mathbb{I}$ can be decomposed into its connected components. We write $\mathbb{I} = \sum_{k\in\mathbb{K}} \mathbb{I}_k$ for this decomposition where $\mathbb{I}_k$'s are the connected components of $\mathbb{I}$. The decomposition can be described abstractly as the left adjoint of the discrete-category functor $\mathsf{Dis}\colon\mathbf{Set} \to \mathbf{Cat}$. This functor takes a set X into the discrete small category with objects the elements of X and it has left and right adjoints:

$$\mathsf{Con} \dashv \mathsf{Dis} \dashv \mathsf{Obj}.$$

The functor $\mathsf{Con}\colon\mathbf{Cat} \to \mathbf{Set}$ takes a category $\mathbb{I}$ into the set of connected components of $\mathbb{I}$ and the functor $\mathsf{Obj}\colon\mathbf{Cat} \to \mathbf{Set}$ acts by mapping a small category into its set of objects. The indexing category $\mathbb{K}$ in $\sum_{k\in\mathbb{K}} \mathbb{I}_k$ is defined to be $\mathsf{Dis}(\mathsf{Con}(\mathbb{I}))$, *i.e.* the discrete category whose objects are identified with the connected components of $\mathbb{I}$ [Par73, Par90].

A connected component $\mathbb{I}_k$ of $\mathbb{I}$ is a full subcategory and there is an inclusion functor $\iota_k\colon\mathbb{I}_k \to \mathbb{I}$. This functor defines by pre-composition the *"restriction"* functor

$$\_ \circ \iota_k\colon[\mathbb{I}, \mathcal{C}] \to [\mathbb{I}_k, \mathcal{C}].$$

A diagram whose index category is given by a unique connected component is called a *connected diagram*.

We can characterise the set of natural transformations from Δc to Δd as

$$[\mathbb{I}, \mathcal{C}](\Delta c, \Delta d) \cong [\mathsf{Con}(\mathbb{I}), \mathcal{C}(c, d)].$$

From the definition of the diagonal functor in §1.3 an arrow $f\colon c \to d$ in $\mathcal{C}$ defines a natural transformation $\Delta f : \Delta c \Rightarrow \Delta d$. Observe that the diagonal functor is faithful but not necessary full. If the category $\mathbb{I}$ is connected then we have

$$[\mathbb{I}, \mathcal{C}](\Delta c, \Delta d) \cong \mathcal{C}(c, d)$$

since $\mathcal{C}(c, d) \cong [1, \mathcal{C}(c, d)]$ and then the diagonal functor is full and faithful.

By the definition of power

$$\mathcal{C}(c, [\mathsf{Con}(\mathbb{I}), d]) \cong [\mathsf{Con}(\mathbb{I}), \mathcal{C}(c, d)] \cong [\mathbb{I}, \mathcal{C}](\Delta c, \Delta d)$$

natural in c which exhibits $[\mathsf{Con}(\mathbb{I}), d]$ as the limit of Δd:

$$\varprojlim_{\mathbb{I}} \Delta d \cong [\mathsf{Con}(\mathbb{I}), d]. \tag{4.3}$$

The counit of this representation

$$\mathsf{Con}(\mathbb{I}) \longrightarrow \mathcal{C}([\mathsf{Con}(\mathbb{I}), d], d)$$

is the limiting cone.

Then by definition of limit if $\mathbb{I}$ is connected the counit above is an isomorphism and any cone $\gamma : \Delta x \Rightarrow \Delta d$ is limiting if and only if γ is an isomorphism. Moreover, as $\Delta\_$ is full and faithful there exists a unique isomorphism $x \overset{f}{\cong} d$ such that $\gamma = \Delta f$. Another observation here is that if $\mathbb{I}$ is connected then $\mathcal{C}$ has limits for all $\mathbb{I}$-indexed constant diagrams.

Lemma 4.3.1 Let $\mathbb{J}$ be a connected small category and $D\colon\mathbb{J} \to \mathcal{C}$ a diagram. For every limiting cone $\gamma : \Delta c \Rightarrow D$ the arrow $\varprojlim_{\mathbb{J}} \gamma$ is an isomorphism.

Proof. By definition $\varprojlim_{\mathbb{J}}\gamma$ is the unique arrow making the diagram

$$\begin{array}{ccc} \Delta\varprojlim_{\mathbb{J}}\Delta c & \xrightarrow{\Delta\varprojlim_{\mathbb{J}}\gamma} & \Delta\varprojlim_{\mathbb{J}} D \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\kappa} \\ \Delta c & \xrightarrow[\gamma]{} & D \end{array}$$

commute, where β and κ are the chosen limiting cones associated to $\varprojlim_{\mathbb{J}}\Delta c$ and $\varprojlim_{\mathbb{J}} D$ respectively. As $\mathbb{J}$ is connected there is an isomorphism $\varprojlim_{\mathbb{J}}\Delta c \overset{f}{\cong} c$ such that $\beta = \Delta f$. As γ and κ are limiting cones by Corollary 4.1.2 there is an isomorphism $g : c \cong \varprojlim_{\mathbb{J}} D$ such that $\gamma = \kappa \circ \Delta g$. Hence,

$$\kappa \circ \Delta g \circ \Delta f = \gamma \circ \Delta f = \kappa \circ \Delta\varprojlim_{\mathbb{J}}\gamma.$$

Since κ is limiting and $\Delta\_$ full and faithful

$$\varprojlim_{\mathbb{J}}\gamma = g \circ f.$$

Thus as f and g are isomorphisms it follows that $\varprojlim_{\mathbb{J}}\gamma$ is an isomorphism as well. □

The following theorem establishes that a natural isomorphism is enough to ensure preservation of limits of connected diagrams.

Theorem 4.3.2 Let $\mathbb{J}$ be a small category and $\mathcal{C}$,$\mathcal{D}$ categories with all $\mathbb{J}$-limits. If the functor $G : \mathcal{C} \to \mathcal{D}$ preserves $\mathbb{J}$-limits then

$$G(\varprojlim_{\mathbb{J}} D) \cong \varprojlim_{\mathbb{J}}(G \circ D)$$

natural in $D \in [\mathbb{J}, \mathcal{C}]$. Moreover, for $\mathbb{J}$ connected the converse also holds.

Proof. If G preserves limits then by definition the mediating arrow

$$G(\varprojlim_{\mathbb{J}} D) \to \varprojlim_{\mathbb{J}}(G \circ D)$$

for each $D \in [\mathbb{J}, \mathcal{C}]$ is an isomorphism and the family is a natural transformation.

Let $\mathbb{J}$ be a connected small category and assume there is an isomorphism

$$\varprojlim_{\mathbb{J}}(G \circ D) \overset{\theta_D}{\cong} G \circ \varprojlim_{\mathbb{J}} D$$

natural in D. Given a diagram $D : \mathbb{J} \to \mathcal{C}$ and a limiting cone $\gamma : \Delta c \Rightarrow D$ there is a unique morphism $m : G(c) \to \varprojlim_{\mathbb{J}} G \circ D$ such that the diagram

$$\begin{array}{ccc} \Delta G(c) & \xrightarrow{\Delta m} & \Delta\varprojlim_{\mathbb{J}}(G \circ D) \\ {\scriptstyle G\gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle\varepsilon} \\ G \circ D & \overset{\mathsf{id}}{\cong} & G \circ D \end{array}$$

commutes, where ε is the chosen limiting cone for $G \circ D$. We verify that m is an isomorphism as required.

The limiting cone $\gamma : \Delta c \Rightarrow D$ induces the naturality square

$$\begin{array}{ccc} G\big(\varprojlim_{\mathbb{J}} \Delta c\big) & \xrightarrow{\cong\theta} & \varprojlim_{\mathbb{J}}(\Delta G(c)) \\ {\scriptstyle G(\varprojlim_{\mathbb{J}}\gamma)}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\mathbb{J}} G\gamma} \\ G(\varprojlim_{\mathbb{J}} D) & \xrightarrow{\cong\theta} & \varprojlim_{\mathbb{J}}(G \circ D). \end{array}$$

Since $\mathbb{J}$ is connected by Lemma 4.3.1 the arrow $\varprojlim_{\mathbb{J}}\gamma$ is an isomorphism and then $G(\varprojlim_{\mathbb{J}}\gamma)$ is an isomorphism as well. From the naturality square above we can conclude $\varprojlim_{\mathbb{J}} G\gamma$ is an isomorphism.

By definition $\varprojlim_{\mathbb{J}} G\gamma$ is the unique mediating arrow making the diagram

$$\begin{array}{ccc} \Delta\varprojlim_{\mathbb{J}}(\Delta G(c)) & \xrightarrow{\Delta\varprojlim_{\mathbb{J}} G\gamma} & \Delta\varprojlim_{\mathbb{J}}(G \circ D) \\ {\scriptstyle \Delta h}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \Delta G(c) & \xrightarrow[G\gamma]{} & G \circ D \end{array}$$

commute where Δh is the chosen limiting cone associated to $\varprojlim_{\mathbb{J}}(G \circ \Delta c)$. Since $\mathbb{J}$ is connected h is an isomorphism. Hence,

$$G\gamma = \varepsilon \circ \Delta\varprojlim_{\mathbb{J}} G\gamma \circ (\Delta h)^{-1}$$

and by uniqueness of the mediating arrow

$$m = \varprojlim_{\mathbb{J}} G\gamma \circ h^{-1}$$

and m is an isomorphism. Thus G preserves the limiting cone γ. □

This proof rests on the key fact that $\varprojlim_{\mathbb{J}}\gamma$ is an isomorphism, or equivalently that the diagonal functor $\Delta\_$ is full and faithful. It is important to stress here that the Theorem above refers to any natural isomorphism and not necessarily to the canonical natural transformation defined by the limit. The theorem establishes that if there exists such a natural isomorphism then the canonical family is indeed a natural isomorphism as well. For some authors the isomorphism above implicitly refers to the canonical one stating a less general and straightforward result (for instance [Joh83, MM92]).

We can relax the conditions of this theorem to consider the case where not all $\mathbb{J}$-limits exist. Take instead a full subcategory $\mathcal{K} \subseteq [\mathbb{J}, \mathcal{C}]$ of diagrams whose limits exist in $\mathcal{C}$ and such that $\mathcal{K}$ includes all constant diagrams.

Corollary 4.3.3 Let $\mathbb{J}$ be a connected small category and $\mathcal{K}$ be a full subcategory of $[\mathbb{J}, \mathcal{C}]$ including all constant diagrams and such that $\mathcal{C}$ is $\mathcal{K}$-complete. Then G is $\mathcal{K}$-continuous if and only if

1. for every $D \in \mathcal{K}$, $G \circ D$ has a limit in $\mathcal{D}$; and
2. there exists an isomorphism $\varprojlim_{\mathbb{J}}(G \circ D) \cong G(\varprojlim_{\mathbb{J}} D)$ natural in $D \in \mathcal{K}$.

Proof. We use the proof of Theorem 4.3.2 within the subcategory $\mathcal{K}$. Notice that the expression $\varprojlim_{\mathbb{J}} D$ is functorial in D but the domain is $\mathcal{K}$ instead of $[\mathbb{J}, \mathcal{C}]$. As the indexing category $\mathbb{J}$ is connected the limits for constants diagrams exist in $\mathcal{C}$. □

Connectivity represents a significant constraint on diagrams. There are, however, many applications where connected limits (and colimits) are central and then the result above (and its dual) can be useful [Par90, Cat99, CW].

In general Theorem 4.3.2 does not hold when the indexing category is not connected. For an example consider the functor category $[\mathbf{2}, \mathbf{1}]$ were $\mathbf{2}$ is the two-objects discrete category. This functor category has a unique object: the constant diagram $\Delta\star$. Now consider the functor $G : \mathbf{1} \to \mathbf{Set}$ selecting a countable infinite set, say the natural numbers $\mathbf{N}$. As $\star \times \star = \star$ in $\mathbf{1}$, where the projections are given by the identity, we have

$$G(\star \times \star) \cong G(\star) \times G(\star)$$

natural in $\star$ since $\mathbf{N} \cong \mathbf{N} \times \mathbf{N}$. The pair $(\mathsf{id}_{\mathbf{N}}, \mathsf{id}_{\mathbf{N}})$, however, is not a product!

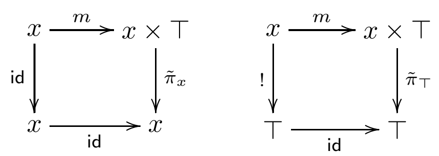
4.4 Products

Theorem 4.3.2 can not be applied to products since the index category in this case is discrete, an extreme example of lack of connectivity. Given a discrete category $\mathbb{K}$ a diagram $D : \mathbb{K} \to \mathcal{C}$ can be regarded as a tuple of objects $\langle x_k \rangle_{k \in \mathbb{K}}$ in $\mathcal{C}$ where $x_k = D(k)$. A cone for this functor is any family of arrows $\langle f_k : x \to x_k \rangle_{k \in \mathbb{K}}$ for some object x. Notice that as the index category is discrete there is no commutativity to check and naturality comes for free. We say that a family $\langle f_k : x \to x_k \rangle_{k \in \mathbb{K}}$ is a product when it is a limiting cone. In this section we study the conditions for a functor to preserve products. In the next section we see how these conditions are enough to ensure preservation of limits in the general case. In a category with terminal object $\top$ we use $! : c \to \top$ to denote the unique arrow from c to $\top$.

Remark 4.4.1 In a category $\mathcal{C}$ with a terminal object $\top$ the pair

$$(\mathsf{id} : x \to x, ! : x \to \top)$$

defines a unique mediating arrow $m : x \to x \times \top$ such that the diagrams



commute where $(\tilde{\pi}_x, \tilde{\pi}_\top)$ is the chosen product. It immediately follows that m is an isomorphism since

$$\tilde{\pi}_x \circ m = \mathsf{id}$$

by the left diagram, and then

$$\tilde{\pi}_x \circ m \circ \tilde{\pi}_x = \tilde{\pi}_x$$

which by uniqueness of the identity as mediating arrow implies

$$m \circ \tilde{\pi}_x = \mathsf{id}.$$

Thus by Corollary 4.1.2 and the squares above the pair

$$(\mathsf{id} : x \to x, ! : x \to \top)$$

is a product.

Proposition 4.4.2 Let $\mathcal{C}$, $\mathcal{D}$ be categories with finite products. The functor $G\colon \mathcal{C} \to \mathcal{D}$ preserves binary products if

1. G preserves terminal objects and
2. there is a section-retraction pair

$$G(x \times y) \underset{r_{x,y}}{\overset{s_{x,y}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} G(x) \times G(y)$$

natural in $x, y \in \mathcal{C}$ (here we mean both s and r are natural).

Proof. There is a morphism $\mathsf{id}_x \times ! \colon x \times y \to x \times \top$ in $\mathcal{C}$. This morphism defines the commuting squares in the diagram

$$\begin{array}{ccccccc}
 & & G(x \times y) & \underset{r_{x,y}}{\overset{s_{x,y}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} & G(x) \times G(y) & & \\
 & \swarrow^{G(\pi_x)} & \downarrow{\scriptstyle G(\mathsf{id}_x \times !)} & & \downarrow{\scriptstyle G(\mathsf{id}_x) \times G(!)} & \searrow^{\pi_{G(x)}} & \\
G(x) & \underset{G(\tilde{\pi}_x)}{\longleftarrow} & G(x \times \top) & \underset{r_{x,\top}}{\overset{s_{x,\top}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} & G(x) \times G(\top) & \underset{\tilde{\pi}_{G(x)}}{\longrightarrow} & G(x).
\end{array}$$

The left triangle commutes since it is obtained by applying G to the commuting triangle

$$\begin{array}{ccc}
 & & x \times y \\
 & \swarrow^{\pi_x} & \downarrow{\scriptstyle \mathsf{id}_x \times !} \\
x & \underset{\tilde{\pi}_x}{\longleftarrow} & x \times \top.
\end{array}$$

By Remark 4.4.1 the morphism $\tilde{\pi}_x$ is an isomorphism and then $G(\tilde{\pi}_x)$ is an isomorphism as well. The right triangle commutes by similar reasons. By assumption $G(\top)$ is a terminal object and from Remark 4.4.1 the arrow $\tilde{\pi}_{G(x)}$ is an isomorphism. Thus the compositions

$$s' = \tilde{\pi}_{G(x)} \circ s_{x,\top} \circ G(\tilde{\pi}_x)^{-1} \quad \text{and} \quad r' = G(\tilde{\pi}_x) \circ r_{x,\top} \circ \tilde{\pi}_{G(x)}^{-1}$$

give a section-retraction pair such that the squares in

$$\begin{array}{ccc}
G(x \times y) & \underset{r_{x,y}}{\overset{s_{x,y}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} & G(x) \times G(y) \\
{\scriptstyle G(\pi_x)}\downarrow & & \downarrow{\scriptstyle \pi_{G(x)}} \\
G(x) & \underset{r'}{\overset{s'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\lhd}}}} & G(x)
\end{array}$$

commute. We can follow the same argument with y instead of x. Then by Proposition 4.1.1 the pair

$$(G(\pi_x), G(\pi_y))$$

is a product. Notice that the mediating arrow defined by $(G(\pi_x), G(\pi_y))$ is an isomorphism and does not necessarily coincide with $s_{x,y}$. $\square$

An instance of this proposition is obtained by replacing the section-retraction pair by a natural isomorphism in x, y. In the example of the category **Count** of countable infinite sets there is no terminal object, moreover any choice for isomorphisms would not give a natural family and then the result above cannot be applied. We generalise the last proposition to $\mathbb{K}$-products where the naturality of the retract is required within a subcategory $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ of tuples.

Theorem 4.4.3 Let $\mathbb{K}$ be a discrete category and $\mathcal{C}$, $\mathcal{D}$ be categories with terminal object. Let $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ be a full subcategory of $\mathbb{K}$-tuples such that for every tuple $x = \langle x_k \rangle_{k\in\mathbb{K}} \in \mathcal{K}$ we have that all the $\mathbb{K}$-tuples obtained by fixing one component of x, say x_i, and replacing the rest by $\top$ are in $\mathcal{K}$. For example, for a $\mathbb{K}$-tuple $\langle x_k \rangle_k \in \mathcal{K}$ then $(x_1, \top, \ldots, \top) \in \mathcal{K}$, $(\top, x_2, \ldots, \top) \in \mathcal{K}$ and so on. The functor $G : \mathcal{C} \to \mathcal{D}$ preserves $\mathbb{K}$-products of tuples in $\mathcal{K}$ if

1. whenever $\prod_{k\in\mathbb{K}} x_k$ exists in $\mathcal{C}$ where $\langle x_k \rangle_k \in \mathcal{K}$ then $\prod_{k\in\mathbb{K}} G(x_k)$ exists in $\mathcal{D}$,
2. G preserves terminal objects and
3. there is a section-retraction pair

$$G(\textstyle\prod_{k\in\mathbb{K}} x_k) \underset{r_{\langle x_k \rangle_k}}{\overset{s_{\langle x_k \rangle_k}}{\rightleftarrows}} \textstyle\prod_{k\in\mathbb{K}} G(x_k)$$

natural in $\langle x_k \rangle_k \in \mathcal{K}$.

Proof. This just generalises the proof of the proposition above to $\mathbb{K}$-products of tuples within $\mathcal{K}$. It follows by fixing one component at a time and mapping all other components to the terminal object $\top$. □

Observe that this result can be applied for the subcategory $diag[\mathbb{K}, \mathcal{C}] \subseteq [\mathbb{K}, \mathcal{C}]$ of tuples for which the product exists in $\mathcal{C}$. This follows from the fact that products for tuples of the form $(\top, \ldots, \top, x_i, \top, \ldots, \top)$ exist in $\mathcal{C}$ and then they are in $diag[\mathbb{K}, \mathcal{C}]$ as required. In the next section we use this result in the special case where the section-retraction pairs are natural isomorphisms.

4.5 General Limits

A limit can be expressed in terms of conceptually simpler ones as for instance a product of equalisers. Another way is based on the connected components of the index category.

Proposition 4.5.1 Let $\mathbb{I} = \sum_{k\in\mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, $k \in \mathbb{K}$, its connected components. There is an isomorphism

$$[\mathbb{I}, \mathcal{C}]\big(H, F\big) \cong \prod_{k\in\mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(H \circ \iota_k, F \circ \iota_k\big)$$

natural in $H, F \in [\mathbb{I}, \mathcal{C}]$ where $\iota_k : \mathbb{I}_k \to \mathbb{I}$ is the inclusion functor.

Proof. The isomorphism takes a natural transformation $\alpha : H \Rightarrow F$ and splits it in the natural transformations $\alpha\,\iota_k : H \circ \iota_k \Rightarrow F \circ \iota_k$. Conversely a collection of natural transformations $\langle \beta_k : H \circ \iota_k \Rightarrow F \circ \iota_k \rangle_{k\in\mathbb{K}}$ gives a natural transformation $\beta : H \Rightarrow F$. This construction is clearly a bijection and it is preserved through pre- and post-composition and thus is natural in both variables. □

Proposition 4.5.2 Let $\mathbb{I} = \sum_{k\in\mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, $k \in \mathbb{K}$, its connected components. Let $D : \mathbb{I} \to \mathcal{C}$ be a small diagram. There is an isomorphism

$$\varprojlim_{\mathbb{I}} D \overset{f}{\cong} \prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$$

such that the diagram

$$\begin{array}{ccc}
\varprojlim_{\mathbb{I}} D & \overset{f}{\cong} & \prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k}(D\circ\iota_k) \\
 & & \big\downarrow{\scriptstyle \pi_k} \\
\big\downarrow{\scriptstyle \varepsilon_i} & & \varprojlim_{\mathbb{I}_k}(D\circ\iota_k) \\
 & & \big\downarrow{\scriptstyle \gamma_i^k} \\
D(i) & \overset{\mathsf{id}}{\cong} & D(i)
\end{array}$$

commutes for every $i \in \mathbb{I}_k$ where ε is the limiting cone associated to $\varprojlim_{\mathbb{I}} D$, γ^k is the limit cone associated to $\varprojlim_{\mathbb{I}_k}(D\circ\iota_k)$ and π_k is the projection associated to the product.

Proof. From Proposition 4.5.1 there is an isomorphism

$$[\mathbb{I},\mathcal{C}]\big(\Delta c, D\big) \cong \prod_{k\in\mathbb{K}} [\mathbb{I}_k,\mathcal{C}]\big(\Delta c, D\circ\iota_k\big)$$

natural in c. Hence,

$$\begin{aligned}
[\mathbb{I},\mathcal{C}]\big(\Delta c, D\big) &\cong \prod_{k\in\mathbb{K}} [\mathbb{I}_k,\mathcal{C}]\big(\Delta c, D\circ\iota_k\big) \\
&\cong \prod_{k\in\mathbb{K}} \mathcal{C}\big(c, \varprojlim_{\mathbb{I}_k}(D\circ\iota_k)\big) \quad \text{by definition of limit,} \\
&\cong \mathcal{C}\big(c, \prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k}(D\circ\iota_k)\big) \quad \text{since hom-functor preserves limits,}
\end{aligned}$$

and thus $\prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k}(D\circ\iota_k)$ is a limit for D. The commutativity of the diagram follows by the definition of the counit of this representation:

$$\mathsf{id} \longmapsto \langle \pi_k \rangle_{k\in\mathbb{K}} \longmapsto \langle \gamma^k \circ \Delta\pi_k \rangle_k \ .$$

□

The task now is to combine the results on products (Theorem 4.4.3) and on connected diagrams (Theorem 4.3.2) to treat preservation of more general limits. In order to do so we need two embeddings of functor categories which preserve naturality. Assuming $\mathcal{C}$ has terminal object $\top$, the first embedding is the right adjoint of $\_ \circ \iota_k$:

$$[\mathbb{I},\mathcal{C}] \underset{\_^{+}}{\overset{\_\circ\iota_k}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{I}_k,\mathcal{C}] \ . \tag{4.4}$$

Given $H : \mathbb{I}_k \to \mathcal{C}$, the functor $H^+ : \mathbb{I} \to \mathcal{C}$ is such that acts as H over the component $\mathbb{I}_k$ and as the constant functor $\Delta\top$ otherwise. The unit of the adjunction above is defined for $D \in [\mathbb{I},\mathcal{C}]$ as

$$(\eta_D)_i = \begin{cases} \mathsf{id}_{D(i)} & \text{if } i \in \mathbb{I}_k \\ ! & \text{otherwise (the unique arrow from } D(i) \text{ to } \top) \end{cases}$$

which is clearly universal.

Proposition 4.5.3 Let $\mathbb{I}_k$ be a connected component of $\mathbb{I}$. Given categories $\mathcal{C}$ and $\mathcal{D}$ with terminal objects and a functor $G : \mathcal{C} \to \mathcal{D}$ that preserves terminal objects. For every connected component $\mathbb{I}_k$ of $\mathbb{I}$:

1. there is an isomorphism $\varprojlim_{\mathbb{I}} H^+ \cong \varprojlim_{\mathbb{I}_k} H$, and
2. there is an isomorphism $\varprojlim_{\mathbb{I}} G \circ H^+ \cong \varprojlim_{\mathbb{I}_k} G \circ H$.

In both cases we mean that if one side of the isomorphism exists then so does the other. The isomorphisms are natural in $H \in \mathcal{K}$ for a subcategory $\mathcal{K} \subseteq [\mathbb{I}_k, \mathcal{C}]$ such that $\mathcal{C}$ is $\mathcal{K}$-complete.

Proof. For 1 consider the chain of isomorphisms

$$\begin{aligned} \mathcal{C}(c, \varprojlim_{\mathbb{I}} H^+) &\cong [\mathbb{I}, \mathcal{C}]\big(\Delta c, H^+\big) && \text{by definition of limit,} \\ &\cong [\mathbb{I}_k, \mathcal{C}]\big((\Delta c) \circ \iota_k, H\big) && \text{by the adjunction (4.4),} \\ &= [\mathbb{I}_k, \mathcal{C}]\big(\Delta c, H\big), \end{aligned}$$

all natural in c and $H \in \mathcal{K}$. Then by definition $\varprojlim_{\mathbb{I}} H^+$ is a limit for H and then there is an isomorphism

$$\varprojlim_{\mathbb{I}} H^+ \cong \varprojlim_{\mathbb{I}_k} H$$

that from parametrised representability (Theorem 2.1.6) is natural in $H \in \mathcal{K}$.

For 2 observe that since G preserves the terminal objects it is possible to define an adjunction as (4.4) with $\mathcal{D}$ as codomain where $G \circ H^+ \cong (G \circ H)^+$. Thus we have

$$\begin{aligned} \mathcal{D}(d, \varprojlim_{\mathbb{I}} G \circ H^+) &\cong [\mathbb{I}, \mathcal{D}]\big(\Delta d, G \circ H^+\big) && \text{by definition of limit,} \\ &\cong [\mathbb{I}, \mathcal{D}]\big(\Delta d, (G \circ H)^+\big) \\ &\cong [\mathbb{I}_k, \mathcal{D}]\big((\Delta d) \circ \iota_k, G \circ H\big) && \text{by the adjunction (4.4),} \\ &= [\mathbb{I}_k, \mathcal{D}]\big(\Delta d, G \circ H\big). \end{aligned}$$

Then $\varprojlim_{\mathbb{I}} G \circ H^+$ is a limit for $G \circ H$ and then there is an isomorphism

$$\varprojlim_{\mathbb{I}} G \circ H^+ \cong \varprojlim_{\mathbb{I}_k} G \circ H$$

natural in H as well. □

There is a less obvious embedding $\mathbf{\Delta} : [\mathbb{K}, \mathcal{C}] \to [\mathbb{I}, \mathcal{C}]$ where $\mathbb{K}$ is the discrete category whose objects are identified with the connected components of $\mathbb{I}$. Given a tuple $\langle x_k \rangle_{k \in \mathbb{K}}$, the functor $\mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} : \mathbb{I} \to \mathcal{C}$ acts as the constant Δx_k over the objects and arrows in $\mathbb{I}_k$.

Proposition 4.5.4 Let $G : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, $k \in \mathbb{K}$, its connected components:

1. there is an isomorphism $\varprojlim_{\mathbb{I}} \mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \cong \prod_{k \in \mathbb{K}} x_k$, and
2. there is an isomorphism $\varprojlim_{\mathbb{I}} G \circ \mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \cong \prod_{k \in \mathbb{K}} G(x_k)$.

In both cases we mean that if one side of the isomorphism exists then so does the other. The isomorphisms are natural in $\langle x_k \rangle_k \in \mathcal{K}$ for a subcategory of $\mathbb{K}$-tuples $\mathcal{K} \subseteq [\mathbb{K}, \mathcal{C}]$ such that $\mathcal{C}$ is $\mathcal{K}$-complete.

Proof. For 1

$$\begin{aligned} \mathcal{C}(c, \varprojlim_{\mathbb{I}} \mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}}) &\cong [\mathbb{I}, \mathcal{C}]\big(\Delta c, \mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}}\big) && \text{by definition of limit,} \\ &\cong \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(\Delta c \circ \iota_k, \mathbf{\Delta}\langle x_k \rangle_{k \in \mathbb{K}} \circ \iota_k\big) && \text{by Proposition 4.5.1,} \\ &= \prod_{k \in \mathbb{K}} [\mathbb{I}_k, \mathcal{C}]\big(\Delta c, \Delta_{\mathbb{I}_k} x_k\big) && \text{by definition of } \mathbf{\Delta}, \\ &= [\mathbb{K}, \mathcal{C}]\big(\Delta c, \langle x_k \rangle_{k \in \mathbb{K}}\big). \end{aligned}$$

Then $\varprojlim_{\mathbb{I}} \mathbf{\Delta}\langle x_k\rangle_{k\in\mathbb{K}}$ is isomorphic to $\prod_{k\in\mathbb{K}} x_k = \varprojlim_{\mathbb{K}} \langle x_k\rangle_{k\in\mathbb{K}}$ and naturality follows from parametrised representability. In a similar way by using the identity

$$G \circ \Delta x = \Delta G(x)$$

we can prove 2. □

Theorem 4.5.5 Let $\mathcal{C}$, $\mathcal{D}$ be complete categories. A functor $G{:}\mathcal{C} \to \mathcal{D}$ is continuous if and only if for any small category $\mathbb{I}$ there is an isomorphism

$$G(\varprojlim_{\mathbb{I}} D) \cong \varprojlim_{\mathbb{I}} (G \circ D) \tag{4.5}$$

natural in $D \in [\mathbb{I}, \mathcal{C}]$.

Proof. The *"only-if"* follows as usual. Now for the *"if"* part first observe that G trivially preserves terminal objects: take $\mathbb{I}$ to be the empty category. Let $\mathbb{I} = \sum_{k\in\mathbb{K}} \mathbb{I}_k$ be a non-empty small category with $\mathbb{I}_k$, $k \in \mathbb{K}$, its connected components. Given a diagram $D : \mathbb{I} \to \mathcal{C}$ and a limiting cone $\varepsilon : \Delta\varprojlim_{\mathbb{I}} D \Rightarrow D$ from Proposition 4.5.2 there is an isomorphism

$$\varprojlim_{\mathbb{I}} D \overset{f}{\cong} \prod_{k\in\mathbb{K}} \varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$$

such that for every $i \in \mathbb{I}$

$$\varepsilon_i = \gamma_i^k \circ \pi_k \circ f$$

where γ^k is the limiting cone associated to $\varprojlim_{\mathbb{I}_k} (D \circ \iota_k)$. In order to prove that $G\varepsilon$ is limiting by Corollary 4.1.2 is enough to verify that the cone defined at component $i \in \mathbb{I}_k$ by the arrow

$$G(\gamma_i^k) \circ G(\pi_k)$$

is limiting. This is equivalent to requiring that $\langle G(\pi_k)\rangle_{k\in\mathbb{K}}$ is a product and that for every $\mathbb{I}_k$ the natural transformation $G\gamma^k$ is a limiting cone.

Given a family of arrows $\langle \alpha_k : x_k \to x'_k\rangle_{k\in\mathbb{K}}$ (*i.e.* a natural transformation $\alpha : \langle x_k\rangle_k \Rightarrow \langle x'_k\rangle_k$ in $[\mathbb{K}, \mathcal{C}]$) there is a diagram

$$\begin{array}{ccccccc}
G(\prod_{k\in\mathbb{K}} x_k) & \xleftarrow{\cong} & G(\varprojlim_{\mathbb{I}} \mathbf{\Delta}\langle x_k\rangle_k) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}} (G \circ \mathbf{\Delta}\langle x_k\rangle_k) & \xrightarrow{\cong} & \prod_{k\in\mathbb{K}} G(x_k) \\
{\scriptstyle G(\langle\alpha_k\rangle_k)}\downarrow & (1) & \downarrow{\scriptstyle G(\varprojlim_{\mathbb{I}} \mathbf{\Delta}\alpha)} & (2) & \downarrow{\scriptstyle \varprojlim_{\mathbb{I}} (G(\mathbf{\Delta}\alpha))} & (3) & \downarrow{\scriptstyle \langle G(\alpha_k)\rangle_k} \\
G(\prod_{k\in\mathbb{K}} x'_k) & \xleftarrow{\cong} & G(\varprojlim_{\mathbb{I}} \mathbf{\Delta}\langle x'_k\rangle_k) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}} (G \circ \mathbf{\Delta}\langle x'_k\rangle_k) & \xrightarrow{\cong} & \prod_{k\in\mathbb{K}} G(x'_k)
\end{array}$$

where (1) and (3) commute by Proposition 4.5.4 and (2) is a naturality square defined by (4.5). Thus by pasting the squares we have that there is an isomorphism

$$G(\prod_{k\in\mathbb{K}} x_k) \cong \prod_{k\in\mathbb{K}} G(x_k)$$

natural in $\langle x_k\rangle_k$. Then by Theorem 4.4.3 G preserves $\mathbb{K}$-products and $\langle G(\pi_k)\rangle_{k\in\mathbb{K}}$ above is a product.

Given a natural transformation $\beta : H \Rightarrow K$ for functors $H, K{:}\mathbb{I}_k \to \mathcal{C}$ there is a diagram

$$\begin{array}{ccccccc}
G(\varprojlim_{\mathbb{I}_k} H) & \xleftarrow{\cong} & G(\varprojlim_{\mathbb{I}} H^+) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}} (G \circ H^+) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}_k} (G \circ H) \\
{\scriptstyle G(\varprojlim_{\mathbb{I}_k} \beta)}\downarrow & (1) & \downarrow{\scriptstyle G(\varprojlim_{\mathbb{I}} \beta^+)} & (2) & \downarrow{\scriptstyle \varprojlim_{\mathbb{I}} (G\beta^+)} & (3) & \downarrow{\scriptstyle \varprojlim_{\mathbb{I}_k} G\beta} \\
G(\varprojlim_{\mathbb{I}_k} K) & \xleftarrow{\cong} & G(\varprojlim_{\mathbb{I}} K^+) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}} (G \circ K^+) & \xrightarrow{\cong} & \varprojlim_{\mathbb{I}_k} (G \circ K)
\end{array}$$

where (1) and (3) commute by Proposition 4.5.3 and (2) is a naturality square defined by (4.5). Thus by pasting the squares we have that there is an isomorphism

$$\varprojlim_{\mathbb{I}_k}(G \circ H) \cong G(\varprojlim_{\mathbb{I}_k} H)$$

natural in $H \in [\mathbb{I}_k, \mathcal{C}]$. Thus by Theorem 4.3.2 G preserves $\mathbb{I}_k$-limits and $G(\gamma^k)$ above is a is a limit for $G(D \circ \iota_k)$. □

Consider the full subcategory of diagrams $diag[\mathbb{I}, \mathcal{C}] \subseteq [\mathbb{I}, \mathcal{C}]$ for which all limits exist in $\mathcal{C}$. For every connected component $\mathbb{I}_k$ of $\mathbb{I}$ define $\mathcal{K}_k \subseteq [\mathbb{I}_k, \mathcal{C}]$ to be the full subcategory induced by the diagrams of the form $D \circ \iota_k$ for some $D \in diag[\mathbb{I}, \mathcal{K}]$. Given a diagram $D \in diag[\mathbb{I}, \mathcal{C}]$ by Proposition 4.5.2 there is a isomorphism

$$\varprojlim_{\mathbb{I}} D \cong \prod_{k \in \mathbb{K}} \varprojlim_{\mathbb{I}_k}(D \circ \iota_k)$$

and thus $D \circ \iota_k$ has limit in $\mathcal{C}$ and then is an object of $diag[\mathbb{I}_k, \mathcal{C}]$. Given a diagram $H \in diag[\mathbb{I}_k, \mathcal{C}]$ by Proposition 4.5.3 we have that $H^+ \in diag[\mathbb{I}, \mathcal{C}]$ and moreover $H^+ \circ \iota_k = H$. This correspondence is one-to-one and then there is an isomorphism of categories

$$\mathcal{K}_k \cong diag[\mathbb{I}_k, \mathcal{C}].$$

Notice that probably not all $\mathbb{K}$-products exist in $\mathcal{C}$, but by Proposition 4.5.2 we can ensure that at least the subcategory $diag[\mathbb{K}, \mathcal{C}]$ of $\mathbb{K}$-tuples whose product exist in $\mathcal{C}$ includes the tuples needed to reconstruct the limits for the diagrams in $diag[\mathbb{I}, \mathcal{D}]$. Then we can relax the completeness condition in Theorem 4.5.5:

Theorem 4.5.6 Let $\mathbb{I} = \sum_{k \in \mathbb{K}} \mathbb{I}_k$ be a small category with $\mathbb{I}_k$, $k \in \mathbb{K}$, its connected components. A functor $G : \mathcal{C} \to \mathcal{D}$ preserves the limits of the diagrams in $diag[\mathbb{I}, \mathcal{C}]$ if

1. $G \circ D$ has a limit in $\mathcal{D}$ for every $D \in diag[\mathbb{I}, \mathcal{C}]$,
2. G preserves terminal objects, and
3. $\varprojlim_{\mathbb{I}}(G \circ D) \cong G(\varprojlim_{\mathbb{I}} D)$ natural in $D \in diag[\mathbb{I}, \mathcal{C}]$.

Proof. The proof follows as in the Theorem 4.5.5 but within the categories $diag[\mathbb{I}_k, \mathcal{C}]$ for each $k \in \mathbb{K}$ and $diag[\mathbb{K}, \mathcal{C}]$. Given that G preserves terminal objects we can apply Propositions 4.5.3 and 4.5.4. □

4.6 Some Examples

4.6.1 Adjunctions and limits

The relation between limits and adjunctions is determined by the preservation properties of left and right adjoints.

Proposition 4.6.1 Given an adjunction and diagram H as in

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D} \xleftarrow{\;H\;} \mathbb{I}.$$

If H has a limit in $\mathcal{D}$ then the functor $G \circ H : \mathbb{I} \to \mathcal{C}$ has a limit in $\mathcal{C}$.

Proof. The limit for $G \circ H$ is just given by $G(\varprojlim_{\mathbb{I}} H)$:

$$\begin{aligned}
\mathcal{C}\big(c, G(\varprojlim_{\mathbb{I}} H)\big) &\cong \mathcal{D}\big(F(c), \varprojlim_{\mathbb{I}} H\big) && \text{by the adjunction,}\\
&\cong \varprojlim_{\mathbb{I}} \mathcal{D}\big(F(c), H(-)\big) && \text{by Propositions 2.2.8 and pointwise computation,}\\
&\cong \varprojlim_{\mathbb{I}} \mathcal{C}\big(c, (G \circ H)(-)\big) && \text{by the adjunction,}\\
&\cong \mathcal{C}\big(c, \varprojlim_{\mathbb{I}} (G \circ H)(-)\big) && \text{by Propositions 2.2.8 and pointwise computation,}\\
&\cong [\mathbb{I}, \mathcal{C}]\big(\Delta c, G \circ H\big) && \text{by definition of limits,}
\end{aligned}$$

all natural in c. □

Theorem 4.6.2 Right adjoints are continuous functors.

Proof. A direct consequence of the proof of the proposition above. The counit of the representation is given by

$$\mathsf{id} \longmapsto \varepsilon \longmapsto \gamma_i \circ \varepsilon \longmapsto \langle G(\gamma_i)\rangle_i$$

where ε is the counit of the adjunction and γ is the universal cone associated to $\varprojlim_{\mathbb{I}} H$. Observe that if $\mathcal{C}$ has terminal object by Theorem 4.5.5 the proposition above is enough to prove the theorem since the chain of isomorphism is natural in $H \in \mathit{diag}[\mathbb{I}, \mathcal{C}]$ as well. □

By duality the discussion on preservation of limits in this chapter exports to colimits. Of course the condition on preservation of terminal objects is substituted by the preservation of initial objects. The dual of Theorem 4.6.2 says that left adjoints preserve colimits.

The Fubini theorem gives a sense for which the formation of limits preserve universal properties. In a category with enough limits there is limit functor, and the Fubini result establishes that this functor preserve limits. This is consistent with the definition of limits as a representation where for a complete category the isomorphism

$$\mathcal{C}(c, \varprojlim_{\mathbb{I}} D) \cong [\mathbb{I}, \mathcal{C}](\Delta c, D)$$

is natural in c and D defining an adjunction $\Delta - \vdash \varprojlim_{\mathbb{I}} -$. As a right adjoint the functor $\varprojlim_{\mathbb{I}} -$ preserves limits. By a dual argument the functor $\mathrm{colim}_{\mathbb{I}} -$ preserves colimits.

4.6.2 Profunctors

For a functor $F : \mathbb{C} \to \mathcal{D}$ where $\mathcal{D}$ is cocomplete there is a functor

$$F_! : [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] \to \mathcal{D}$$

where $F_!$ is the left Kan extension of F along the Yoneda embedding, *i.e.*

$$\begin{aligned}
F_! &= \lambda G. \textstyle\int^x [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\mathbb{C}(-, x), G) \otimes F(x) && \text{by (3.14),}\\
&= \lambda G. \textstyle\int^x \int_y [\mathbb{C}(y, x), G(y)] \otimes F(x) && \text{by naturality formula,}\\
&= \lambda G. \textstyle\int^x G(x) \otimes F(x) && \text{by the Yoneda lemma.}
\end{aligned}$$

The functor $F_!$ preserves colimits: for a diagram $D : \mathbb{I} \to \mathbf{Set}$ for convenience we write $\mathrm{colim}_{\mathbb{I}} D$ as $\int^y D(y)$ where the contravariant argument is dummy. The result follows from

$$
\begin{aligned}
F_!(\textstyle\int^y D(y)) &= \textstyle\int^x (\int^y D(y))(x) \otimes F(x) && \text{by definition of } F_!,\\
&\cong \textstyle\int^x \int^y G_x(y) \otimes F(x) && \text{by pointwise computation where } G = \boldsymbol{\lambda}^{-1}(D),\\
&\cong \textstyle\int^y \int^x G_x(y) \otimes F(x) && \text{by Fubini for coends,}\\
&= \textstyle\int^y F_!(G^y) && \text{by definition of } F_!,\\
&= \textstyle\int^y F_!(D(y)).
\end{aligned}
$$

In fact the functor $F_!$ is a left adjoint where the right adjoint is given by

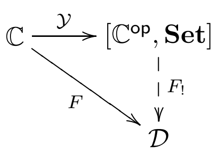
$$F^! = \lambda y.\lambda x.\mathcal{D}(F(x), y).$$

In the special case where $\mathcal{D}$ is a presheaf the functor $F^!$ has a right adjoint itself given by the right Kan extension of F. This is an example of geometric morphism between topoi [MM92, Joh02].

By the density formula there is an isomorphism

$$F \cong \lambda x.\textstyle\int^y F(y) \otimes \mathbb{C}(y, x) \cong F_! \circ \mathcal{Y}.$$

In fact $F_!$ is unique with this property exhibiting the category of presheaves $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ as a concrete colimit completion of $\mathbb{C}$. More formally, for any functor $F : \mathbb{C} \to \mathcal{D}$ where $\mathcal{D}$ is a category with all colimits $F_!$ is the *unique* up to isomorphism colimit-preserving functor such that the diagram



commutes. It remains to check uniqueness. Suppose that $H : [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] \to \mathcal{D}$ is a colimit-preserving functor such that $H \circ \mathcal{Y} \cong F$. Then for $G : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$

$$
\begin{aligned}
H(G) &\cong H\left(\lambda y.\textstyle\int^x G(x) \otimes \mathbb{C}(y, x)\right) && \text{by density formula,}\\
&\cong H\left(\textstyle\int^x G(x) \otimes \mathcal{Y}(x)\right) && \text{by componentwise computation,}\\
&\cong \textstyle\int^x G(x) \otimes H(\mathcal{Y}(x)) && \text{since } H \text{ preserves colimits,}\\
&\cong \textstyle\int^x G(x) \otimes F(x) && \text{since } H \circ \mathcal{Y} \cong F\\
&\cong F_!(G) && \text{by definition of } F_!,
\end{aligned}
$$

then $H \cong F_!$.

Presheaf categories can be organised themselves in a 2-category. The objects of this category are presheaves and since they are free cocompletions the obvious choices for the arrows are the cocontinuous functors and the two-cells are the corresponding natural transformations. This category is referred in the literature as **Cocont** [Cat99, CW]. As presheaves are colimit completions, a functor

$$[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] \longrightarrow [\mathbb{D}^{\mathsf{op}}, \mathbf{Set}]$$

corresponds to a functor

$$\mathbb{C} \longrightarrow [\mathbb{D}^{\mathsf{op}}, \mathbf{Set}]$$

and then to a functor

$$\mathbb{C} \times \mathbb{D}^{\mathsf{op}} \longrightarrow \mathbf{Set}$$

a so-called *profunctor* (distributor [Bor94a, Bén00], bimodule [Wag96]) from $\mathbb{C}$ to $\mathbb{D}$. Profunctors are morphism of the bicategory category **Prof** where objects are small categories. The composition of profunctors is defined by the coend formula

$$G \circ F = \lambda c, i. \int^{x} F(c, x) \otimes G(x, i).$$

Since universal properties are unique up to isomorphism from this definition the composition is not strictly associative. A categorical structure of this form is called a *bicategory* [Bén78, Lei].

If presheaves are often regarded as sets with extra structure, then profunctors correspond to relations [Law73, Bor94a]. In computer science there is another analogy of **Prof** as a category of non-deterministic domains [Cat99, CW] which comes equipped with a built-in notion of bisimulation given by open maps [JNW96]. In this approach constructions on presheaves correspond to power domain constructions. A central result in this theory of domains for concurrency is that connected colimit-preserving functors between presheaves preserve open-map bisimulation.

Chapter 5

A Term Calculus for Functors

In this chapter we present a typed language for categories based on the functorial expressions used along the previous chapters. The syntax for functors extends the simply typed lambda calculus with new binders for ends and coends and term constructors for hom-functors, powers and copowers. The types of the language correspond to locally small categories and a derivation for a well-typed expression is interpreted as a functor in **CAT**. The calculus supports a syntactic treatment of duality which allows us to derive a functor F from its dual version F^{op}. This formalisation of duality results in a non-standard notion of substitution for terms. The proof system includes explicit structural rules to manipulate contexts. As a consequence well-typed expressions can be proved in the theory through different derivations. We show the syntax is coherent with respect to the interpretation up to natural isomorphism.

5.1 The Language

5.1.1 Basic Types

Types in the language denote locally small categories. We consider a basic set of type constructors necessary to present the core syntax for terms. In the presentation of the language we deliberately confuse "types" with "categories". There is a type **Set** to denote the category of sets and functions. Given the categories $\mathcal{C}$ and $\mathcal{D}$

- the opposite category $\mathcal{C}^{\mathsf{op}}$ is the category whose objects are the objects of $\mathcal{C}$ and whose arrows are the arrows in $\mathcal{C}$ reversed, *i.e.* $\mathcal{C}^{\mathsf{op}}(a, b) = \mathcal{C}(b, a)$ for any pair of objects a, b; and
- the functor category $[\mathcal{C}, \mathcal{D}]$ is the category of functors from $\mathcal{C}$ to $\mathcal{D}$ and the natural transformation between them where $\mathcal{C}$ must be small.

The syntax for types $\mathcal{C}, \mathcal{D}, \ldots$ as locally small categories is

$$\mathcal{C}, \mathcal{D}, \ldots ::= \mathbf{Set} \mid \mathcal{C}^{\mathsf{op}} \mid [\mathcal{C}, \mathcal{D}].$$

In order to properly introduce functor categories we must be able to decide whether a category is small. Similarly the rules for limits and ends shall require to decide if a category is complete. We assume that when talking about types there are judgements of form

$$\mathcal{C}\ \mathsf{small} \quad \text{and} \quad \mathcal{D}\ \mathsf{complete}.$$

5.1.2 Syntax and Type Theory

The language for expressions extends the simply typed λ-calculus *à la Church* [Bar92] with new constructions and binders. The treatment of contravariance and duality determines a non-standard definition of substitution. The system is shown to be closed under substitution. Finally we define a normal form for derivations where the use of duality is permitted only in the essential cases.

The raw expressions $E_1, E_2, \cdots$ of the language for functors are defined by

$$E ::= x \mid 1 \mid \lambda x^{\mathcal{C}}.E \mid E_1 E_2 \mid \mathcal{C}(E_1, E_2) \mid [E_1, E_2] \mid E_1 \otimes E_2 \mid$$
$$\int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E \mid \int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E$$

where $\mathcal{C}$, $\mathcal{D}$ are types, and x is drawn from a countably infinite set of variables. The constant 1 stands for the singleton set. The "integral" binders denote ends and coends as expected. There are also special syntactic constructors for hom-expressions, powers and copowers.

Not all possible expressions in the language give rise to functors. Furthermore the formation of functor categories and the computation of ends are restricted to small categories. These constraints impose a discipline to determine the well-typed expressions, *i.e.* expressions which are functorial in their free variables.

The syntactic judgement or sequent

$$x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}$$

says that the expression E of type $\mathcal{C}$ is functorial in the free variables occurring in the context $x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n$. In informal mathematical usage one says $E(x_1, \cdots, x_n)$ is functorial in $x_1, \cdots, x_n$. Thus an expression-in-context has two possible readings: as an object when the free variables are interpreted as objects, and as an arrow when the free variables are interpreted as arrows. Of course this is consistent with the definition of functor.

A context is a finite *sequence* of variable declarations. The typing rules are given in a *context-independent* presentation [TS00, Remark 3.1.5], which simplifies the form for the rules which involve contravariance. As usual we assume that different contexts have a disjoint set of variables, and the notation $\Gamma, z : \mathcal{C}$ assumes that the variable z is not in Γ.

Basic Axioms

singleton

$$\frac{}{\vdash 1 : \mathbf{Set}}$$

identity

$$\frac{}{x : \mathcal{C} \vdash x : \mathcal{C}}$$

Structural Rules

weakening

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n, x_{n+1} : \mathcal{C}_{n+1} \vdash E : \mathcal{C}}$$

exchange

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_i : \mathcal{C}_i, x_{i+1} : \mathcal{C}_{i+1}, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_{i+1} : \mathcal{C}_{i+1}, x_i : \mathcal{C}_i, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}$$

contraction

$$\frac{\Gamma, x : \mathcal{C}, y : \mathcal{C} \vdash E : \mathcal{D}}{\Gamma, z : \mathcal{C} \vdash E[z/x, z/y] : \mathcal{D}}$$

The structural rules play an important role in the proof of coherence and later in the enriched setting. The expression $E[z/x, z/y]$ in the rule **contraction** denotes the expression E where all occurrences of x and y are replaced by z with the usual treatment of bound variables. A special notion for substitution of terms for variables is later required for the most general case. The structural rules can be absorbed if the axioms above are replaced by

$$\frac{x : \mathcal{C} \in \Gamma}{\Gamma \vdash x : \mathcal{C}} \quad \text{and} \quad \frac{}{\Gamma \vdash 1 : \mathbf{Set}},$$

(refer to [Gir87, TS00, RS92, Bar92]). In the case of exchange is enough to consider the contexts to be multisets of variables declarations instead of sequences.

Hom-sets and Powers

The type **Set** has a special role as the base type for hom-expressions:

hom

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \qquad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}.}$$

In general Γ^{op} denotes the context obtained from Γ where each occurrence of a type $\mathcal{C}$ is replaced by $\mathcal{C}^{\mathsf{op}}$. We insist, also in the syntax, that $(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} = \mathcal{C}$. For instance, given the context

$$\Gamma = x : \mathcal{C}, y : \mathcal{D}^{\mathsf{op}}$$

we have

$$\Gamma^{\mathsf{op}} = x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{D}.$$

This rule captures the usual practice in category theory where the contravariance in the first argument of $\mathcal{C}(E_1, E_2)$ is reflected in the free variables rather than dualising the expression E_1.

If $\mathcal{C}$ is complete then it has all powers:

power

$$\frac{\Gamma_1 \vdash E_1 : \mathbf{Set} \qquad \Gamma_2 \vdash E_2 : \mathcal{C} \qquad \mathcal{C} \text{ complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] : \mathcal{C}.}$$

This rule asserts that the power is functorial in all free variables. In the special case of the type **Set** we have both

$$\frac{\begin{matrix}\vdots & & \vdots \\ \Gamma_1 \vdash E_1 : \mathbf{Set} & & \Gamma_2 \vdash E_2 : \mathbf{Set}\end{matrix}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathbf{Set}(E_1, E_2) : \mathbf{Set}}$$

and

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma_1 \vdash E_1 : \mathbf{Set}\end{array} \quad \begin{array}{c}\vdots \\ \Gamma_2 \vdash E_2 : \mathbf{Set}\end{array} \quad \mathbf{Set}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] : \mathbf{Set}}$$

to keep with practice we identify these two expressions in the syntax

$$\mathbf{Set}(E_1, E_2) = [E_1, E_2].$$

Functor Categories

The rule for lambda abstraction introduces functor categories:

lambda

$$\dfrac{\Gamma, x : \mathcal{C} \vdash E : \mathcal{D} \qquad \mathcal{C}\ \mathsf{small}}{\Gamma \vdash \lambda x^{\mathcal{C}}.E : [\mathcal{C}, \mathcal{D}].}$$

The side condition "$\mathcal{C}$ small" ensures the new type stays in the realm of locally small categories. Sometimes in order to simplify the notation we write $\mathbb{C}$ to mean the category is proved to be small. In general the type $[\mathcal{C}, \mathcal{D}]$ implicitly assumes that $\mathcal{C}$ small.

There is a symmetric rule for eliminating functor categories:

application

$$\dfrac{\Gamma_1 \vdash E_1 : [\mathcal{C}, \mathcal{D}] \quad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 : \mathcal{D}.}$$

Ends

The "integral" binder captures two variables at once in the expression under its scope. In the terminology of binder signatures [Plo90, FPT99] the integral is an operation of arity $\langle 2 \rangle$ (whereas lambda abstraction has arity $\langle 1 \rangle$).

end

$$\dfrac{\Gamma, x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash E : \mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E : \mathcal{D}.}$$

The order of the variables in the binder is immaterial, thus there is no difference between $\int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E$ and $\int_{y^{\mathcal{C}}, x^{\mathcal{C}^{\mathsf{op}}}} E$. This notation for ends, a bit more formal that the notation used in the previous chapters, allows us to keep track of variance and the type of the bound variables. The rule **end** says that the process of taking ends (or limits) does not disturb functoriality on the variables which remain free. The completeness requirement over $\mathcal{D}$ ensures that for all instances of the free parameters the end exists. The integral notation is also used for denoting limits by omitting the dummy argument, we use for instance $\int_{y^{\mathcal{C}}} E$. We could here introduce the rule for coends, instead we show how the rules above together with a simple rule for duality is enough to derive the formation for coends and copowers.

From the point of view of the logic the rule **end** can be regarded as a structural rule since at the level of types it only involves the manipulation of the assumptions. On the other hand it introduces a new element in the syntax. A similarly situation is found in some linear calculus where the structural rules are represented explicitly in the syntax [BBdPH93, Bie95].

Duality

Like reflection in a mirror, dualising affects our view of things and the way we describe them. Informally a judgement $\Gamma \vdash E : \mathcal{D}$ denotes a functor whose dual is described by a dual form of judgement:

dual

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}}$$

where E^* is obtained by turning ends into coends and coends into ends in E, and similarly for prowers and copowers.

Definition 5.1.1 The meta-operation $(\_)^*$ on raw expressions is defined inductively as follows:

$$\begin{aligned}
x^* &= x \qquad (x \text{ is a variable}) & 1^* &= 1\\
(\lambda x^{\mathcal{C}}.E)^* &= \lambda x^{\mathcal{C}^{\mathsf{op}}}.E^* & (E_1E_2)^* &= E_1^*E_2^*\\
\mathcal{C}(E_1,E_2)^* &= \mathcal{C}(E_1^*,E_2^*) & &\\
[E_1,E_2]^* &= E_1 \otimes E_2^* & (E_1\otimes E_2)^* &= [E_1,E_2^*]\\
(\textstyle\int_{x^{\mathcal{C}^{\mathsf{op}}},y^{\mathcal{C}}} E)^* &= \textstyle\int^{y^{\mathcal{C}^{\mathsf{op}}},x^{\mathcal{C}}} E^* & (\textstyle\int^{x^{\mathcal{C}^{\mathsf{op}}},y^{\mathcal{C}}} E)^* &= \textstyle\int_{y^{\mathcal{C}^{\mathsf{op}}},x^{\mathcal{C}}} E^*.
\end{aligned}$$

For example, taking the judgement

$$z : \mathcal{A} \vdash \textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}},y^{\mathbb{C}}} E : \mathcal{D}$$

the application of the rule **dual** yields

$$z : \mathcal{A}^{\mathsf{op}} \vdash \textstyle\int^{y^{\mathbb{C}^{\mathsf{op}}},x^{\mathbb{C}}} E^* : \mathcal{D}^{\mathsf{op}}$$

where E is an expression with free variables amongst x, y, z. For another example consider the judgement

$$x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash \mathcal{C}(x,y) : \mathbf{Set}$$

by applying the rule **dual** we obtain

$$x : \mathcal{C}, y : \mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x,y) : \mathbf{Set}^{\mathsf{op}}$$

where the covariance of x and contravariance of y reflect the change of orientation of the morphisms in $\mathbf{Set}^{\mathsf{op}}$.

Proposition 5.1.2 Given a raw expression E we have

$$\big(E^*\big)^* = E.$$

Proof. By induction on the structure of E. The two base cases, x and 1, are straightforward. The same goes for application and hom-expressions where the inductive hypothesis is applied directly. For lambda abstraction we need to adjust the type of the bound variable by using $(\mathbb{C}^{\mathsf{op}})^{\mathsf{op}} = \mathbb{C}$. For ends it follows that

$$\big((\textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}},y^{\mathbb{C}}} E)^*\big)^* = (\textstyle\int^{y^{\mathbb{C}^{\mathsf{op}}},x^{\mathbb{C}}} E^*)^* = \textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}},y^{\mathbb{C}}} (E^*)^*.$$

By inductive hypothesis $(E^*)^* = E$ and

$$\big((\textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}},y^{\mathbb{C}}} E)^*\big)^* = \textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}},y^{\mathbb{C}}} E.$$

In a similar way the proposition is proved for coends, powers and copowers. □

A simpler aspect of duality involves assertions on types. As there is a judgement for "completeness", by duality there is a judgement for "cocompleteness" and the rules:

$$\frac{\mathcal{D}\ \mathsf{complete}}{\mathcal{D}^{\mathsf{op}}\ \mathsf{cocomplete}} \qquad \frac{\mathcal{D}\ \mathsf{cocomplete}}{\mathcal{D}^{\mathsf{op}}\ \mathsf{complete}.}$$

Coends and Copowers

We can now derive the rule **end*** for typing coend expressions:

$$\dfrac{\dfrac{\Gamma, x{:}\mathcal{C}^{\mathsf{op}}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}}{\Gamma^{\mathsf{op}}, y{:}\mathcal{C}^{\mathsf{op}}, x{:}\mathcal{C} \vdash E^*{:}\mathcal{D}^{\mathsf{op}}}\,\textbf{dual + exchange} \quad \mathcal{C}\ \mathsf{small} \quad \dfrac{\mathcal{D}\ \mathsf{cocomplete}}{\mathcal{D}^{\mathsf{op}}\ \mathsf{complete}}}{\dfrac{\Gamma^{\mathsf{op}} \vdash \int_{y^{\mathcal{C}^{\mathsf{op}}}, x^{\mathcal{C}}} E^*{:}\mathcal{D}^{\mathsf{op}}}{\Gamma \vdash \int^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E{:}\mathcal{D}.}\,\textbf{dual}}\,\textbf{end}$$

Observe that in the last application of **dual** we use the identities

$$(\Gamma^{\mathsf{op}})^{\mathsf{op}} = \Gamma \qquad (\mathcal{D}^{\mathsf{op}})^{\mathsf{op}} = \mathcal{D} \qquad \left(\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} E^*\right)^* = \int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E$$

the last one justified by Proposition 5.1.2.

Similarly we can derive the rule **power*** for copowers:

$$\dfrac{\Gamma_1 \vdash E_1{:}\mathbf{Set} \quad \dfrac{\Gamma_2 \vdash E_2{:}\mathcal{D}}{\Gamma_2^{\mathsf{op}} \vdash E_2^*{:}\mathcal{D}^{\mathsf{op}}}\,\textbf{dual} \quad \dfrac{\mathcal{D}\ \mathsf{cocomplete}}{D^{\mathsf{op}}\ \mathsf{complete}}}{\dfrac{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}} \vdash [E_1, E_2^*]{:}\mathcal{D}^{\mathsf{op}}}{\Gamma_1, \Gamma_2 \vdash E_1 \otimes E_2{:}\mathcal{D}.}\,\textbf{dual}}\,\textbf{power}$$

Through duality the derived rule **power*** respects the expected functorial behaviour on the free variables in E_1 (see 3.4). This complete the typing rules for all the possible expressions.

5.1.3 Substitution

The simply typed lambda calculus is closed under substitution. This result corresponds via the Curry-Howard correspondence [How90, SU99] to the cut-elimination for the minimal intuitionistic sequent calculus.

In our setting due to contravariance substitution takes a novel definition. As an example let us consider the usual context-independent rule for substitution in lambda calculus (see [Bar92, VW01, TS00]):

$$\frac{\Gamma_1, x{:}\mathcal{C} \vdash E_1{:}\mathcal{D} \quad \Gamma_2 \vdash E_2{:}\mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1[E_2/x]{:}\mathcal{D}} \tag{5.1}$$

where $E_1[E_2/x]$ represents the expression E_1 where all the occurrences of x are replaced by E_2 (taking care of renaming of bound variables if necessary). As the following example shows this definition is not suitable in the presence of contravariance. Consider the following instances for the premises of the rule (5.1)

$$y{:}\mathcal{C}, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x,y){:}\mathbf{Set} \qquad \Gamma' \vdash \int_{z^{\mathbb{D}^{\mathsf{op}}}, w^{\mathbb{D}}} E{:}\mathcal{C}^{\mathsf{op}}.$$

The corresponding conclusion is

$$y{:}\mathcal{C}, \Gamma' \vdash \mathcal{C}(\int_{z^{\mathbb{D}^{\mathsf{op}}}, w^{\mathbb{D}}} E, y){:}\mathbf{Set}.$$

Notice that this does not keep with practice in category theory where an expression occurring on the left of a hom-expression is implicitly dualised. Concretely, the sequent

$$y{:}\mathcal{C}, \Gamma' \vdash \mathcal{C}(\int^{w^{\mathbb{D}^{\mathsf{op}}}, z^{\mathbb{D}}} E^*, y){:}\mathbf{Set}$$

conforms this view where the contravariance of the first argument is reflected by keeping the now contravariant context Γ'.

Definition 5.1.3 (Substitution) Given raw expressions E_1 and E_2 the expression $E_1[E_2/x]$ is defined inductively over the structure of E_1 as follows:

$$\begin{aligned}
x[E_2/x] &= E_2 \\
y[E_2/x] &= y && x \neq y, \\
1[E_2/x] &= 1 \\
(\lambda x^{\mathcal{C}}.E)[E_2/x] &= \lambda x^{\mathcal{C}}.E \\
(\lambda y^{\mathcal{C}}.E)[E_2/x] &= \lambda x^{\mathcal{C}}.E[E_2/x] && x \neq y, \\
(E'E'')[E_2/x] &= (E'[E_2/x])(E''[E_2/x]) \\
\mathcal{C}(E', E'')[E_2/x] &= \mathcal{C}\big(E'[E_2^*/x], E''[E_2/x]\big) \\
(\textstyle\int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E)[E_2/x] &= \textstyle\int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E && x = w \text{ or } x = y, \\
(\textstyle\int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E)[E_2/x] &= \textstyle\int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E[E_2/x] && x \neq w \text{ and } x \neq y, \\
(\textstyle\int^{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E)[E_2/x] &= \textstyle\int^{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E && x = w \text{ or } x = y, \\
(\textstyle\int^{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E)[E_2/x] &= \textstyle\int^{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E[E_2/x] && x \neq w \text{ and } x \neq y, \\
(E' \otimes E'')[E_2/x] &= E'[E_2/x] \otimes E''[E_2/x] \\
([E', E''])[E_2/x] &= \big[E'[E_2^*/x], E''[E_2/x]\big]
\end{aligned}$$

The special cases are for hom-expressions and powers where the contravariance on the first argument is reflected by dualising the expression to be substituted in. This definition can be extended in the usual way to simultaneous substitution over multiple variables.

As a consequence of Proposition 5.1.2 the substitution $E_1[E_2/x]$ replaces the occurrences of the free variable x on the left in an odd number of hom-expressions or powers by E_2^* and other occurrences by E_2. For example

$$\begin{aligned}
&[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}(y, x), \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}^{\mathsf{op}}(x, y))[E/x] \\
&\qquad = [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big((\lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}(y, x))[E^*/x], (\lambda y^{\mathbb{C}}.\mathbb{C}^{\mathsf{op}}(x, y))[E/x]\big) \\
&\qquad = [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}(y, x[E^*/x]), \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}^{\mathsf{op}}(x[E^*/x], y)\big) \\
&\qquad = [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](\lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}(y, E^*), \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}^{\mathsf{op}}(E^*, y))
\end{aligned}$$

where both occurrences of x are on the left of a hom-expression once.[1] Another interesting example is

$$\begin{aligned}
[\mathcal{C}(c, x), x][E/x] &= \big[\mathcal{C}(c, x)[E^*/x], x[E/x]\big] \\
&= [\mathcal{C}(c, x[E^*/x]), E] \\
&= [\mathcal{C}(c, E^*), E]
\end{aligned}$$

showing that different occurrences for the same variable can be substituted by different expressions.

Proposition 5.1.4 Given raw expressions E_1 and E_2 we have

$$\big(E_1[E_2/x]\big)^* = E_1^*[E_2^*/x].$$

[1]The expression in this example does not type in the calculus, see page 81.

Proof. By induction on the structure of E_1. The base cases are given by 1 which follows trivially and by variables, the only interesting case being

$$(x[E_2/x])^* = E_2^* = x[E_2^*/x] = x^*[E_2^*/x].$$

For the inductive step we consider the special cases, *i.e.* hom-expressions, powers and integrals, for the other constructions the result follows in a similar manner:

- $E_1 = \mathcal{C}(E', E'')$

$$\begin{aligned}\Big(\mathcal{C}\big(E', E''\big)[E_2/x]\Big)^* &= \Big(\mathcal{C}\big(E'[E_2^*/x], E''[E_2/x]\big)\Big)^* \\ &= \mathcal{C}\big((E'[E_2^*/x])^*, (E''[E_2/x])^*\big) \\ &= \mathcal{C}\big((E')^*[E_2/x], (E'')^*[E_2^*/x]\big) && \text{by hypothesis and Proposition 5.1.2,} \\ &= \mathcal{C}\big((E')^*, (E'')^*\big)[E_2^*/x] && \text{by Proposition 5.1.2,} \\ &= \big(\mathcal{C}(E', E'')\big)^*[E_2^*/x]\end{aligned}$$

- $E_1 = [E', E'']$

$$\begin{aligned}\Big([E', E''][E_2/x]\Big)^* &= \big[E'[E_2^*/x], E''[E_2/x]\big]^* \\ &= E'[E_2^*/x] \otimes (E''[E_2/x])^* \\ &= E'[E_2^*/x] \otimes (E'')^*[E_2^*/x] && \text{by hypothesis,} \\ &= \big(E' \otimes (E'')^*\big)[E_2^*/x] \\ &= ([E', E''])^*[E_2^*/x] && \text{by Proposition 5.1.2.}\end{aligned}$$

- $E_1 \equiv \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E'$

$$\begin{aligned}\big((\textstyle\int_{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E')[E_2/x]\big)^* &= \big(\textstyle\int_{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E'[E_2/x]\big)^* \\ &= \textstyle\int^{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} (E'[E_2/x])^* \\ &= \textstyle\int^{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} (E')^*[E_2^*/x] && \text{by induction hypothesis,} \\ &= (\textstyle\int^{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} (E')^*)[E_2^*/x] \\ &= \big(\textstyle\int_{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E'\big)^*[E_2^*/x].\end{aligned}$$

Dually the proposition holds for $E_1 \equiv \int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E'$.

□

By Proposition 5.1.2 an immediate corollary is that

$$\big(E_1[E_2^*/x]\big)^* = E_1^*[E_2/x] \quad \text{and} \quad \big(E_1^*[E_2/x]\big)^* = E_1[E_2^*/x].$$

Theorem 5.1.5 The rule

$$\frac{\Gamma, x\!:\!\mathcal{C}, \Theta \vdash E_1\!:\!\mathcal{D} \quad \Xi \vdash E_2\!:\!\mathcal{C}}{\Gamma, \Xi, \Theta \vdash E_1[E_2/x]\!:\!\mathcal{D}}$$

where $E_1[E_2/x]$ is defined as in 5.1.3 is *admissible* in the type theory.

Proof. It follows along the lines the proof of cut-elimination for sequent calculus by Girard [Gir87]. The proof proceeds by propagating the use of substitution toward the axioms in a derivation by means of local transformations. This is done inductively on the derivation for the sequent

$$\begin{array}{c}\vdots\\ \Gamma, x{:}\mathcal{C}, \Theta \vdash E_1{:}\mathcal{D}.\end{array} \tag{5.2}$$

The base case is given by the axioms, for **identity** the result is proved by the derivation

$$\begin{array}{c}\vdots\\ \Xi \vdash E_2{:}\mathcal{C}\end{array}$$

since $x[E_2/x] = E_2$. In the case of **singleton** the result is trivial.

For the induction step in most of the cases the hypothesis can be applied directly. The exception is the rule for contraction, the standard procedure here is to strengthen the induction hypothesis to consider a multiple simultaneous substitution or "multicut" [TS00, VW01] (for other solutions to this problem refer to [Bus98, vP01]). Let us concentrate on the novel rules. Consider the case where the last rule is

- **hom** (and $E_1 = \mathcal{A}(E', E'')$)

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Delta_1 \vdash E'{:}\mathcal{A} & \Delta_2 \vdash E''{:}\mathcal{A}\end{array}}{\Gamma, x{:}\mathcal{C}, \Theta \vdash \mathcal{A}(E', E''){:}\mathbf{Set}}\ \mathbf{hom}$$

  where $\Gamma, x{:}\mathcal{C}, \Theta = \Delta_1^{\mathsf{op}}, \Delta_2$. Because the rule is context-independent there are only two cases to consider, either x is declared in Δ_1 or in Δ_2. In the first case there is a context Δ_1' such that

$$\Delta_1 = \Gamma^{\mathsf{op}}, x{:}\mathcal{C}^{\mathsf{op}}, \Delta_1'$$

  and by induction hypothesis there is a derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma^{\mathsf{op}}, x{:}\mathcal{C}^{\mathsf{op}}, \Delta_1' \vdash E'{:}\mathcal{A}\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Xi \vdash E_2{:}\mathcal{C}\end{array}}{\Xi^{\mathsf{op}} \vdash E_2^*{:}\mathcal{C}^{\mathsf{op}}}\ \mathbf{dual}}{\Gamma^{\mathsf{op}}, \Xi^{\mathsf{op}}, \Delta_1' \vdash E'[E_2^*/x]{:}\mathcal{A}}\ \mathbf{substitution} \quad \begin{array}{c}\vdots\\ \Delta_2 \vdash E''{:}\mathcal{A}\end{array}}{\Gamma, \Xi, (\Delta_1')^{\mathsf{op}}, \Delta_2 \vdash \mathcal{A}(E'[E_2^*/x], E''){:}\mathbf{Set}}\ \mathbf{hom}$$

  As x is not free in E'' we have that

$$E'' = E''[E_2/x]$$

  and then

$$\mathcal{A}(E'[E_2^*/x], E'') = \mathcal{A}(E'[E_2^*/x], E''[E_2/x]) = \mathcal{A}(E', E'')[E_2/x] = E_1[E_2/x].$$

  The second case, *i.e.* x is declared in Δ_2, follows similarly and in fact it is simpler since it does not require the use of **dual**. The case where the last rule is **power** is analogous.

- **dual**

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma^{\mathsf{op}}, x\!:\!\mathcal{C}^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^*\!:\!\mathcal{D}^{\mathsf{op}}\end{array}}{\Gamma, x\!:\!\mathcal{C}, \Theta \vdash E_1\!:\!\mathcal{D}.}\ \textbf{dual}$$

This form for the derivation (5.2) is justified by Proposition 5.1.2. By induction hypothesis there is a derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma^{\mathsf{op}}, x\!:\!\mathcal{C}^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^*\!:\!\mathcal{D}^{\mathsf{op}}\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Xi \vdash E_2\!:\!\mathcal{C}\end{array}}{\Xi^{\mathsf{op}} \vdash E_2^*\!:\!\mathcal{C}^{\mathsf{op}}}\ \textbf{dual}}{\Gamma^{\mathsf{op}}, \Xi^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^*[E_2^*/x]\!:\!\mathcal{D}^{\mathsf{op}}}\ \textbf{substitution}}{\Gamma, \Xi, \Theta \vdash \left(E_1^*[E_2^*/x]\right)^*\!:\!\mathcal{D}.}\ \textbf{dual}$$

As a corollary of Propositions 5.1.2 and 5.1.4 we have

$$\left(E_1^*[E_2^*/x]\right)^* = E_1[E_2/x].$$

□

5.1.4 Examples

Informally the model of the calculus is that types represent locally small categories and expressions functors. This is enough to see that some of the raw expressions cannot represent functors and then, as expected, the type theory cannot give a derivation for them.

The simplest example of a expression with no type is $\mathcal{C}(x,x)$.[2] Trying to produce a derivation for this term would fail in the application of the rule to introduce the hom-expression since it would require x to be declare with a covariant and contravariant type at the same time. This is actually prevented by the context-independent presentation of the rule for hom-expressions.

The Yoneda functor embeds a small category $\mathbb{C}$ into the presheaf category $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$. An expression for this functor is derivable in the type theory:

$$\dfrac{\dfrac{\dfrac{}{y\!:\!\mathbb{C} \vdash y\!:\!\mathbb{C}}\ \textbf{identity} \quad \dfrac{}{x\!:\!\mathbb{C} \vdash x\!:\!\mathbb{C}}\ \textbf{identity}}{x\!:\!\mathbb{C}, y\!:\!\mathbb{C}^{\mathsf{op}} \vdash \mathbb{C}(y,x)\!:\!\mathbf{Set}}\ \textbf{hom + exchange}}{x\!:\!\mathbb{C} \vdash \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathbb{C}(y,x)\!:\![\mathbb{C}^{\mathsf{op}}, \mathbf{Set}].}\ \textbf{lambda}$$

Limits are ends where the extra contravariant argument is “dummy”. We shall be able to formalise this result in the calculus after introducing the natural isomorphisms. The “dummy” argument can be modelled through **weakening**:

$$\dfrac{\dfrac{\Gamma, x\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}}{\Gamma, y\!:\!\mathbb{C}^{\mathsf{op}}, x\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}}\ \textbf{weakening+exchange}}{\Gamma \vdash \int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} E\!:\!\mathcal{D}}\ \textbf{end}$$

[2]For a non-trivial category $\mathcal{C}$ (see the section on products).

where $\mathcal{D}$ is complete and $\mathbb{C}$ is small. As y is a fresh variable we write this end as just $\int_{x^{\mathbb{C}}} E$ to emphasise that is a limit.

Now we show that the judgement

$$\text{and}\quad y:\mathbb{C}, x:\mathbb{C}^{\mathsf{op}} \vdash [\mathbb{C}(y,x),x]:\mathbb{C}^{\mathsf{op}}$$

whose expression was used in previous examples is well-typed:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y:\mathbb{C}\vdash y:\mathbb{C}}\,\textbf{identity}\qquad \dfrac{}{w:\mathbb{C}\vdash w:\mathbb{C}}\,\textbf{identity}}{y:\mathbb{C}^{\mathsf{op}}, w:\mathbb{C}\vdash \mathbb{C}(y,w):\mathbf{Set}}\,\textbf{hom}\qquad \dfrac{}{z:\mathbb{C}^{\mathsf{op}}\vdash z:\mathbb{C}^{\mathsf{op}}}\,\textbf{identity}}{y:\mathbb{C}, w:\mathbb{C}^{\mathsf{op}}, z:\mathbb{C}^{\mathsf{op}}\vdash [\mathbb{C}(y,w),z]:\mathbb{C}^{\mathsf{op}}}\,\textbf{power}}{y:\mathbb{C}, x:\mathbb{C}^{\mathsf{op}}\vdash [\mathbb{C}(y,x),x]:\mathbb{C}^{\mathsf{op}}.}\,\textbf{contraction} \tag{5.3}$$

An example of a non-derivation is given by[3]

$$x:\mathbb{I}^{\mathsf{op}}\vdash [\mathbb{I}^{\mathsf{op}},\mathbf{Set}](\lambda i.\mathbb{I}(i,x),\lambda i.\mathbb{I}^{\mathsf{op}}(x,i)):\mathbf{Set},$$

an attempt to find a derivation yields

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{i:\mathbb{I}\vdash i:\mathbb{I}}\,\textbf{identity}\qquad \dfrac{}{y:\mathbb{I}\vdash y:\mathbb{I}}\,\textbf{identity}}{y:\mathbb{I}, i:\mathbb{I}^{\mathsf{op}}\vdash \mathbb{I}(i,y):\mathbf{Set}}\,\textbf{hom+exchange}}{y:\mathbb{I}\vdash \lambda i.\mathbb{I}(i,y):[\mathbb{I}^{\mathsf{op}},\mathbf{Set}]}\,\textbf{lambda}\qquad \dfrac{\dfrac{z:\mathbb{I}\vdash z:\mathbb{I}^{\mathsf{op}}\qquad \dfrac{}{i:\mathbb{I}^{\mathsf{op}}\vdash i:\mathbb{I}^{\mathsf{op}}}\,\textbf{identity}}{z:\mathbb{I}^{\mathsf{op}}, i:\mathbb{I}^{\mathsf{op}}\vdash \mathbb{I}^{\mathsf{op}}(z,i):\mathbf{Set}}\,\textbf{hom}}{z:\mathbb{I}^{\mathsf{op}}\vdash \lambda i.\mathbb{I}^{\mathsf{op}}(z,i):[\mathbb{I}^{\mathsf{op}},\mathbf{Set}]}\,\textbf{lambda}}{y:\mathbb{I}^{\mathsf{op}}, z:\mathbb{I}^{\mathsf{op}}\vdash [\mathbb{I}^{\mathsf{op}},\mathbf{Set}](\lambda i.\mathbb{I}(i,y),\lambda i.\mathbb{I}^{\mathsf{op}}(z,i)):\mathbf{Set}}\,\textbf{hom}}{x:\mathbb{I}^{\mathsf{op}}\vdash [\mathbb{I}^{\mathsf{op}},\mathbf{Set}](\lambda i.\mathbb{I}(i,x),\lambda i.\mathbb{I}^{\mathsf{op}}(x,i)):\mathbf{Set}}\,\textbf{contraction}$$

where the judgement $z:\mathbb{I}\vdash z:\mathbb{I}^{\mathsf{op}}$ cannot be simplified further.

From the last derivation for the expression $[\mathbb{C}(y,x),x]$ and a derivation

$$\begin{array}{c}\vdots\\ \Gamma\vdash E:\mathbb{C}^{\mathsf{op}}\end{array}$$

we obtain by means of substitution the sequent

$$x:\mathbb{C},\Gamma\vdash [\mathbb{C}(y,E^*),E]:\mathbb{C}^{\mathsf{op}}.$$

It is remarkable that in this example the substitution replaces x by different expressions depending on the variance of the occurrence of x. It follows a derivation for the same sequent without using substitution.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y:\mathbb{C}\vdash y:\mathbb{C}}\,\textbf{identity}\qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash E:\mathbb{C}^{\mathsf{op}}\end{array}}{\Gamma^{\mathsf{op}}\vdash E^*:\mathbb{C}}\,\textbf{dual}}{y:\mathbb{C}^{\mathsf{op}},\Gamma^{\mathsf{op}}\vdash \mathbb{C}(y,E^*):\mathbf{Set}}\,\textbf{hom}\qquad \begin{array}{c}\vdots\\ \Gamma'\vdash E':\mathbb{C}^{\mathsf{op}}\end{array}}{y:\mathbb{C}^{\mathsf{op}},\Gamma,\Gamma'\vdash [\mathbb{C}(y,E^*),E']:\mathbb{C}^{\mathsf{op}}}\,\textbf{power}}{y:\mathbb{C}^{\mathsf{op}},\Gamma\vdash [\mathbb{C}(y,E^*),E]:\mathbb{C}^{\mathsf{op}}}\,\textbf{contraction} + \textbf{exchange}\ \text{(many times)}$$

[3] This was originally presented as a legal derivation in the calculus. Later, Philip Matchett implemented the calculus in the theorem prover ISABELLE to find out that the derivation was incorrect.

The context-independent presentation for rules forces us to work modulo renaming of variables (α-conversion). In this example the sequent $\Gamma' \vdash E' : \mathbb{C}^{\mathsf{op}}$ represents the sequent $\Gamma \vdash E : \mathbb{C}^{\mathsf{op}}$ where free variables were rename to avoid any clash. Observe that in the last part of the derivation we obtain the desired result by applying contraction many times. This apparent *ad-hoc* manipulation of contexts can be formalised by using indexes for variables at the cost of complicating the notation of sequents (see [Jac99]).

5.2 A Normal Form for Derivations

Due to the presence of structural rules and the rule for duality, the same sequent can be proved through different derivations. It is easy to see that by spurious use of the rule **dual**, for example, we can produce different proofs for any well-typed sequent. In this section we give a general form of derivation which is unique modulo the relative order of the application of the structural rules. This amounts to identifying the non-essential uses of duality and to restrict the use of the structural rules. Then we show that any derivation can be translated into that form. This conforms the syntactic side of the coherence result.

We say that the applications of two rules in a derivation are *adjacent* if they are only separated by the occurrence of structural rules. The proofs in this section proceed inductively on the structure of derivations where we analyse the situation created by each possible rule. In most of the cases we shall neglect the occurrences of **exchange**. In this spirit we say that an occurrence of a rule is *above* or *below* the occurrence of another rule if they are only separated by applications of **exchange**. We sometimes use the same terminology to refer to blocks of rules.

5.2.1 W-normal Derivations

A derivation is in *W-normal* form if the **weakening**'s occur in blocks above

1. the final sequent, or
2. the introduction of a binder (**lambda** or **end**) where the fresh variable is bound, for example

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash E : \mathcal{D}\end{array}}{\Gamma, x : \mathbb{C} \vdash E : \mathcal{D}}\ \textbf{weakening}}{\Gamma \vdash \lambda x^{\mathbb{C}}.E : [\mathbb{C}, \mathcal{D}]}\ \textbf{lambda}$$

   or in the introduction of a limit through an end with a dummy argument.

In this way any variable in the context which is not just introduced by an adjacent occurrence of **weakening** is either introduced by a use of **contraction** or can be traced to an axiom.

Proposition 5.2.1 Every derivation can be transformed into a derivation in W-normal form.

Proof. By means of local transformations we show how all the occurrences of the rule **weakening** violating the conditions above can be permuted below other rules until they reach a legal position. This operation removes combinations of adjacent **weakening** and **contraction** which are made redundant. To simplify the explanation we fix z of type $\mathcal{A}$ to be the fresh variable.

- **contraction**

  There are two different cases:

  1. z is contracted, then the derivation

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ \gamma\\ \Gamma, w{:}\mathcal{A} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, w{:}\mathcal{A}, z{:}\mathcal{A} \vdash E{:}\mathcal{D}}\ \textbf{weakening}}{\Gamma, y{:}\mathcal{A} \vdash E[y/w, y/z]{:}\mathcal{D}}\ \textbf{contraction}$$

  is converted to just

$$\begin{array}{c}\vdots\ \gamma[y/w]\\ \Gamma, y{:}\mathcal{A} \vdash E[y/w]{:}\mathcal{D}\end{array}$$

  since z is new and not free in E (this final sequent is equal to the original modulo the renaming given by $[y/w]$);

  2. z is not contracted, and then the derivation

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma, x{:}\mathcal{C}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, z{:}\mathcal{A}, x{:}\mathcal{C}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}}\ \textbf{weakening+exchange}\text{'s}}{\Gamma, z{:}\mathcal{A}, w{:}\mathcal{C} \vdash E[w/x, w/y]{:}\mathcal{D}}\ \textbf{contraction}$$

  is converted to

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma, x{:}\mathcal{C}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, w{:}\mathcal{C} \vdash E[w/x, w/y]{:}\mathcal{D}}\ \textbf{contraction}}{\Gamma, z{:}\mathcal{A}, w{:}\mathcal{C} \vdash E[w/x, w/y]{:}\mathcal{D}.}\ \textbf{weakening+exchange}\text{'s}$$

- **hom**

  There are two cases depending on which premise of the rule **hom** the occurrence of **weakening** happens. We consider the case involving contravariance, the other is symmetric. The derivation

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash E_1{:}\mathcal{C}\end{array}}{\Gamma_1, z{:}\mathcal{A} \vdash E_1{:}\mathcal{C}}\ \textbf{weakening} \qquad \begin{array}{c}\vdots\\ \Gamma_2 \vdash E_2{:}\mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, z{:}\mathcal{A}^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2){:}\mathbf{Set}}\ \textbf{hom}$$

is converted to

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots \\ \Gamma_1 \vdash E_1 : \mathcal{C} & & \Gamma_2 \vdash E_2 : \mathcal{C}\end{matrix}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}}\;\textbf{hom}}{\Gamma_1^{\mathsf{op}}, z : \mathcal{A}^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}.}\;\textbf{weakening+exchange}\text{'s}$$

The cases for the rules **power** and **application** are analogous.

- **end**

  The derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D}\end{matrix}}{\Gamma, z : \mathcal{A}, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D}}\;\textbf{weakening+exchange}\text{'s}}{\Gamma, z : \mathcal{A} \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E : \mathcal{D}}\;\textbf{end}$$

  is converted to

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D}\end{matrix}}{\Gamma \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E : \mathcal{D}}\;\textbf{end}}{\Gamma, z : \mathcal{A} \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E : \mathcal{D}.}\;\textbf{weakening}$$

  The cases where $z = x$ or $z = y$ comply with the definition of W-normal form and the **weakening** is not propagated. The case for **lambda** is analogous.

- **dual**

  The derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash E : \mathcal{D}\end{matrix}}{\Gamma, z : \mathcal{A} \vdash E : \mathcal{D}}\;\textbf{weakening}}{\Gamma^{\mathsf{op}}, z : \mathcal{A}^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}}\;\textbf{dual}$$

  is converted to

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash E : \mathcal{D}\end{matrix}}{\Gamma^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}}\;\textbf{dual}}{\Gamma^{\mathsf{op}}, z : \mathcal{A}^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}.}\;\textbf{weakening}$$

□

Observe that by definition in a W-normal derivation all meaningless use of **contraction**[4] are removed. The definition of this normal form is inspired in the W-normal deductions for intuitionistic sequent calculus, the system **G1i** in [TS00].

[4] A variable introduced by a weakening than then is removed by a contraction.

5.2.2 CW-normal Derivations

A derivation is in *CW-normal* form if it is in W-normal form and the **contraction**'s occur adjacent above

1. the final sequent, or
2. the introduction of a binder (**lambda** or **end**) where the new variable introduced by the contraction is bound, for example

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, x:\mathbb{C}, y:\mathbb{C} \vdash E:\mathcal{D}\end{array}}{\Gamma, z:\mathbb{C} \vdash E[z/x, z/y]:\mathcal{D}}\ \textbf{contraction}}{\Gamma \vdash \lambda z^{\mathbb{C}}.E[z/x, z/y]:[\mathbb{C}, \mathcal{D}]}\ \textbf{lambda}$$

for another example consider the derivation (5.3) extended with an application of **lambda**.

In this way any variable in the context which is not introduced by an adjacent **weakening** or **contraction** can be traced to an axiom.

Proposition 5.2.2 Every derivation can be transformed into a derivation in CW-normal form.

Proof. First obtain the W-normal form. Then we show how every occurrence of **contraction** left which violates the conditions above can be permuted below other rules until it reaches a legal position.

- **weakening**

  This is dual to the case for **contraction** considered in the proof of Proposition 5.2.1.

- **hom**

  As usual there are two cases to consider depending on which of the premises the **contraction** takes place. We show the transformation for the contravariant case, the other follows similarly. The derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1, x:\mathcal{A}, y:\mathcal{A} \vdash E_1:\mathcal{C}\end{array}}{\Gamma_1, z:\mathcal{A} \vdash E_1[z/x, z/y]:\mathcal{C}}\ \textbf{contraction} \qquad \begin{array}{c}\vdots\\ \Gamma_2 \vdash E_2:\mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, z:\mathcal{A}^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1[z/x, z/y], E_2):\mathbf{Set}}\ \textbf{hom}$$

  is converted to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1, x:\mathcal{A}, y:\mathcal{A} \vdash E_1:\mathcal{C}\end{array} \quad \begin{array}{c}\vdots\\ \Gamma_2 \vdash E_2:\mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x:\mathcal{A}^{\mathsf{op}}, y:\mathcal{A}^{\mathsf{op}} \vdash \mathcal{C}(E_1, E_2):\mathbf{Set}}\ \textbf{hom+exchange}\text{'s}}{\Gamma_1^{\mathsf{op}}, z:\mathcal{A}^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2)[z/x, z/y]:\mathbf{Set}}\ \textbf{contraction+exchange}\text{'s}$$

and since x and y are not free in E_2 we have

$$\mathcal{C}(E_1, E_2)[z/x, z/y] = \mathcal{C}(E_1[z/x, z/y], E_2).$$

The cases for the rules **power** and **application** are analogous.

- **end**

The derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, w{:}\mathcal{A}, z{:}\mathcal{A}, x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, s{:}\mathcal{A}, x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash E[s/w, s/z]{:}\mathcal{D}}\ \textbf{contraction+exchange}\text{'s}}{\Gamma, s{:}\mathcal{A} \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[s/w, s/z]{:}\mathcal{D}}\ \textbf{end}$$

is converted to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, w{:}\mathcal{A}, z{:}\mathcal{A}, x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, w{:}\mathcal{A}, z{:}\mathcal{A} \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E{:}\mathcal{D}}\ \textbf{end}}{\Gamma, s{:}\mathcal{A} \vdash (\int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E)[s/w, s/z]{:}\mathcal{D}.}\ \textbf{contraction}$$

As all variables in the sequents are different then

$$(\textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E)[s/w, s/z] = \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[s/w, s/z].$$

The case for **lambda** is analogous.

- **dual**

The derivation

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, x{:}\mathcal{A}, y{:}\mathcal{A} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, z{:}\mathcal{A} \vdash E[z/x, z/y]{:}\mathcal{D}}\ \textbf{contraction}}{\Gamma^{\mathsf{op}}, z{:}\mathcal{A}^{\mathsf{op}} \vdash (E[z/x, z/y])^*{:}\mathcal{D}^{\mathsf{op}}}\ \textbf{dual}$$

is converted to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, x{:}\mathcal{A}, y{:}\mathcal{A} \vdash E{:}\mathcal{D}\end{array}}{\Gamma^{\mathsf{op}}, x{:}\mathcal{A}^{\mathsf{op}}, y{:}\mathcal{A}^{\mathsf{op}} \vdash E^*{:}\mathcal{D}^{\mathsf{op}}}\ \textbf{dual}}{\Gamma^{\mathsf{op}}, z{:}\mathcal{A}^{\mathsf{op}} \vdash E^*[z/x, z/y]{:}\mathcal{D}^{\mathsf{op}}}\ \textbf{contraction}$$

where by Proposition 5.1.4

$$(E[z/x, z/y])^* = E^*[z^*/x, z^*/y] = E^*[z/x, z/y].$$

□

5.2.3 DCW-normal Derivations

Every constructor in the syntax is introduced by a rule but the binder for coends and the binary operator for copowers. These two cases call for the use of the rule for duality in a type checking procedure. Thus coends become ends and copowers become powers. In general from Proposition 5.1.2 we have

$$\int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E = \left(\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} E^*\right)^* \quad \text{and} \quad E_1 \otimes E_2 = ([E_1, E_2^*])^*.$$

Another less interesting example which demands the use of **dual** is in the derivation for a hom-expression of type $\mathbf{Set}^{\mathsf{op}}$.

As mentioned above we identify in the syntax a dual category $(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$ with the category $\mathcal{C}$. From the definition of the meta-operation $(\_)^*$ we need to identify as well a functor category $[\mathbb{C}, \mathcal{D}]^{\mathsf{op}}$ with $[\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]$. This is justified by the derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, x\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}\end{matrix}}{\Gamma \vdash \lambda x^{\mathbb{C}}.E\!:\![\mathbb{C},\mathcal{D}]}\ \textbf{lambda}}{\Gamma^{\mathsf{op}} \vdash \lambda x^{\mathbb{C}^{\mathsf{op}}}.E^*\!:\![\mathbb{C}^{\mathsf{op}},\mathcal{D}^{\mathsf{op}}]}\ \textbf{dual}$$

where by definition $\left(\lambda x^{\mathbb{C}}.E\right)^* = \lambda x^{\mathbb{C}^{\mathsf{op}}}.E^*$. The following gives another derivation for the same sequent

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, x\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}\end{matrix}}{\Gamma^{\mathsf{op}}, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash E^*\!:\!\mathcal{D}^{\mathsf{op}}}\ \textbf{dual}}{\Gamma^{\mathsf{op}} \vdash \lambda x^{\mathbb{C}^{\mathsf{op}}}.E^*\!:\![\mathbb{C}^{\mathsf{op}},\mathcal{D}^{\mathsf{op}}].}\ \textbf{lambda} \tag{5.4}$$

In this case with no implicit coercion on the types.

Definition 5.2.3 An application of **dual** in

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash E\!:\!\mathcal{D}\end{matrix}}{\Gamma^{\mathsf{op}} \vdash E^*\!:\!\mathcal{D}^{\mathsf{op}}}\ \textbf{dual}$$

is *essential* if the outermost syntax constructor in E is 1, an end binder, a hom-expression or a power.

For a non-example consider an instance of (5.4) where Γ is empty and $E = x$. Then the application of **dual** is not essential. Indeed the derivation

$$\dfrac{\dfrac{}{x\!:\!\mathbb{C} \vdash x\!:\!\mathbb{C}}\ \textbf{identity}}{x\!:\!\mathbb{C}^{\mathsf{op}} \vdash x\!:\!\mathbb{C}^{\mathsf{op}}}\ \textbf{dual}$$

can be replaced by just

$$\dfrac{}{x\!:\!\mathbb{C}^{\mathsf{op}} \vdash x\!:\!\mathbb{C}^{\mathsf{op}}.}\ \textbf{identity}$$

A derivation is in *DCW-normal* form if it is in CW-normal form and all the occurrences of the rule **dual** are essential.

Proposition 5.2.4 Every derivation can be transformed into a derivation in DCW-normal form.

Proof. We first obtain the CW-normal derivation. Then by means of local transformations the non-essential occurrences of **dual** are permuted above other rules until they reach a legal position or encounter:

- another occurrence of the rule **dual**, or
- an axiom (and then it is trivially solved).

By Proposition 5.1.2 double applications of the rule for duality can be safely removed. Thus a derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash E : \mathcal{D}\end{matrix}}{\Gamma^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}}\ \textbf{dual}}{\Gamma \vdash E : \mathcal{D}}\ \textbf{dual}$$

is converted to just

$$\begin{matrix}\vdots\\ \Gamma \vdash E : \mathcal{D}.\end{matrix}$$

At the beginning of this section we show an example of how to permute the occurrences of **dual** above **lambda**. For the other cases:

- **application**

  A derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \vdash E_1 : [\mathbb{C}, \mathcal{D}]\end{matrix} \quad \begin{matrix}\vdots\\ \Gamma_2 \vdash E_2 : \mathbb{C}\end{matrix}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 : \mathcal{D}}\ \textbf{application}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}} \vdash E_1^* E_2^* : \mathcal{D}^{\mathsf{op}}}\ \textbf{dual}$$

  is converted to

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \vdash E_1 : [\mathbb{C}, \mathcal{D}]\end{matrix}}{\Gamma_1^{\mathsf{op}} \vdash E_1^* : [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]}\ \textbf{dual} \quad \dfrac{\begin{matrix}\vdots\\ \Gamma_2 \vdash E_2 : \mathbb{C}\end{matrix}}{\Gamma_2^{\mathsf{op}} \vdash E_2^* : \mathbb{C}^{\mathsf{op}}}\ \textbf{dual}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}} \vdash E_1^* E_2^* : \mathcal{D}^{\mathsf{op}}}\ \textbf{application}$$

  where we need to use $[\mathbb{C}, \mathcal{D}]^{\mathsf{op}} = [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]$.

- the cases for **weakening** and **contraction** are just dual to the cases in the proofs of Propositions 5.2.1 and 5.2.2.

The remaining rules produce essential occurrences of **dual**. □

In the DCW-normal form the use of **weakening** and **contraction** is delayed as much as possible whereas the use of **dual** is permitted only in the essential cases. The DCW-normal form of a derivation δ is unique up to the relative order of structural rules and renaming of variables. If indexes are used for variables then the DCW-normal form is in fact unique. Henceforth by "normal form" we mean the DCW-normal form.

5.3 Categorical Model

A categorical model for a logic is a category where each derivation is interpreted as a morphism and where equations refer to commutative diagrams [Bie95].

In our setting a derivation

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E : \mathcal{C} \end{array}$$

is interpreted as a morphism in **CAT**, the category of locally small categories, *i.e.* as a functor

$$\left[\!\!\left[\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E : \mathcal{C} \end{array} \right]\!\!\right] : [\![\Gamma]\!] \to [\![\mathcal{C}]\!].$$

5.3.1 Types and Contexts

Every type denotes a locally small category. The type **Set** is just the category of sets and functions. The notion of context is that of *typed cartesian context* [FPT99], this is reflected in the interpretation for the structural rules. A context

$$\Gamma = x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n$$

denotes the product of categories

$$[\![\Gamma]\!] = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n.$$

The empty context is just the category **1**.

That a type $(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}$ corresponds to $\mathcal{C}$ can now be justified at the level of the interpretation. Here we do not even need to talk about isomorphism but just equality. Similarly for the modality taking a context Γ into Γ^{op}, it is clear that we can say

$$(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n)^{\mathsf{op}} = \mathcal{C}_1^{\mathsf{op}} \times \ldots \times \mathcal{C}_n^{\mathsf{op}}.$$

We understand the interpretation of contexts as an n-product which represents at once all possible isomorphic expressions obtained via associativity.

5.3.2 Interpretation for Derivations

The interpretation for derivations is defined inductively by using the typing rules in § 5.1.2. Each rule is interpreted as an operation taking the functors denoted by the premises into the functor denoted by the whole derivation. We write $[\![\delta \rhd \Gamma \vdash E : \mathcal{C}]\!]$ for the interpretation of a derivation

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E : \mathcal{C}. \end{array}$$

To simplify the notation we omit the semantics brackets in most cases in particular when referring to categories and types.

- The derivation

  $$\frac{}{\vdash 1 : \mathbf{Set}}\ \textbf{singleton}$$

  is interpreted as

  $$\mathbf{1} \xrightarrow{\ 1\ } \mathbf{Set}$$

  where **1** is the singleton category and the functor 1 selects the singleton (a chosen one) in the category of sets.

- The derivation

$$\frac{}{x\!:\!\mathcal{C} \vdash x\!:\!\mathcal{C}}\ \textbf{identity}$$

is interpreted as

$$\mathcal{C} \xrightarrow{\mathsf{id}_{\mathcal{C}}} \mathcal{C}.$$

- The derivation

$$\frac{\begin{array}{c}\vdots\ \delta \\ \Gamma = \mathcal{C}_1, \cdots, x_n\!:\!\mathcal{C}_n \vdash E\!:\!\mathcal{C}\end{array}}{\mathcal{C}_1, \cdots, x_n\!:\!\mathcal{C}_n, x_{n+1}\!:\!\mathcal{C}_{n+1} \vdash E\!:\!\mathcal{C}}\ \textbf{weakening}$$

is interpreted as

$$\big(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n\big) \times \mathcal{C}_{n+1} \xrightarrow{\pi} \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \xrightarrow{[\![\delta \rhd \Gamma \vdash E:\mathcal{C}]\!]} \mathcal{C}$$

where π is the universal arrow $\langle \pi_{\mathcal{C}_1}, \dots, \pi_{\mathcal{C}_n} \rangle$ defined by the projections of $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n$.

- The derivation

$$\frac{\begin{array}{c}\vdots\ \delta \\ \Gamma = x_1\!:\!\mathcal{C}_1, \cdots, x_i\!:\!\mathcal{C}_i, x_{i+1}\!:\!\mathcal{C}_{i+1}, \cdots, x_n\!:\!\mathcal{C}_n \vdash E\!:\!\mathcal{C}\end{array}}{x_1\!:\!\mathcal{C}_1, \cdots, x_{i+1}\!:\!\mathcal{C}_{i+1}, x_i\!:\!\mathcal{C}_i, \cdots, x_n\!:\!\mathcal{C}_n \vdash E\!:\!\mathcal{C}}\ \textbf{exchange}$$

is interpreted through the isomorphisms of categories

$$\Gamma_1 \times \mathcal{C}_{i+1} \times \mathcal{C}_i \times \Gamma_2 \xrightarrow{\langle \pi_{\Gamma_1}, \pi_{\mathcal{C}_i+1}, \pi_{\mathcal{C}_i}, \pi_{\Gamma_2} \rangle} \Gamma_1 \times \mathcal{C}_i \times \mathcal{C}_{i+1} \times \Gamma_2 \cong \Gamma \xrightarrow{[\![\delta \rhd \Gamma \vdash E:\mathcal{C}]\!]} \mathcal{C}.$$

- The derivation

$$\frac{\begin{array}{c}\vdots\ \delta \\ \overbrace{\Gamma, x\!:\!\mathcal{C}}^{\Delta}, y\!:\!\mathcal{C} \vdash E\!:\!\mathcal{D}\end{array}}{\Gamma, z\!:\!\mathcal{C} \vdash E[z/x, z/y]\!:\!\mathcal{D}}\ \textbf{contraction}$$

is interpreted as

$$\Gamma \times \mathcal{C} \xrightarrow{\langle \pi_\Gamma, \pi_{\mathcal{C}}, \pi_{\mathcal{C}} \rangle} \Gamma \times \mathcal{C} \times \mathcal{C} \xrightarrow{[\![\delta \rhd \Delta \vdash E:\mathcal{D}]\!]} \mathcal{D}$$

where $\langle \pi_\Gamma, \pi_{\mathcal{C}}, \pi_{\mathcal{C}} \rangle$ is the diagonal functor parametrised on Γ.

- The derivation

$$\frac{\begin{array}{cc}\vdots\ \delta_1 & \vdots\ \delta_2 \\ \Gamma_1 \vdash E_1\!:\!\mathcal{C} & \Gamma_2 \vdash E_2\!:\!\mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2)\!:\!\mathbf{Set}}\ \textbf{hom}$$

is interpreted as

$$\Gamma_1^{\mathsf{op}} \times \Gamma_2 \xrightarrow{[\![\delta_1 \rhd \Gamma_1 \vdash E_1:\mathcal{C}]\!]^{\mathsf{op}} \times [\![\delta_2 \rhd \Gamma_2 \vdash E_2:\mathcal{C}]\!]} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}(=,-)} \mathbf{Set}.$$

Observe that contravariance on the first argument is modelled through the dualising 2-functor $(\_)^{\mathsf{op}}$ (see § 1.3).

- The derivation

$$\dfrac{\begin{array}{cc}\vdots\ \delta_1 & \vdots\ \delta_2 \\ \Gamma_1 \vdash E_1 : \mathbf{Set} & \Gamma_2 \vdash E_2 : \mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] : \mathcal{C}}\ \textbf{power}$$

is interpreted as

$$\Gamma_1^{\mathsf{op}} \times \Gamma_2 \xrightarrow{[\![\delta_1 \rhd \Gamma_1 \vdash E_1 : \mathbf{Set}]\!]^{\mathsf{op}} \times [\![\delta_2 \rhd \Gamma_2 \vdash E_2 : \mathcal{C}]\!]} \mathbf{Set}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{[=,-]} \mathcal{C}$$

where $[=,-]$ is the power functor (see § 3.4).

- The derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta \\ \Gamma, x : \mathbb{C} \vdash E : \mathcal{D}\end{array}}{\Gamma \vdash \lambda x^{\mathbb{C}}.E : [\mathcal{C}, \mathcal{D}]}\ \textbf{lambda}$$

is interpreted as

$$\Gamma \xrightarrow{\boldsymbol{\lambda}([\![\delta \rhd \Gamma, x : \mathbb{C} \vdash E : \mathcal{D}]\!])} [\mathbb{C}, \mathcal{D}]$$

where $\boldsymbol{\lambda}$ is the isomorphism defined by (2.14).

- The derivation

$$\dfrac{\begin{array}{cc}\vdots\ \delta_1 & \vdots\ \delta_2 \\ \Gamma_1 \vdash E_1 : [\mathbb{C}, \mathcal{D}] & \Gamma_2 \vdash E_2 : \mathbb{C}\end{array}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 : \mathcal{D}}\ \textbf{application}$$

is interpreted as

$$\Gamma_1 \times \Gamma_2 \xrightarrow{[\![\delta_1 \rhd \Gamma_1 \vdash E_1 : [\mathbb{C}, \mathcal{D}]]\!] \times [\![\delta_2 \rhd \Gamma_2 \vdash E_2 : \mathbb{C}]\!]} [\mathbb{C}, \mathcal{D}] \times \mathbb{C} \xrightarrow{eval} \mathcal{D}$$

where *eval* is the counit of $\boldsymbol{\lambda}$.

- The derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta \\ \Gamma, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D}\end{array}}{\Gamma \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E : \mathcal{D}}\ \textbf{end}$$

is interpreted as

$$\Gamma \xrightarrow{\lambda r. \int_w F^r(w,w)} \mathcal{D}$$

where

$$F = [\![\delta \rhd \Gamma, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D}]\!] : \Gamma \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathcal{D}.$$

By parametrised representability since $\mathcal{D}$ is complete we can define a mapping

$$\gamma \mapsto \int_w F^{\gamma}(w,w)$$

which extends to a functor.

- The derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash E : \mathcal{D}\end{array}}{\Gamma^{\mathsf{op}} \vdash E^{*} : \mathcal{D}^{\mathsf{op}}}\ \textbf{dual}$$

is interpreted as

$$\Gamma^{\mathsf{op}} \xrightarrow{[\![\delta \triangleright \Gamma \vdash E:\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}.$$

This is obtained by applying the dualising 2-functor $(\_)^{\mathsf{op}}$ (see § 1.3).

5.3.3 Example

In section 5.1.4 we show a derivation for the sequent

$$y:\mathbb{C}, x:\mathbb{C}^{\mathsf{op}} \vdash [\mathbb{C}(y,x),x]:\mathbb{C}^{\mathsf{op}}.$$

The last rule in the derivation is

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ y:\mathbb{C}, w:\mathbb{C}^{\mathsf{op}}, z:\mathbb{C}^{\mathsf{op}} \vdash [\mathbb{C}(y,w),z]:\mathbb{C}^{\mathsf{op}}\end{array}}{y:\mathbb{C}, x:\mathbb{C}^{\mathsf{op}} \vdash [\mathbb{C}(y,x),x]:\mathbb{C}^{\mathsf{op}}.}\ \textbf{contraction}$$

The interpretation for δ results in a functor

$$\lambda y,w,z.[\mathbb{C}(y,w),z]:\mathbb{C} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C}^{\mathsf{op}} \to \mathbb{C}^{\mathsf{op}}$$

whose action on a morphism $(c,d,e) \xrightarrow{(f,g,h)} (c',d',e')$ is the unique mediating arrow defined by the powers and the projections in

$$\begin{array}{ccc}
[\mathbb{C}(c,d),e] = \prod_{k \in \mathbb{C}(c,d)} e & \xrightarrow{\pi_{g \circ k \circ f}} & e\\
\Big\downarrow{\scriptstyle \langle h \circ \pi_{g \circ k \circ f}\rangle_{k \in \mathbb{C}(c',d')}} & & \Big\downarrow{\scriptstyle h}\\
[\mathbb{C}(c',d'),e'] = \prod_{k \in \mathbb{C}(c',d')} e' & \xrightarrow{\pi_k} & e'.
\end{array}$$

The application of **contraction** in the last step of the derivation fixes g to be $h^{\mathsf{op}}:e' \to e$ in $\mathbb{C}$.

5.3.4 Semantics of Substitution

As it is standard in categorical logic the derivable rule for substitution is interpreted as composition.

Proposition 5.3.1 The interpretation for a derivation

$$\dfrac{\begin{array}{cc}\vdots\ \delta_1 & \vdots\ \delta_2\\ \Gamma, x:\mathcal{C}, \Theta \vdash E_1:\mathcal{D} & \Xi \vdash E_2:\mathcal{C}\end{array}}{\Gamma, \Xi, \Theta \vdash E_1[E_2/x]:\mathcal{D}}\ \textbf{substitution} \qquad (5.5)$$

is the functor

$$\Gamma \times \Xi \times \Theta \xrightarrow{\Gamma \times [\![\delta_2 \rhd \Xi \vdash E_2 : \mathcal{C}]\!] \times \Theta} \Gamma \times \mathcal{C} \times \Theta \xrightarrow{[\![\delta_1 \rhd \Gamma, x:\mathcal{C}, \Theta \vdash E_1 : \mathcal{D}]\!]} \mathcal{D}$$

Proof. The proof proceeds as the proof of Theorem 5.1.5 by induction on the derivation δ_1. Here we only analyse the cases for **hom** and **dual** (as given in the proof of Theorem 5.1.5). The case for **power** is analogous to **hom**; the case for **end** follows as for lambda abstraction in a standard manner.

- **hom**

  Here we consider the interpretation of (5.5) where δ_1 is

$$\dfrac{\begin{matrix} \vdots\ \delta' & \vdots\ \delta'' \\ \Delta_1 \vdash E' : \mathcal{A} & \Delta_2 \vdash E'' : \mathcal{A} \end{matrix}}{\Gamma, x:\mathcal{C}, \Theta \vdash \mathcal{A}(E', E'') : \mathbf{Set}}\ \mathbf{hom}$$

  and x is defined in Δ_1, *i.e.* there exists a context Δ_1' such that

$$\Delta_1 = \Gamma^{\mathsf{op}}, x:\mathcal{C}^{\mathsf{op}}, \Delta_1'.$$

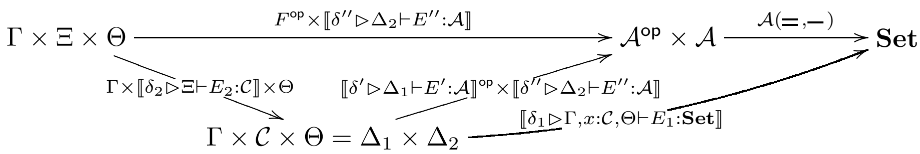
  From the proof of Theorem 5.1.5) by definition the interpretation of (5.5) is

$$\Gamma \times \Xi \times (\Delta_1')^{\mathsf{op}} \times \Delta_2 \xrightarrow{F^{\mathsf{op}} \times [\![\delta'' \rhd \Delta_2 \vdash E'' : \mathcal{A}]\!]} \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \xrightarrow{\mathcal{A}(=,-)} \mathbf{Set}$$

  where the functor F is the composition

$$\Gamma^{\mathsf{op}} \times \Xi^{\mathsf{op}} \times \Delta_1' \xrightarrow{\Gamma^{\mathsf{op}} \times [\![\delta_2 \rhd \Xi \vdash E_2 : \mathcal{C}]\!]^{\mathsf{op}} \times \Delta_1'} \Gamma^{\mathsf{op}} \times \mathcal{C}^{\mathsf{op}} \times \Delta_1' \xrightarrow{[\![\delta' \rhd \Delta_1 \vdash E' : \mathcal{A}]\!]} \mathcal{A}.$$

  As $\Theta = (\Delta_1')^{\mathsf{op}}, \Delta_2$ we have by functoriality of $(-)^{\mathsf{op}}$ that the triangles in



  commute.

- **dual**

  The interpretation of (5.5) where δ_1 is the derivation

$$\dfrac{\begin{matrix} \vdots\ \delta' \\ \Gamma^{\mathsf{op}}, x:\mathcal{C}^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^* : \mathcal{D}^{\mathsf{op}} \end{matrix}}{\Gamma, x:\mathcal{C}, \Theta \vdash E_1 : \mathcal{D}}\ \mathbf{dual}$$

  is by functoriality of $(-)^{\mathsf{op}}$ and induction defined to be

$$\Gamma \times \Xi \times \Theta \xrightarrow{\Gamma \times [\![\delta_2 \rhd \Xi \vdash E_2 : \mathcal{C}]\!] \times \Theta} \Gamma \times \mathcal{C} \times \Theta \xrightarrow{[\![\delta' \rhd \Gamma^{\mathsf{op}}, x:\mathcal{C}^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^* : \mathcal{D}^{\mathsf{op}}]\!]^{\mathsf{op}}} \mathcal{D}$$

  and by definition we have

$$[\![\delta_1 \rhd \Gamma, x:\mathcal{C}, \Theta \vdash E_1 : \mathcal{D}]\!] = [\![\delta' \rhd \Gamma^{\mathsf{op}}, x : \mathcal{C}^{\mathsf{op}}, \Theta^{\mathsf{op}} \vdash E_1^* : \mathcal{D}^{\mathsf{op}}]\!]^{\mathsf{op}}.$$

□

5.4 Coherence

We need to verify that all possible derivations for a judgement yield the same interpretation: a so-called "coherence" result. The importance of coherence is noticed for example by Breazu-Tannen, Coquand, Gunter and Scedrov [BTCGS91]. Coherence was an important issue in the effort to discover the syntax associated to linear logic [Blu91, Wad92b, BBdPH93]. In [Wad92a] Wadler discusses the incoherent system obtained by taking the Abramsky's syntax for linear logic together with the semantics due to Seely.

Due to the presence of limits and ends in the calculus, coherence amounts to showing that different derivations for the same judgement yield isomorphic functors in **CAT**. The cause for having different derivations are mainly the order of the application of the structural rules and duality. The application of these rules do not leave any trace in the terms and then the final sequent does not register the order in which they were applied. The syntax for expressions could be extended with constructors for the structural rules and duality in the spirit of [BBdPH93]. Then every well-typed term would uniquely encode a derivation. The main result in this section shows that this extreme solution is not really necessary.

Proposition 5.2.4 shows that a well-typed sequent can always be proved through a derivation in a special form. The proof proceeds by showing how through local transformations a given derivation can be transformed into the normal form. Then it is enough to show that every transformation step preserves the interpretation (up to isomorphism) to conclude that the system is coherent with respect to the semantics defined in the previous section. This is basically the strategy used in [BTCGS91].

Permuting weakenings. Here we show that the local transformations in the proof of Proposition 5.2.1 permuting occurrences of **weakening** below other rules preserve the interpretation. As structural rules manipulate the contexts it is not surprising that most of the cases rely on the universal property of products and the fact that the $(\_)^{\mathsf{op}}$ preserve products.

- **contraction**

  For the first case we want to show that the functor

$$\Gamma \times \mathcal{A} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A} \rangle} \Gamma \times \mathcal{A} \times \mathcal{A} \xrightarrow{\pi} \Gamma \times \mathcal{A} \xrightarrow{[\![\delta \rhd \Gamma, w{:}\mathcal{A} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}$$

  is just

$$\Gamma \times \mathcal{A} \xrightarrow{[\![\delta \rhd \Gamma, w{:}\mathcal{A} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}.$$

  From the universal property of products it follows that the diagram

$$\Gamma \times \mathcal{A} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A} \rangle} \Gamma \times \mathcal{A} \times \mathcal{A} \xrightarrow{\pi} \Gamma \times \mathcal{A} \qquad \Gamma \times \mathcal{A} \xrightarrow{\mathsf{id}} \Gamma \times \mathcal{A}$$

  commutes as required.

  For the second possibility we want to show that the functors

$$\Gamma \times \mathcal{A} \times \mathcal{C} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{C}, \pi_\mathcal{C} \rangle} \Gamma \times \mathcal{A} \times \mathcal{C} \times \mathcal{C} \xrightarrow{\pi} \Gamma \times \mathcal{C} \times \mathcal{C} \xrightarrow{[\![\delta \rhd \Gamma, x{:}\mathcal{C}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}$$

  and

$$\Gamma \times \mathcal{A} \times \mathcal{C} \xrightarrow{\pi} \Gamma \times \mathcal{C} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{C}, \pi_\mathcal{C} \rangle} \Gamma \times \mathcal{C} \times \mathcal{C} \xrightarrow{[\![\delta \rhd \Gamma, x{:}\mathcal{C}, y{:}\mathcal{C} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}$$

are the same (we omit the isomorphisms related to the use of **exchange**). By the universal property of product we have that the diagram

$$\begin{array}{ccc} \Gamma \times \mathcal{A} \times \mathcal{C} & \xrightarrow{\pi} & \Gamma \times \mathcal{C} \\ \downarrow{\scriptstyle \langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{C}, \pi_\mathcal{C} \rangle} & & \downarrow{\scriptstyle \langle \pi_\Gamma, \pi_\mathcal{C}, \pi_\mathcal{C} \rangle} \\ \Gamma \times \mathcal{A} \times \mathcal{C} \times \mathcal{C} & \xrightarrow[\pi]{} & \Gamma \times \mathcal{C} \times \mathcal{C} \end{array}$$

commutes, and so the functors above are equal.

- **hom**

  Since $(\_)^{\mathsf{op}}$ is a functor and moreover π^{op} is indeed a projection we have that the functors

$$\Gamma_1^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \Gamma_2 \xrightarrow{(\llbracket \delta_1 \rhd \Gamma_1 \vdash E_1 : \mathcal{C} \rrbracket \circ \pi)^{\mathsf{op}} \times \llbracket \delta_2 \rhd \Gamma_2 \vdash E_2 : \mathcal{C} \rrbracket} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}(=,-)} \mathbf{Set}$$

  and

$$\Gamma_1^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \Gamma_2 \xrightarrow{\pi} \Gamma_1^{\mathsf{op}} \times \Gamma_2 \xrightarrow{\llbracket \delta_1 \rhd \Gamma_1 \vdash E_1 : \mathcal{C} \rrbracket^{\mathsf{op}} \times \llbracket \delta_2 \rhd \Gamma_2 \vdash E_2 : \mathcal{C} \rrbracket} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}(=,-)} \mathbf{Set}$$

  are equal.

- **end**

  For a functor

$$F = \llbracket \delta \rhd \Gamma, x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C} \vdash E : \mathcal{D} \rrbracket : \Gamma \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathcal{D}$$

  the transformation takes a functor

$$\Gamma \times \mathcal{A} \xrightarrow{\lambda r, s. \int_w G^{r,s}(w,w)} \mathcal{D}$$

  and yields the functor

$$\Gamma \times \mathcal{A} \xrightarrow{\pi} \Gamma \xrightarrow{\lambda r. \int_w F^r(w,w)} \mathcal{D}$$

  where G is the composition

$$\Gamma \times \mathcal{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \xrightarrow{\pi} \Gamma \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \xrightarrow{F} \mathcal{D}.$$

  By definition G is "dummy" in the argument of type $\mathcal{A}$. Then these functors are *isomorphic* since for $r \in \Gamma$ and $s \in \mathcal{A}$

$$\begin{aligned} \mathcal{D}\big(d, \big((\lambda r. \textstyle\int_w F^r(w,w)) \circ \pi\big)(r,s)\big) &= \mathcal{D}(d, \textstyle\int_w F^r(w,w)) \\ &\cong \mathbf{Dinat}\big(\Delta d, F^r\big) \\ &= \mathbf{Dinat}\big(\Delta d, \big((\lambda r. F^r) \circ \pi\big)(r, s, =, -)\big) \\ &= \mathbf{Dinat}\big(\Delta d, G^{r,s}\big). \end{aligned}$$

  By parametrised representability the isomorphism obtained is natural in r, s.

- **lambda**

  In the proof of Proposition 5.2.1 the case for lambda abstraction is not considered explicitly since it is analogous to that for the end binder. The interpretation given to the application of **lambda**, however, is obtained through a different categorical construction. Here we want to show that the functors defined by

$$\Gamma \times \mathcal{A} \xrightarrow{\boldsymbol{\lambda}(F)} [\mathbb{C}, \mathcal{D}]$$

where

$$F = \Gamma \times \mathcal{A} \times \mathbb{C} \xrightarrow{\pi} \Gamma \times \mathbb{C} \xrightarrow{[\![\delta \triangleright \Gamma, x{:}\mathbb{C} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}$$

and

$$\Gamma \times \mathcal{A} \xrightarrow{\pi} \Gamma \xrightarrow{\boldsymbol{\lambda}([\![\delta \triangleright \Gamma, x{:}\mathbb{C} \vdash E{:}\mathcal{D}]\!])} [\mathbb{C}, \mathcal{D}]$$

are actually the same. This follows from

$$\dfrac{\dfrac{\Gamma \times \mathcal{A} \xrightarrow{\pi} \Gamma \xrightarrow{\boldsymbol{\lambda}([\![\delta \triangleright \Gamma, x{:}\mathbb{C} \vdash E{:}\mathcal{D}]\!])} [\mathbb{C}, \mathcal{D}]}{\Gamma \times \mathcal{A} \times \mathbb{C} \underset{\pi}{\overset{\pi \times \mathbb{C}}{\underset{=}{\longrightarrow}}} \Gamma \times \mathbb{C} \xrightarrow{[\![\delta \triangleright \Gamma, x{:}\mathbb{C} \vdash E{:}\mathcal{D}]\!]} \mathcal{D}}}{\Gamma \times \mathcal{A} \xrightarrow{\boldsymbol{\lambda}(F)} [\mathbb{C}, \mathcal{D}].}$$

Again we omit the isomorphism arising from the use of **exchange**.

- **dual**

  As $(\_)^{\mathsf{op}}$ is a functor we need to verify that the functors

  $$\Gamma^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi^{\mathsf{op}}} \Gamma^{\mathsf{op}} \xrightarrow{[\![\delta \triangleright \Gamma \vdash E{:}\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}$$

  and

  $$\Gamma^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi} \Gamma^{\mathsf{op}} \xrightarrow{[\![\delta \triangleright \Gamma \vdash E{:}\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}$$

  are equal. Indeed this follows by the observation that π^{op} is the projection π.

Permuting contractions. The transformations in the proof of Proposition 5.2.2 permute occurrences of **contraction** below other rules. Here we show that each permutation preserves the interpretation. As for W-normal derivations most of the cases follow from the universal property of products and that $(\_)^{\mathsf{op}}$ preserve products.

- **weakening**

  This case is symmetric to the one in the previous section.

- **hom**

  Given the functors

  $$F = [\![\delta_1 \triangleright \Gamma_1, x{:}\mathcal{A}, y{:}\mathcal{A} \vdash E_1{:}\mathcal{C}]\!] : \Gamma_1 \times \mathcal{A} \times \mathcal{A} \to \mathcal{C}$$

  and

  $$G = [\![\delta_2 \triangleright \Gamma_2 \vdash E_2{:}\mathcal{C}]\!] : \Gamma_2 \to \mathcal{C},$$

  it follows from the universal property of products and since $(\_)^{\mathsf{op}}$ preserves products that the diagram of functors

  $$\begin{array}{ccc} \Gamma_1^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \Gamma_2 & & \\ {\scriptstyle \langle \pi_{\Gamma_1^{\mathsf{op}}}, \pi_{\mathcal{A}^{\mathsf{op}}}, \pi_{\mathcal{A}^{\mathsf{op}}}, \pi_{\Gamma_2} \rangle} \downarrow & \searrow^{(F \circ \langle \pi_{\Gamma_1}, \pi_{\mathcal{A}}, \pi_{\mathcal{A}}, \pi_{\Gamma_2} \rangle)^{\mathsf{op}} \times G} & \\ \Gamma_1^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \Gamma_2 & \xrightarrow[F^{\mathsf{op}} \times G]{} & \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \end{array}$$

  commutes as required.

- **end**

  Given the interpretation

  $$F = [\![\delta \rhd \Gamma, w{:}\mathcal{A}, z{:}\mathcal{A}, x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash E{:}\mathcal{D}]\!] : \Gamma \times \mathcal{A} \times \mathcal{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathcal{D}$$

  we need to verify the functors

  $$\Gamma \times \mathcal{A} \xrightarrow{\lambda r,s.\int_w G^{r,s}(w,w)} \mathcal{D}$$

  where

  $$G = F \circ \langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A}, \pi_{\mathbb{C}^{\mathsf{op}}}, \pi_\mathbb{C} \rangle : \Gamma \times \mathcal{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathcal{D}$$

  and

  $$\Gamma \times \mathcal{A} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A} \rangle} \Gamma \times \mathcal{A} \times \mathcal{A} \xrightarrow{\lambda r,s,t.\int_w F^{r,s,t}(w,w)} \mathcal{D}$$

  are *isomorphic*. Thus for every $r \in \Gamma$ and $s \in \mathcal{A}$ we have

  $$\begin{aligned}\mathcal{D}\big(d, \big((\lambda r,s,t.\textstyle\int_w F^{r,s,t}(w,w)) \circ \langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A} \rangle\big)(r,s)\big) &= \mathcal{D}(d, \textstyle\int_w F^{r,s,s}(w,w)) \\ &\cong \mathbf{Dinat}\big(\Delta d, F^{r,s,s}\big) \\ &= \mathbf{Dinat}\big(\Delta d, G^{r,s}\big)\end{aligned}$$

  which by parametrised representability is natural in r, s. The case for **lambda** is similar to the case for Proposition 5.2.1.

- **dual**

  This follows by functoriality of $(\_)^{\mathsf{op}}$ and the fact that this functor preserves products. Thus we have that the functors

  $$\Gamma^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \xrightarrow{\langle \pi_\Gamma, \pi_\mathcal{A}, \pi_\mathcal{A} \rangle^{\mathsf{op}}} \Gamma^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \xrightarrow{[\![\delta \rhd \Gamma, x:\mathcal{A}, y:\mathcal{A} \vdash E:\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}$$

  and

  $$\Gamma^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \xrightarrow{\langle \pi_{\Gamma^{\mathsf{op}}}, \pi_{\mathcal{A}^{\mathsf{op}}}, \pi_{\mathcal{A}^{\mathsf{op}}} \rangle} \Gamma^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \times \mathcal{A}^{\mathsf{op}} \xrightarrow{[\![\delta \rhd \Gamma, x:\mathcal{A}, y:\mathcal{A} \vdash E:\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}$$

  are equal.

Removing non-essential duals. An occurrence of **dual** violating the condition defined by the normal form is permuted above occurrences of **lambda**, **application**, **weakening** and **contraction** until it reaches a legal position, meets another occurrence of **dual** or a sequent whose expression is a variable. That two applications of **dual** one after the other are removed is justified in the semantics by the fact that the 2-functor $(\_)^{\mathsf{op}}$ is an involution by definition. The case for variables is a consequence of preservation of products.

- **lambda**

  This follows from the adjoint property of $\boldsymbol{\lambda}$ (2.14) and the fact that the functor $(\_)^{\mathsf{op}}$ is full and faithful:

  $$\frac{\dfrac{\dfrac{\dfrac{\Gamma^{\mathsf{op}} \xrightarrow{\boldsymbol{\lambda}([\![\delta \rhd \Gamma, x:\mathbb{C} \vdash E:\mathcal{D}]\!]^{\mathsf{op}})} [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]}{\Gamma^{\mathsf{op}} \times \mathbb{C}^{\mathsf{op}} \xrightarrow{[\![\delta \rhd \Gamma, x:\mathbb{C} \vdash E:\mathcal{D}]\!]^{\mathsf{op}}} \mathcal{D}^{\mathsf{op}}}}{\Gamma \times \mathbb{C} \xrightarrow{[\![\delta \rhd \Gamma, x:\mathbb{C} \vdash E:\mathcal{D}]\!]} \mathcal{D}}}{\Gamma \xrightarrow{\boldsymbol{\lambda}([\![\delta \rhd \Gamma, x:\mathbb{C} \vdash E:\mathcal{D}]\!])} [\mathbb{C}, \mathcal{D}]}}{\Gamma^{\mathsf{op}} \xrightarrow{(\boldsymbol{\lambda}([\![\delta \rhd \Gamma, x:\mathbb{C} \vdash E:\mathcal{D}]\!]))^{\mathsf{op}}} [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}].}$$

- **application**

  This is justified by the fact that $(\_)^{\mathsf{op}}$ preserves products and the exponentials in **CAT**. Thus the functor $eval^{\mathsf{op}}$ is in fact the counit for the adjunction defined by $\boldsymbol{\lambda}$.

- **weakening** and **contraction**

  These cases are symmetric to the cases from Propositions 5.2.1 and 5.2.2 respectively.

By Proposition 5.2.4 a derivation $\delta \rhd \Gamma \vdash E\!:\!\mathcal{D}$ can be translated into a derivation $\delta' \rhd \Gamma' \vdash E'\!:\!\mathcal{D}$ in the normal form where Γ' and E' indicate possible renaming of variables. The renaming of variables in the context does not affect the interpretation, *i.e.* $[\![\Gamma]\!] = [\![\Gamma']\!]$. Then from the discussion above the functors

$$[\![\delta \rhd \Gamma \vdash E\!:\!\mathcal{D}]\!]\!:\!\Gamma \to \mathcal{D} \quad \text{and} \quad [\![\delta' \rhd \Gamma' \vdash E'\!:\!\mathcal{D}]\!]\!:\!\Gamma \to \mathcal{D}$$

are isomorphic. The syntactic structure of a well-typed judgement defines a set of derivations in normal form that differ only in the order of the application of the structural rules (and is free of spourious uses of duality). The interpretation for those derivations in normal form is not affected (up to isomorphism) by the application of structural rules including duality. Therefore, different derivations for a sequent yield the same interpretation up to isomorphism.

Theorem 5.4.1 The categorical interpretation is coherent up to isomorphism.

It is essential to interpret the types $[\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]$ and $[\mathbb{C}, \mathcal{D}]^{\mathsf{op}}$ as the category of functors whose domain is $\mathbb{C}^{\mathsf{op}}$ and codomain is $\mathcal{D}^{\mathsf{op}}$. In the syntax this is represented by the definition clause

$$(\lambda x^{\mathbb{C}}.E)^* = \lambda x^{\mathbb{C}^{\mathsf{op}}}.E^*$$

of the meta-operator $(\_)^*$.

Remark 5.4.2 In general a particular interpretation of a well-typed sequent $\Gamma \vdash E\!:\!\mathcal{D}$ assumes a derivation δ. Otherwise, if no derivation is mentioned, the canonical one is assumed and taken to give the interpretation.

5.5 Without Duality

The typing rules for coends and powers are derived from the rules for ends and powers together with duality. That reflects a common practice in category theory. We could have introduced primitive rules for coends and copowers instead:

coend

$$\frac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash E\!:\!\mathcal{D} \qquad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{cocomplete}}{\Gamma \vdash \int^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E\!:\!\mathcal{D}}$$

copower

$$\frac{\Gamma_1 \vdash E_1\!:\!\mathbf{Set} \qquad \Gamma_2 \vdash E_2\!:\!\mathcal{C} \qquad \mathcal{C}\ \mathsf{cocomplete}}{\Gamma_1, \Gamma_2 \vdash E_1 \otimes E_2\!:\!\mathcal{C}.}$$

It is sensible to ask whether these rules are enough to absorb the rule **dual**. Because of the presence of the basic types like **Set** the answer is negative. Consider for example the derivation

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \vdash E_1 : \mathcal{C} & \Gamma_2 \vdash E_2 : \mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}}\ \mathbf{hom}}{\Gamma_1, \Gamma_2^{\mathsf{op}} \vdash \mathcal{C}(E_1^*, E_2^*) : \mathbf{Set}^{\mathsf{op}}}\ \mathbf{dual}$$

where the application of **dual** provides the only way in which we can introduce $\mathbf{Set}^{\mathsf{op}}$.

In order to obtain a system closed under duality we need rules to introduce the dual forms of the basic types:

hom*

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \qquad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1, \Gamma_2^{\mathsf{op}} \vdash \mathcal{C}(E_1^*, E_2^*) : \mathbf{Set}^{\mathsf{op}}}$$

singleton*

$$\frac{}{\vdash 1 : \mathbf{Set}^{\mathsf{op}}}.$$

Thus every essential occurrence of **dual** (as defined in 5.2.3) can be replaced by one of the new rules. For the non-essential occurrences they can be removed by following exactly the same procedure as in the proof for DCW-normal derivations.

The definition of the operation $(\_)^*$ over raw expressions may appear arbitrary for the case of hom-expressions. In fact it all amounts to defining a suitable syntax for the functor

$$\mathcal{C} \times \mathcal{C}^{\mathsf{op}} \xrightarrow{(\mathcal{C}(=,-))^{\mathsf{op}}} \mathbf{Set}^{\mathsf{op}}.$$

Consider the alternative definition

$$\mathcal{C}(E_1, E_2)^* = \mathcal{C}(E_1, E_2) \tag{5.6}$$

where the type $\mathbf{Set}^{\mathsf{op}}$ and the change of variance in the context breaks the ambiguity with the hom-expressions of type **Set**, for example

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \vdash E_1 : \mathcal{C} & \Gamma_2 \vdash E_2 : \mathcal{C}\end{array}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}}\ \mathbf{hom}}{\Gamma_1, \Gamma_2^{\mathsf{op}} \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}^{\mathsf{op}}.}\ \mathbf{dual}$$

In (5.6) the contravariant argument, the one on the right, is assumed to be dualised. That contrasts with our original definition of $(\_)^*$ where the contravariant argument takes the correct form while the covariant one is "implicitly" dualised. One disadvantage of (5.6) is that the definition of substitution over raw terms is not longer valid. To adjust substitution demands to know in advance the type of a hom-expression (*i.e.* **Set** or $\mathbf{Set}^{\mathsf{op}}$) and then could not be carried out on raw expressions in general. In few words if the hom-expression is of type $\mathbf{Set}^{\mathsf{op}}$ the substitution should act as

$$\mathcal{C}(E_1, E_2)[E/x] = \mathcal{C}(E_1[E/x], E_2[E^*/x]).$$

To avoid this inconvenience we prefer

$$(\mathcal{C}(E_1, E_2))^* = \mathcal{C}(E_1^*, E_2^*)$$

as a syntax for $\mathcal{C}(=,-)^{\mathsf{op}}$ rather than (5.6).

5.6 Products

The cartesian structure of **CAT** is already present in the semantics of contexts. An expression-in-context corresponds to a functor with as many arguments as variables in the context. To fully incorporate functors with multiple variables into the calculus we need finite products for types

$$\mathcal{C} ::= \ldots \mid \mathbf{1} \mid \mathcal{C}_1 \times \mathcal{C}_2$$

and the rules

$$\frac{\mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{small}}{\mathcal{C} \times \mathcal{D}\ \mathsf{small}} \qquad \frac{}{\mathbf{1}\ \mathsf{small}} \qquad \frac{}{\mathbf{1}\ \mathsf{complete.}}$$

The corresponding syntax and the rules for products are the usual ones from the typed lambda calculus with pairs:

unit

$$\frac{}{\vdash \langle\ \rangle : \mathbf{1}}$$

pair

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \quad \Gamma_2 \vdash E_2 : \mathcal{D}}{\Gamma_1, \Gamma_2 \vdash \langle E_1, E_2 \rangle : \mathcal{C} \times \mathcal{D}}$$

first

$$\frac{\Gamma \vdash E : \mathcal{C} \times \mathcal{D}}{\Gamma \vdash \mathit{fst}^{\mathcal{C}\times\mathcal{D}}(E) : \mathcal{C}}$$

second

$$\frac{\Gamma \vdash E : \mathcal{C} \times \mathcal{D}}{\Gamma \vdash \mathit{snd}^{\mathcal{C}\times\mathcal{D}}(E) : \mathcal{D}}$$

Duality over a product type acts by dualising each factor, this is consistent with the definition of Γ^{op} for a context Γ. The meta-operation $(\_)^*$ acts over the new syntax as

$$\begin{aligned}
(\langle\ \rangle)^* &= \langle\ \rangle \\
\langle E_1, E_2 \rangle^* &= \langle E_1^*, E_2^* \rangle \\
\mathit{fst}^{\mathcal{C}\times\mathcal{D}}(E)^* &= \mathit{fst}^{\mathcal{C}^{\mathsf{op}}\times\mathcal{D}^{\mathsf{op}}}(E^*) \\
\mathit{snd}^{\mathcal{C}\times\mathcal{D}}(E)^* &= \mathit{snd}^{\mathcal{C}^{\mathsf{op}}\times\mathcal{D}^{\mathsf{op}}}(E^*).
\end{aligned}$$

This definition is justified in the semantics by the fact that the functor $(\_)^{\mathsf{op}}$ preserves products. By checking the cases introduced by these new rules we can verify that Propositions 5.1.2 and 5.1.4, and Theorem 5.1.5 are still valid.

The proofs of Propositions 5.2.1, 5.2.2 and 5.2.4 related to the normal forms can be extended to contemplate the rules for products as well. The empty product defines a new basic type $\mathbf{1}$, we identify in the syntax $\mathbf{1}^{\mathsf{op}}$ with $\mathbf{1}$. Thus the expression

$\mathbf{1}(x,x)$ is well-typed

$$\dfrac{\dfrac{}{z\!:\!\mathbf{1} \vdash z\!:\!\mathbf{1}}\ \textbf{identity} \quad \dfrac{}{w\!:\!\mathbf{1} \vdash w\!:\!\mathbf{1}}\ \textbf{identity}}{\dfrac{z\!:\!\mathbf{1}, w\!:\!\mathbf{1} \vdash \mathbf{1}(z,w)\!:\!\mathbf{Set}}{x\!:\!\mathbf{1} \vdash \mathbf{1}(x,x)\!:\!\mathbf{Set}.}\ \textbf{contraction}}\ \textbf{hom}$$

The interpretation for the new derivations is defined through the cartesian structure of **CAT** as usual:

- **unit**

  The new type $\mathbf{1}$ represents the final object in **CAT**, *i.e.* the singleton category and the derivation

  $$\dfrac{}{\vdash \langle\ \rangle\!:\!\mathbf{1}}$$

  is just the unique endomorphism

  $$\mathbf{1} \xrightarrow{\mathsf{id}} \mathbf{1}.$$

- **pair**

  The derivation

  $$\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Gamma_1 \vdash E_1\!:\!\mathcal{C}\end{matrix} \quad \begin{matrix}\vdots\ \delta_2 \\ \Gamma_2 \vdash E_2\!:\!\mathcal{D}\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \langle E_1, E_2\rangle\!:\!\mathcal{C} \times \mathcal{D}}\ \textbf{pair}$$

  is interpreted as

  $$\Gamma_1 \times \Gamma_2 \xrightarrow{[\![\delta_1 \rhd \Gamma_1 \vdash E_1:\mathcal{C}]\!] \times [\![\delta_2 \rhd \Gamma_2 \vdash E_2:\mathcal{D}]\!]} \mathcal{C} \times \mathcal{D}.$$

- **first**

  The derivation

  $$\dfrac{\begin{matrix}\vdots\ \delta \\ \Gamma \vdash E\!:\!\mathcal{C} \times \mathcal{D}\end{matrix}}{\Gamma \vdash \mathit{fst}^{\mathcal{C}\times\mathcal{D}}(E)\!:\!\mathcal{C}}\ \textbf{first}$$

  is interpreted as

  $$\Gamma \xrightarrow{[\![\delta \rhd \Gamma \vdash E:\mathcal{C}\times\mathcal{D}]\!]} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi} \mathcal{C}.$$

  Similarly for the rule **second**.

If there is a derivation for the expression-in-context

$$\Gamma_1, x\!:\!\mathcal{A}, y\!:\!\mathcal{B}, \Gamma_2 \vdash E\!:\!\mathcal{D} \tag{5.7}$$

then there is a derivation for

$$\Gamma_1, v\!:\!\mathcal{A} \times \mathcal{B}, \Gamma_2 \vdash E[\mathit{fst}^{\mathcal{A}\times\mathcal{B}}(v)/x, \mathit{snd}^{\mathcal{A}\times\mathcal{B}}(v)/y]\!:\!\mathcal{D}.$$

This follows from substitution since from the rules above we have the necessary derivations

$$\dfrac{\dfrac{}{v\!:\!\mathcal{A}\times\mathcal{B}\vdash v\!:\!\mathcal{A}\times\mathcal{B}}\ \textbf{identity}}{v\!:\!\mathcal{A}\times\mathcal{B}\vdash \mathit{fst}^{\mathcal{A}\times\mathcal{B}}(v)\!:\!\mathcal{A}}\ \textbf{first} \qquad \dfrac{\dfrac{}{v\!:\!\mathcal{A}\times\mathcal{B}\vdash v\!:\!\mathcal{A}\times\mathcal{B}}\ \textbf{identity}}{v\!:\!\mathcal{A}\times\mathcal{B}\vdash \mathit{snd}^{\mathcal{A}\times\mathcal{B}}(v)\!:\!\mathcal{B}.}\ \textbf{second}$$

In a more general setting this correspondence is a consequence of the relation between the natural deduction systems and the intuitionistic sequent calculus (see [TD88, TS00]).

As **CAT** is a bicartesian category, *i.e.* a category with both finite product and coproducts together with the obvious distributivity laws, we could extend the syntax with "disjoint unions" constructors. Indeed as **CAT** is complete and cocomplete we could be tempted to develop a richer theory for types where if not all limits and colimits are consider at least includes some kind of fixed point construction. This, however, escapes from the main focus of this dissertation and it is not developed further.

Chapter 6

Natural Isomorphisms

We extend the language for functors with judgements for natural isomorphisms. A derivation for an isomorphism judgement is interpreted as a natural isomorphism between functors. As with the functoriality judgements this interpretation is defined by induction but in this case the system is not coherent. This theory of isomorphisms provides a formal presentation of the repertoire of natural isomorphism used in chapters 2 and 3. We also extend the syntax to represent weighted limits and show how under the assumption of completeness the new expressions is naturally isomorphic to an end formula (Proposition 3.5.3). At the end of this chapter we propose another approach to contravariance where the meta-operation $(\_)^*$ becomes part of the syntax.

6.1 Isomorphism Judgements

The natural isomorphisms extend the equational theory of the lambda calculus with rules for Fubini, the naturality formula, the Yoneda Lemma and the definitions for end and power. There are also two rules for duality from which we derive rules for coends and copowers. As representability is the key concept in our reasoning most of the rules in fact formalise natural isomorphisms between expressions of type **Set**.

The syntactic judgement

$$x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}$$

says that the expressions E_1 and E_2 are naturally isomorphic in the variables $x_1, \cdots, x_n$. The rules for isomorphisms shall ensure that in a derivation for the sequent above both E_1 and E_2 are functorial in those variables as well, *i.e.*

$$x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E_1 : \mathcal{C} \quad \text{and} \quad x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E_2 : \mathcal{C}$$

are well-typed.

6.1.1 Equational Theory of the Lambda Calculus

As the language for functors extend the syntax of the lambda calculus we inherit its equational theory [Bar84, Mit96]. These rules basically allow us to replace "equals for equals" in expressions.

Structural Rules

As for the term-formation rules we need rules to manipulate contexts:

weakening

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n, x_{n+1} : \mathcal{C}_{n+1} \vdash E_1 \cong E_2 : \mathcal{C}}$$

exchange

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_i : \mathcal{C}_i, x_{i+1} : \mathcal{C}_{i+1}, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_{i+1} : \mathcal{C}_{i+1}, x_i : \mathcal{C}_i, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}$$

contraction

$$\frac{\Gamma, x : \mathcal{C}, y : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D}}{\Gamma, z : \mathcal{C} \vdash E_1[z/x, z/y] \cong E_2[z/x, z/y] : \mathcal{D}}$$

It is obvious from an inductive argument that the expressions in the conclusions are functorial if the premises are. In general we shall not bother to mention the use of these rules when reasoning about natural isomorphisms.

Conversion Rules

To define the evaluation of functors we have:

β-conversion

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 : \mathcal{D} \quad \Gamma_2 \vdash E_2 : \mathcal{C} \quad \mathcal{C} \text{ small}}{\Gamma_1, \Gamma_2 \vdash (\lambda x^{\mathcal{C}}.E_1)E_2 \cong E_1[E_2/x] : \mathcal{D}}$$

where the substitution $E_1[E_2/x]$ is defined as in 5.1.3. From the premises both expressions in the final sequent are well-typed. The left-hand side follows by using the rules **lambda** and **application**, and the right-hand side from Theorem 5.1.5.

For extensionality of functors we have the rule:

η-conversion

$$\frac{\Gamma \vdash E : [\mathcal{C}, \mathcal{D}]}{\Gamma \vdash \lambda x^{\mathcal{C}}.Ex \cong E : [\mathcal{C}, \mathcal{D}]} \quad x \text{ is not free in } E.$$

Notice that $\mathcal{C}$ is necessary small. The expression E is functorial since is proved by the premise of the rule. That the expression $\lambda x^{\mathcal{C}}.Ex$ is functorial follows from

$$\dfrac{\dfrac{\Gamma \vdash E : [\mathcal{C}, \mathcal{D}] \quad \dfrac{}{x : \mathcal{C} \vdash x : \mathcal{C}}\ \textbf{identity}}{\Gamma, x : \mathcal{C} \vdash Ex : \mathcal{D}}\ \textbf{application}}{\Gamma \vdash \lambda x^{\mathcal{C}}.Ex : [\mathcal{C}, \mathcal{D}].}\ \textbf{lambda}$$

We sometimes refer to these rules under the general designation of **conversion** rules.

Equivalence Rules

The next rules make the relation $\cong$ an equivalence:

reflexivity

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma \vdash E \cong E : \mathcal{D}}$$

symmetry

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{D}}{\Gamma \vdash E_2 \cong E_1 : \mathcal{D}}$$

transitivity

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{D} \quad \Gamma \vdash E_2 \cong E_3 : \mathcal{D}}{\Gamma \vdash E_1 \cong E_3 : \mathcal{D}.}$$

In the case of **reflexivity** is obvious that the expression E is functorial since it is required by the premise of the rule. For the rules **symmetry** and **transitivity** that the expressions involved are functorial follows by an inductive argument. The rule for **transitivity** plays an important role in the calculus since most of the proofs proceed by chaining together natural isomorphisms.

Congruence Rules

Now we extend the relation $\cong$ to be compatible with the construction of the different expressions in the language, this is achieve by the so-called *congruence rules*. Thus for every piece of syntax there is a corresponding congruence rule:

hom cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : \mathcal{C} \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) \cong \mathcal{C}(E_1', E_2') : \mathbf{Set}}$$

power cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : \mathbf{Set} \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C} \quad \mathcal{C} \text{ complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] \cong [E_1', E_2'] : \mathcal{C}}$$

ξ-conversion lambda

$$\frac{\Gamma, x : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D} \quad \mathcal{C} \text{ small}}{\Gamma \vdash \lambda x^{\mathcal{C}}.E_1 \cong \lambda x^{\mathcal{C}}.E_2 : [\mathcal{C}, \mathcal{D}]}$$

application cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : [\mathcal{C}, \mathcal{D}] \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 \cong E_1' E_2' : [\mathcal{C}, \mathcal{D}]}$$

ξ-conversion end

$$\frac{\Gamma, x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D} \quad \mathcal{C} \text{ small} \quad \mathcal{D} \text{ complete}}{\Gamma \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E_1 \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E_2 : \mathcal{D}.}$$

It is obvious in all these rules that the expressions in the conclusions are indeed functorial if one assumes that the premises are. Because of the constraint on the formation of functor categories ξ-conversion for lambda abstraction is not enough to derive the other congruence rules (see [LS86]). For example the rule **hom cong** can not be derived from **β-conversion** if the type $\mathcal{C}$ is not small (similarly for **power cong**).

This series is complete with the rule for duality introduced in the next section – but it cannot be consider a congruence rules since the $(\_)^*$ is not part of the syntax. These rules justify to replace "equals for equals" in expressions regardless questions of bound variables [Bar84, Proposition 21.19]. To simplify the notation in the proofs we often do not mention the congruence rules, or just refer to them under the generic name of **congruence**.

6.1.2 Representables and Ends

The following rule allows us to move from an arbitrary type $\mathcal{C}$ into **Set**, the place where representable functors live:

representable

$$\frac{\Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1) \cong \mathcal{C}(x, E_2){:}\mathbf{Set}}{\Gamma \vdash E_1 \cong E_2{:}\mathcal{C}.}$$

Notice that from the assumption on the manipulation of contexts the variable x cannot be free in E_1 or E_2. From the judgement

$$\Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1) \cong \mathcal{C}(x, E_2){:}\mathbf{Set}$$

we inductively have that both

$$\Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1){:}\mathbf{Set} \quad \text{and} \quad \Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_2){:}\mathbf{Set}$$

are derivable and then as x is not free in E_1 or E_2 we have that the sequents

$$\Gamma \vdash E_1{:}\mathcal{C} \quad \text{and} \quad \Gamma \vdash E_2{:}\mathcal{C}$$

are derivable in the type theory.

The rule **representable** can actually be read bottom-up as well, though that is derivable from the congruence rule for hom-expressions:

$$\dfrac{\dfrac{\dfrac{}{x{:}\mathcal{C} \vdash x{:}\mathcal{C}}\ \textbf{identity}}{x{:}\mathcal{C} \vdash x \cong x{:}\mathcal{C}}\ \textbf{reflexivity} \qquad \Gamma \vdash E_1 \cong E_2{:}\mathcal{C}}{\Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1) \cong \mathcal{C}(x, E_1){:}\mathbf{Set}.}\ \textbf{hom cong}$$

Then we have the derivable rule

$$\frac{\Gamma \vdash E_1 \cong E_2{:}\mathcal{C}}{\Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1) \cong \mathcal{C}(x, E_2){:}\mathbf{Set}}$$

with the side condition that x is not free in E_1 or E_2.

The following rule formalises the naturality formula:

nat formula

$$\frac{\Gamma_1, x{:}\mathcal{C} \vdash E_1{:}\mathcal{D} \quad \Gamma_2, y{:}\mathcal{C} \vdash E_2{:}\mathcal{D} \quad \mathcal{C}\ \mathsf{small}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [\mathcal{C}, \mathcal{D}](\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.E_2) \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(E_1, E_2){:}\mathbf{Set}.}$$

From the premises of this rule we have

$$\dfrac{\dfrac{\Gamma_1, x\!:\!\mathcal{C} \vdash E_1\!:\!\mathcal{D}}{\Gamma_1 \vdash \lambda x^{\mathcal{C}}.E_1\!:\![\mathcal{C},\mathcal{D}]}\ \textbf{lambda} \qquad \dfrac{\Gamma_2, y\!:\!\mathcal{C} \vdash E_2\!:\!\mathcal{D}}{\Gamma_2 \vdash \lambda y^{\mathcal{C}}.E_2\!:\![\mathcal{C},\mathcal{D}]}\ \textbf{lambda}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [\mathcal{C},\mathcal{D}](\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.E_2)\!:\!\mathbf{Set}}\ \textbf{hom}$$

and

$$\dfrac{\dfrac{\Gamma_1, x\!:\!\mathcal{C} \vdash E_1\!:\!\mathcal{D} \quad \Gamma_2, y\!:\!\mathcal{C} \vdash E_2\!:\!\mathcal{D}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash \mathcal{D}(E_1,E_2)\!:\!\mathbf{Set}}\ \textbf{hom+exchange}\text{'s}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(E_1,E_2)\!:\!\mathbf{Set}.}\ \textbf{end}$$

The rule **nat formula** exemplifies the importance of the *Church*-style presentation of the syntax. The variable x is bound by the lambda binder as covariant and by the integral binder as contravariant. This is a consequence of the assumption that the left argument of a hom-expression is implicitly dualised. We have previously used this information in an informal manner, for instance in the last step of the proof of Proposition 3.5.5 where we informally write

$$\int_x [G(x), [F(x), Z]] \cong [\mathbb{C}, \mathbf{Set}]\big(G, [F(-), Z]\big).$$

In order to define the functor category we need to keep track of the types of the *occurrences* of x. In the formal calculus there are different variables for the different occurrences of x in the expression above.

For interchange of ends we have the rule:

Fubini

$$\dfrac{\Gamma, x_1\!:\!\mathcal{C}^{\mathsf{op}}, x_2\!:\!\mathcal{C}, y_1\!:\!\mathcal{D}^{\mathsf{op}}, y_2\!:\!\mathcal{D} \vdash E\!:\!\mathcal{E} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{small} \quad \mathcal{E}\ \mathsf{complete}}{\Gamma \vdash \int_{x_1^{\mathcal{C}^{\mathsf{op}}}, x_2^{\mathcal{C}}} \int_{y_1^{\mathcal{D}^{\mathsf{op}}}, y_2^{\mathcal{D}}} E \cong \int_{y_1^{\mathcal{D}^{\mathsf{op}}}, y_2^{\mathcal{D}}} \int_{x_1^{\mathcal{C}^{\mathsf{op}}}, x_2^{\mathcal{C}}} E\!:\!\mathcal{E}}$$

The functoriality of both expressions follow from the premises by application of the rule **end** twice and the use of some **exchange**'s.

Through representables we can give the definition of end as:

def end

$$\dfrac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash E\!:\!\mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E) \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(w, E)\!:\!\mathbf{Set}}$$

where from the assumption on contexts w cannot be free in E. From the premises we have

$$\dfrac{\dfrac{}{w\!:\!\mathcal{D} \vdash w\!:\!\mathcal{D}}\ \textbf{identity} \quad \dfrac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash E\!:\!\mathcal{D}}{\Gamma \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E\!:\!\mathcal{D}}\ \textbf{end}}{\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E)\!:\!\mathbf{Set}}\ \textbf{hom+exchange}\text{'s}$$

and

$$\dfrac{\dfrac{}{w\!:\!\mathcal{D} \vdash w\!:\!\mathcal{D}}\,\textbf{identity} \qquad \Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash E\!:\!\mathcal{D}}{\dfrac{w\!:\!\mathcal{D}^{\mathsf{op}}, \Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash \mathcal{D}(w,E)\!:\!\mathbf{Set}}{\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(w,E)\!:\!\mathbf{Set}.}\,\textbf{end+exchange}\text{'s}}\,\textbf{hom}$$

Similarly we can give the definition of power:

def power

$$\frac{\Gamma_1 \vdash E_1\!:\!\mathbf{Set} \quad \Gamma_2 \vdash E_2\!:\!\mathcal{D} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(x,[E_1,E_2]) \cong [E_1, \mathcal{D}(x,E_2)]\!:\!\mathbf{Set}}$$

where again x is not free in E. From the premises we have

$$\dfrac{\dfrac{}{x\!:\!\mathcal{D} \vdash x\!:\!\mathcal{D}}\,\textbf{identity} \qquad \dfrac{\Gamma_1 \vdash E_1\!:\!\mathbf{Set} \quad \Gamma_2 \vdash E_2\!:\!\mathcal{D}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1,E_2]\!:\!\mathcal{D}}\,\textbf{power}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(x,[E_1,E_2])\!:\!\mathbf{Set}}\,\textbf{hom+exchange}\text{'s}$$

and

$$\dfrac{\Gamma_1 \vdash E_1\!:\!\mathbf{Set} \qquad \dfrac{\dfrac{}{x\!:\!\mathcal{D} \vdash x\!:\!\mathcal{D}}\,\textbf{identity} \quad \Gamma_2 \vdash E_2\!:\!\mathcal{D}}{\Gamma_2, x\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(x,E_2)\!:\!\mathbf{Set}}\,\textbf{hom+exchange}\text{'s}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x\!:\!\mathcal{D}^{\mathsf{op}} \vdash \big[E_1, \mathcal{D}(x,E_2)\big]\!:\!\mathbf{Set}.}\,\textbf{power}$$

In both cases, **def end** and **def power**, a universal property on an arbitrary complete category $\mathcal{D}$ is expressed as the same universal property but in **Set**.

There is as well a definition for the singleton set:

def singleton

$$\frac{\Gamma \vdash E\!:\!\mathcal{D} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma \vdash [1,E] \cong E\!:\!\mathcal{D}}$$

where functoriality of E follows from the premise and 1 is proved by the rule **singleton**.

The Yoneda lemma is given as an end by the rule:

Yoneda

$$\frac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}} \vdash E\!:\!\mathbf{Set} \quad \mathcal{C}\ \mathsf{small}}{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}} \vdash E \cong \int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \big[\mathcal{C}(y,x), E[w/x]\big]\!:\!\mathbf{Set}}\ x, w \text{ are not free in } E.$$

The left-hand side of the conclusion is functorial since it is just the premise. For the right-hand expression we have

$$\cfrac{\cfrac{\cfrac{}{x\!:\!\mathcal{C} \vdash x\!:\!\mathcal{C}}\,\textbf{identity} \qquad \cfrac{}{y\!:\!\mathcal{C} \vdash y\!:\!\mathcal{C}}\,\textbf{identity}}{x\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash \mathcal{C}(x,y)\!:\!\mathbf{Set}}\,\textbf{hom} \qquad \Gamma, w\!:\!\mathcal{C}^{\mathsf{op}} \vdash E[w/x]\!:\!\mathbf{Set}}{\cfrac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}}, w\!:\!\mathcal{C}^{\mathsf{op}}, y\!:\!\mathcal{C} \vdash \big[\mathcal{C}(y,x), E[w/x]\big]\!:\!\mathbf{Set}}{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}} \vdash \int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \big[\mathcal{C}(y,x), E[w/x]\big]\!:\!\mathbf{Set}.}\,\textbf{end}}\,\textbf{power+exchange}\text{'s}$$

The conclusion of the rule **Yoneda** says that the natural isomorphisms given by the Yoneda Lemma is natural in x. For a small category $\mathbb{C}$ the derivation

$$\cfrac{\cfrac{\cfrac{}{f\!:\![\mathbb{C}^{\mathsf{op}},\mathbf{Set}] \vdash f\!:\![\mathbb{C}^{\mathsf{op}},\mathbf{Set}]}\,\textbf{identity} \qquad \cfrac{}{x\!:\!\mathbb{C}^{\mathsf{op}} \vdash x\!:\!\mathbb{C}^{\mathsf{op}}}\,\textbf{identity}}{f\!:\![\mathbb{C}^{\mathsf{op}},\mathbf{Set}], x\!:\!\mathbb{C}^{\mathsf{op}} \vdash fx\!:\!\mathbf{Set}}\,\textbf{application}}{f\!:\![\mathbb{C}^{\mathsf{op}},\mathbf{Set}], x\!:\!\mathbb{C}^{\mathsf{op}} \vdash fx \cong \int_{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}}[\mathcal{C}(y,x), fw]\!:\!\mathbf{Set}}\,\textbf{Yoneda}$$

gives the most general case where we have also naturality in f.

6.1.3 Duality

The definitions for coend and copower are obtained from that one of end and power by using the rule which models the definition of opposite category (see §1.3):

opposite cat

$$\frac{\Gamma_1 \vdash E_1\!:\!\mathcal{C} \quad \Gamma_2 \vdash E_2\!:\!\mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1,E_2) \cong \mathcal{C}^{\mathsf{op}}(E_2^*, E_1^*)\!:\!\mathbf{Set}}$$

where the meta-operation $(\_)^*$ is defined as in 5.1.1. The functoriality of the left-hand side follows by applying **hom** to the premises. For the right-hand side we have

$$\cfrac{\cfrac{\Gamma_2 \vdash E_2\!:\!\mathcal{C}}{\Gamma_2^{\mathsf{op}} \vdash E_2^*\!:\!\mathcal{C}^{\mathsf{op}}}\,\textbf{dual} \qquad \cfrac{\Gamma_1 \vdash E_1\!:\!\mathcal{C}}{\Gamma_1^{\mathsf{op}} \vdash E_1^*\!:\!\mathcal{C}^{\mathsf{op}}}\,\textbf{dual}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}^{\mathsf{op}}(E_2^*, E_1^*)\!:\!\mathbf{Set}.}\,\textbf{hom+exchange}\text{'s}$$

Now we show how derive the rule for the definition of coends. Given a small category $\mathbb{C}$, a cocomplete category $\mathcal{D}$ and a judgement

$$\Gamma, x\!:\!\mathbb{C}^{\mathsf{op}}, y\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}$$

we have

$$\cfrac{\cfrac{\Gamma, x\!:\!\mathbb{C}^{\mathsf{op}}, y\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}}{\Gamma \vdash \int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E\!:\!\mathcal{D}}\,\textbf{end}^* \qquad \cfrac{}{w\!:\!\mathcal{D} \vdash w\!:\!\mathcal{D}}\,\textbf{identity}}{\Gamma^{\mathsf{op}}, w\!:\!\mathcal{D} \vdash \mathcal{D}(\int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E, w) \cong \mathcal{D}^{\mathsf{op}}(w, \int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} E^*)\!:\!\mathbf{Set}.}\,\textbf{opposite cat}$$

It follows through the chain of isomorphisms

$$\begin{aligned} \mathcal{D}^{\mathsf{op}}(w, \textstyle\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} E^*) &\cong \textstyle\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} \mathcal{D}^{\mathsf{op}}(w, E^*) && \textbf{def end} \\ &\cong \textstyle\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} \mathcal{D}(E, w) && \textbf{opposite cat+congruence}. \end{aligned}$$

Thus we have the derivable rule:

(def end)*

$$\frac{\Gamma, x{:}\mathcal{C}^{\mathsf{op}}, y{:}\mathcal{C} \vdash E{:}\mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{cocomplete}}{\Gamma^{\mathsf{op}}, w{:}\mathcal{D} \vdash \mathcal{D}(\int^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E, w) \cong \int_{y^{\mathcal{C}^{\mathsf{op}}}, x^{\mathcal{C}}} \mathcal{D}(E, w){:}\mathbf{Set}.}$$

Notice the adjustment in the type of the bound variables to reflect contravariance when moving outside of the left argument in a hom-expression.

Similarly we can derive the rule for the definition of copowers. Given a cocomplete category $\mathcal{D}$ and judgements

$$\Gamma_1 \vdash E_1{:}\mathbf{Set} \quad \text{and} \quad \Gamma_2 \vdash E_2{:}\mathcal{D}$$

we have

$$\cfrac{\cfrac{\Gamma_1 \vdash E_1{:}\mathbf{Set} \quad \Gamma_2 \vdash E_2{:}\mathcal{D}}{\Gamma_1, \Gamma_2 \vdash E_1 \otimes E_2{:}\mathcal{D}}\ \textbf{power}^* \qquad \cfrac{}{w{:}\mathcal{D} \vdash w{:}\mathcal{D}}\ \textbf{identity}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}}, w{:}\mathcal{D} \vdash \mathcal{D}(E_1 \otimes E_2, w) \cong \mathcal{D}^{\mathsf{op}}(w, [E_1, E_2^*]){:}\mathbf{Set}.}\ \textbf{opposite cat}$$

The definition of copower is given by

$$\begin{aligned} \mathcal{D}^{\mathsf{op}}(w, [E_1, E_2^*]) &\cong \big[E_1, \mathcal{D}^{\mathsf{op}}(w, E_2^*)\big] && \textbf{def power} \\ &\cong \big[E_1, \mathcal{D}(E_2, w)\big] && \textbf{opposite cat+congruence}. \end{aligned}$$

Thus we have the derivable rule:

(def power)*

$$\frac{\Gamma_1 \vdash E_1{:}\mathbf{Set} \quad \Gamma_2 \vdash E_2{:}\mathcal{D} \quad \mathcal{D}\ \mathsf{cocomplete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}}, w{:}\mathcal{D} \vdash \mathcal{D}(E_1 \otimes E_2, w) \cong \big[E_1, \mathcal{D}(E_2, w)\big]{:}\mathbf{Set}}$$

where w is neither free in E_1 nor E_2.

From the identity $(\mathbb{C}^{\mathsf{op}})^{\mathsf{op}} = \mathbb{C}$ we obtain an instance of the rule **Yoneda** which deals with the covariant case:

$$\frac{\Gamma, x{:}\mathbb{C} \vdash E{:}\mathbf{Set}}{\Gamma, x{:}\mathbb{C} \vdash E \cong \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \big[\mathbb{C}^{\mathsf{op}}(y, x), E[w/x]\big]{:}\mathbf{Set},}\ \textbf{Yoneda}$$

on the other hand we have the derivation

$$\cfrac{\cfrac{}{x{:}\mathbb{C} \vdash x{:}\mathbb{C}}\ \textbf{identity} \qquad \cfrac{}{y{:}\mathbb{C} \vdash y{:}\mathbb{C}}\ \textbf{identity}}{x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash \mathbb{C}^{\mathsf{op}}(y, x) \cong \mathbb{C}(x, y){:}\mathbf{Set}}\ \textbf{opposite cat}$$

and by **congruence** we can conclude

$$\Gamma, x{:}\mathbb{C} \vdash \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \big[\mathbb{C}^{\mathsf{op}}(y, x), E[w/x]\big] \cong \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \big[\mathbb{C}(x, y), E[w/x]\big]{:}\mathbf{Set}.$$

Finally by transitivity we obtain the derivable rule

Yoneda*

$$\frac{\Gamma, x{:}\mathcal{C} \vdash E{:}\mathbf{Set} \quad \mathcal{C}\ \mathsf{small}}{\Gamma, x{:}\mathcal{C} \vdash E \cong \int_{y^{\mathcal{C}^{\mathsf{op}}}, w^{\mathcal{C}}} \big[\mathcal{C}(x, y), E[w/x]\big]{:}\mathbf{Set}.}$$

The operation of duality respects natural isomorphisms, this is expressed by:

dual iso

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{C}}{\Gamma^{\mathsf{op}} \vdash E_1^* \cong E_2^* : \mathcal{C}^{\mathsf{op}}.}$$

The functoriality judgements $\Gamma \vdash E_1 : \mathcal{C}$ and $\Gamma \vdash E_2 : \mathcal{C}$ are derivable by induction. By applying **dual** to both sequents we have that the expressions in the conclusions of the rule are functorial as well.

The rules **opposite cat** and **dual iso** allow us to derive the dual form of the rule **representable**. Given the judgement

$$\Gamma, x : \mathcal{C} \vdash \mathcal{C}(E_1, x) \cong \mathcal{C}(E_2, x) : \mathbf{Set}$$

there are by definition functoriality judgements for E_1 and E_2 and then we can derive by using **opposite cat**

$$\Gamma, x : \mathcal{C} \vdash \mathcal{C}(E_1, x) \cong \mathcal{C}^{\mathsf{op}}(x, E_1^*) : \mathbf{Set} \quad \text{and} \quad \Gamma, x : \mathcal{C} \vdash \mathcal{C}(E_2, x) \cong \mathcal{C}^{\mathsf{op}}(x, E_2^*) : \mathbf{Set}.$$

Then by transitivity we have

$$\cfrac{\cfrac{\cfrac{}{\Gamma, x : \mathcal{C} \vdash \mathcal{C}^{\mathsf{op}}(x, E_1^*) \cong \mathcal{C}^{\mathsf{op}}(x, E_2^*) : \mathbf{Set}}\ \textbf{transitivity}}{\Gamma \vdash E_1^* \cong E_2^* : \mathcal{C}^{\mathsf{op}}}\ \textbf{representable}}{\Gamma^{\mathsf{op}} \vdash E_1 \cong E_2 : \mathcal{C}.}\ \textbf{dual iso}$$

Then we have the derivable rule

representable*

$$\frac{\Gamma, x : \mathcal{C} \vdash \mathcal{C}(E_1, x) \cong \mathcal{C}(E_2, x) : \mathbf{Set}}{\Gamma^{\mathsf{op}} \vdash E_1 \cong E_2 : \mathcal{C}.}$$

Again the reverse reading of this rule is derivable from the congruence rule for hom-expressions.

As for the functoriality judgements we can prove that the rule

substitution

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 \cong E_1' : \mathcal{D} \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1[E_2/x] \cong E_1'[E_2'/x] : \mathcal{D}}$$

is admissable in the calculus.

6.1.4 Products

Associated to the rules in § 5.6 we have the usual conversions from the lambda calculus with pairs

unit iso

$$\frac{\Gamma \vdash E : \mathbf{1}}{\Gamma \vdash E \cong \langle\ \rangle : \mathbf{1}}$$

β-conversion fst

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \quad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash \mathit{fst}^{\mathcal{C}\times\mathcal{D}}(\langle E_1, E_2\rangle) \cong E_1 : \mathcal{D}}$$

β-conversion snd

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \quad \Gamma_2 \vdash E_2 : \mathcal{D}}{\Gamma_1, \Gamma_2 \vdash \mathit{snd}^{\mathcal{C}\times\mathcal{D}}(\langle E_1, E_2\rangle) \cong E_2 : \mathcal{D}}$$

η-conversion pair

$$\frac{\Gamma \vdash E : \mathcal{C} \times \mathcal{D}}{\Gamma \vdash \langle \mathit{fst}^{\mathcal{C}\times\mathcal{D}}(E), \mathit{snd}^{\mathcal{C}\times\mathcal{D}}(E)\rangle \cong E : \mathcal{D}}$$

together with the congruence rules not listed here. A novel rule arises from internalising small products as domains of functor categories. Thus there is a rule for the adjunction (2.14):

Curry

$$\frac{\Gamma_1, x : \mathcal{C} \times \mathcal{D} \vdash E_1 : \mathcal{B} \quad \Gamma_2, y : \mathcal{C} \times \mathcal{D} \vdash E_2 : \mathcal{B} \quad \mathcal{C} \times \mathcal{D} \text{ small}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [\mathcal{C} \times \mathcal{D}, \mathcal{B}]\big(\lambda x.E_1, \lambda y.E_2\big) \cong [\mathcal{C}, [\mathcal{D}, \mathcal{B}]]\big(\lambda v.\lambda w.E_1[\langle v, w\rangle/x], \lambda v.\lambda w.E_2[\langle v, w\rangle/y]\big) : \mathbf{Set}}$$

where some types are omitted to simplify the notation.

The "swapping" of arguments of bifunctors is derived from **Curry** as sketched by the chain of isomorphisms

$$\begin{aligned}
&[\mathcal{C}, [\mathcal{D}, \mathcal{B}]]\big(\lambda v.\lambda w.E_1[\langle v, w\rangle/x], \lambda v.\lambda w.E_2[\langle v, w\rangle/y]\big) \\
&\qquad \cong \textstyle\int_{v_1,v_2} [\mathcal{D}, \mathcal{B}]\big(\lambda w.E_1[\langle v_1, w\rangle/x], \lambda w.E_2[\langle v_2, w\rangle/y]\big) && \textbf{nat formula} \\
&\qquad \cong \textstyle\int_{v_1,v_2} \int_{w_1,w_2} \mathcal{B}\big(E_1[\langle v_1, w_1\rangle/x], E_2[\langle v_2, w_2\rangle/y]\big) && \textbf{nat formula} \\
&\qquad \cong \textstyle\int_{w_1,w_2} \int_{v_1,v_2} \mathcal{B}\big(E_1[\langle v_1, w_1\rangle/x], E_2[\langle v_2, w_2\rangle/y]\big) && \textbf{Fubini} \\
&\qquad \cong \textstyle\int_{v_1,v_2} [\mathcal{C}, \mathcal{B}]\big(\lambda v.E_1[\langle v, w\rangle/x], \lambda v.E_2[\langle v, w\rangle/y]\big) && \textbf{nat formula} \\
&\qquad [\mathcal{D}, [\mathcal{C}, \mathcal{B}]]\big(\lambda w.\lambda v.E_1[\langle v, w\rangle/x], \lambda w.\lambda v.E_2[\langle v, w\rangle/y]\big) && \textbf{nat formula}.
\end{aligned}$$

Thus from **transitivity** and **Curry** we have the derivable rule:

swap

$$\frac{\Gamma_1, x : \mathcal{C} \times \mathcal{D} \vdash E_1 : \mathcal{B} \quad \Gamma_2, y : \mathcal{C} \times \mathcal{D} \vdash E_2 : \mathcal{B} \quad \mathcal{C} \times \mathcal{D} \text{ small}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [\mathcal{C} \times \mathcal{D}, \mathcal{B}]\big(\lambda x.E_1, \lambda y.E_2\big) \cong [\mathcal{D} \times \mathcal{C}, \mathcal{B}]\big(\lambda v.E_1[\langle \mathit{snd}(v), \mathit{fst}(v)\rangle/x], \lambda w.E_1[\langle \mathit{snd}(w), \mathit{fst}(w)\rangle/x]\big) : \mathbf{Set}.}$$

6.1.5 Examples

6.1.6 Limits

Limits are ends where the functor involved is extended with a "dummy" argument. The definition of limits as given by a representation is derived from the rule **def end**. Assume the judgements

$$\Gamma, y : \mathbb{C} \vdash E : \mathcal{D} \quad \mathbb{C} \text{ small} \quad \mathcal{D} \text{ complete}.$$

We first derive

$$\dfrac{\dfrac{\dfrac{}{w\!:\!\mathcal{D} \vdash w\!:\!\mathcal{D}}\ \textbf{identity}}{w\!:\!\mathcal{D}, x\!:\!\mathbb{C} \vdash w\!:\!\mathcal{D}}\ \textbf{weakening} \qquad \Gamma, y\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}}{\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash [\mathbb{C},\mathcal{D}](\lambda x^{\mathbb{C}}.w, \lambda y^{\mathbb{C}}.E) \cong \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} \mathcal{D}(w,E)\!:\!\mathbf{Set}.}\ \textbf{nat formula}$$

Then,

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, y\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}\end{matrix}}{\Gamma, x\!:\!\mathbb{C}^{\mathsf{op}}, y\!:\!\mathbb{C} \vdash E\!:\!\mathcal{D}}\ \textbf{weakening+exchange}}{\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} \mathcal{D}(w,E) \cong \mathcal{D}(w, \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E)\!:\!\mathbf{Set}.}\ \textbf{def end}$$

Finally by transitivity

$$\Gamma, w\!:\!\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E) \cong [\mathbb{C},\mathcal{D}](\lambda x^{\mathbb{C}}.w, \lambda y^{\mathbb{C}}.E)\!:\!\mathbf{Set}.$$

As x does not occur free in E we just write $\int_{y^{\mathbb{C}}} E$.

6.1.7 Density Formula and the Yoneda Lemma

In § 3.5.3 we show three different forms in which a presheaf can be expressed as a colimit. Here we formalise the proof for the density formula in the calculus and show that in fact their expression is (in some sense) the dual for the Yoneda lemma.

The interpretation for a derivation

$$\begin{matrix}\vdots\ \delta \\ \Gamma, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash E\!:\!\mathbf{Set}\end{matrix} \tag{6.1}$$

results in a functor $[\![\delta \rhd \Gamma, x : \mathbb{C}^{\mathsf{op}} \vdash E : \mathbf{Set}]\!] : \Gamma \times \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$, a presheaf with parameters. From (6.1) we can derive the sequent

$$\Gamma, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash \int^{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[w/x] \otimes \mathbb{C}(x,y)$$

the so-called density formula for E. Then we have the chain of isomorphisms

$$\begin{aligned}
\left[\int^{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[w/x] \otimes \mathbb{C}(x,y), z\right] &\cong \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \left[E[w/x] \otimes \mathbb{C}(x,y), z\right] && \textbf{(def end)}^* \\
&\cong \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \left[\mathbb{C}(x,y), [E[w/x], z]\right] && \textbf{(def power)}^* + \textbf{congruence} \\
&\cong [E, z] && \textbf{Yoneda}^*
\end{aligned}$$

and then by transitivity there is a derivation

$$\dfrac{\begin{matrix}\vdots \\ \Gamma^{\mathsf{op}}, x\!:\!\mathbb{C}, z\!:\!\mathbf{Set} \vdash \left[\int^{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[w/x] \otimes \mathbb{C}(x,y), z\right] \cong [E,z]\!:\!\mathbf{Set}\end{matrix}}{\Gamma, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash \int^{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E[w/x] \otimes \mathbb{C}(x,y) \cong E\!:\!\mathbf{Set}.}\ \textbf{representable}^*$$

Then by further use of **symmetry** and **congruence** we have the derivable rule

density

$$\frac{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}} \vdash E\!:\!\mathbf{Set} \quad \mathcal{C}\ \mathsf{small}}{\Gamma, x\!:\!\mathcal{C}^{\mathsf{op}} \vdash E \cong \int^{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{C}(x,y) \otimes E[w/x]\!:\!\mathbf{Set}}$$

In order to derive **density** we need the Yoneda lemma in the calculus, but the relation goes beyond that. In fact the dual form of the density formula gives a form of the Yoneda lemma for $\mathbf{Set}^{\mathsf{op}}$:

$$\cfrac{\cfrac{\Gamma, x\!:\!\mathbb{C} \vdash E\!:\!\mathbf{Set}^{\mathsf{op}}}{\Gamma^{\mathsf{op}}, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash E^*\!:\!\mathbf{Set}}\ \mathbf{dual}}{\cfrac{\Gamma^{\mathsf{op}}, x\!:\!\mathbb{C}^{\mathsf{op}} \vdash E^* \cong \int^{w^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} \mathbb{C}(x,y) \otimes E^*[w/x]\!:\!\mathbf{Set}}{\Gamma, x\!:\!\mathbb{C} \vdash E \cong \int_{y^{\mathbb{C}^{\mathsf{op}}}, w^{\mathbb{C}}} \left[\mathbb{C}(x,y), E[w/x]\right]\!:\!\mathbf{Set}^{\mathsf{op}}.}\ \mathbf{dual\ iso}}\ \mathbf{density}$$

6.2 Interpretation

As proved in the presentation of the rules for natural isomorphisms a derivation

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E_1 \cong E_2\!:\!\mathcal{D} \end{array}$$

in turn determines derivations

$$\begin{array}{c} \vdots\ \delta_1 \\ \Gamma \vdash E_1\!:\!\mathcal{D} \end{array} \quad \text{and} \quad \begin{array}{c} \vdots\ \delta_2 \\ \Gamma \vdash E_2\!:\!\mathcal{D}. \end{array}$$

An interpretation for δ is a natural isomorphism

$$[\![\delta \rhd \Gamma \vdash E_1 \cong E_2\!:\!\mathcal{D}]\!] : [\![\delta_1 \rhd \Gamma \vdash E_1\!:\!\mathcal{D}]\!] \cong [\![\delta_2 \rhd \Gamma \vdash E_2\!:\!\mathcal{D}]\!].$$

We show how to build this natural isomorphism incrementally by considering the effect of each of the rules.

The interpretation for natural isomorphisms is not coherent with respect to the syntax. Different derivations may produce different witnesses that two functors are naturally isomorphic. In general the judgement $\Gamma \vdash E_1 \cong E_2 : \mathcal{D}$ is understood as asserting that there is a derivation and thus the existence of a natural isomorphisms between the corresponding functors.

Notice that as a consequence of the coherence for functoriality this interpretation ensures the existence of a natural isomorphisms between any other interpretations of $\Gamma \vdash E_1 : \mathcal{D}$ and $\Gamma \vdash E_2 : \mathcal{D}$, in particular between the functors obtained by the derivations in normal form.

To simplify the notation we sometimes represent functorial judgements by just their expressions, where contexts and type information can be recovered from the isomorphism judgement.

6.2.1 Lambda Calculus

The interpretation for the equational theory of the lambda calculus in a cartesian closed category is well-known and studied elsewhere [Sza78, LS86, Cro93, AC98, Jac99, Tay99, Pit00]. Here we just revise the theory for the particular case where the categorical model is giving by **CAT**.

The semantics of the structural rules follow a similar pattern that the case for functoriality. For instance if the interpretation of the premise **weakening** yields the natural isomorphism

$$[\![\Gamma \vdash E_1 : \mathcal{D}]\!] \overset{\theta}{\cong} [\![\Gamma \vdash E_2 : \mathcal{D}]\!]$$

then the interpretation of the complete derivation is

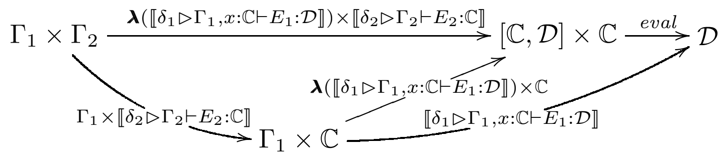
$$[\![\Gamma, x : \mathcal{C} \vdash E_1 : \mathcal{D}]\!] \overset{\theta \circ \mathsf{id}_\pi}{\cong} [\![\Gamma, x : \mathcal{C} \vdash E_2 : \mathcal{D}]\!]$$

where $[\![\Gamma, x : \mathcal{C} \vdash E_i : \mathcal{D}]\!] = [\![\Gamma \vdash E_i : \mathcal{D}]\!] \circ \pi$ (for $i = 1, 2$), π is the projection and id_π is the identity natural isomorphism associated. Similarly for **exchange** and **contraction**.

For **β-conversion** we use the semantics of substitution (see § 5.3.4) as composition in **CAT**. Given a derivation

$$\dfrac{\begin{array}{cc} \vdots\ \delta_1 & \vdots\ \delta_2 \\ \Gamma_1, x : \mathbb{C} \vdash E_1 : \mathcal{D} & \Gamma_2 \vdash E_2 : \mathcal{C} \end{array}}{\Gamma_1, \Gamma_2 \vdash (\lambda x^{\mathbb{C}}.E_1)E_2 \cong E_1[E_2/x] : \mathcal{D}}\ \textbf{β-conversion}$$

as *eval* is the counit of the adjunction defined by $\boldsymbol{\lambda}$ (2.14) then the triangles in the diagram



commute. The lower composition is the semantics for the substitution.

The natural isomorphism defined by **η-conversion** is just the one-to-one correspondence:

$$\dfrac{\Gamma \times \mathbb{C} \xrightarrow{[\![\delta \rhd \Gamma_1 \vdash E : [\mathbb{C}, \mathcal{D}]]\!] \times \mathbb{C}} [\mathbb{C}, \mathcal{D}] \times \mathbb{C} \xrightarrow{eval} \mathcal{D}}{\Gamma \xrightarrow{[\![\delta \rhd \Gamma_1 \vdash E : [\mathbb{C}, \mathcal{D}]]\!]} \mathcal{D}}$$

for some derivation δ.

For the equivalence rules, **reflexivity** is interpreted as the identity natural transformation over the functor defined by the canonical derivation of E, **symmetry** reverses the natural isomorphism in the premise and **transitivity** is the result of composing the natural isomorphisms defined by the interpretation of the premises. Finally the congruence rules correspond to the application of a functor to an argument. As all the expressions involved are functorial by construction, then we can ensure that the application of those expressions to particular isomorphic instances preserves the isomorphism: the slogan is *"functors preserve isomorphisms"*.

6.2.2 Axioms

The rules used to model the definitions of end and power are indeed axioms at the level of isomorphism judgements. They are basically interpreted through a representation for a **Set**-valued functor. For instance the interpretation of a derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma, x{:}\mathbb{C}^{\mathsf{op}}, y{:}\mathbb{C} \vdash E{:}\mathcal{D}\end{array}}{\Gamma, w{:}\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E) \cong \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} \mathcal{D}(w, E){:}\mathbf{Set}}$$

is a consequence of the definition for ends 3.2.1 together with the fact that representables preserve ends (Proposition 2.2.8).

The situation is similar for the rules **nat formula**, **Fubini** and **Yoneda**. The first one introduces the isomorphism defined by the naturality formula (3.4), the second one is interpreted through the Fubini result for ends (see§ 3.2.5) and the last one is just the Yoneda Lemma as an end formula 2.1.3. The rule **opposite cat** is as well an axiom which basically captures the definition of opposite category (see § 1.3).

6.2.3 Representability and Duality

For the remaining rules we have:

- **representable**

  The derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma, x{:}\mathcal{C}^{\mathsf{op}} \vdash \mathcal{C}(x, E_1) \cong \mathcal{C}(x, E_2){:}\mathbf{Set}\end{array}}{\Gamma \vdash E_1 \cong E_2{:}\mathcal{C}}\ \textbf{representable}$$

  is interpreted as

$$[\![E_1]\!] \overset{\theta_{[\![E_1]\!]}(\mathsf{id}_{[\![E_1]\!]})}{\cong} [\![E_2]\!]$$

  where $[\![\mathcal{C}(x, E_1)]\!] \overset{\theta}{\cong} [\![\mathcal{C}(x, E_2)]\!]$ (this follows from Proposition 2.1.5).

- **dual iso**

  The derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash E_1 \cong E_2 : \mathcal{C}\end{array}}{\Gamma^{\mathsf{op}} \vdash E_1^* \cong E_2^*{:}\mathcal{C}^{\mathsf{op}}}\ \textbf{dual iso}$$

  is interpreted as the action of the 2-functor $(\_)^{\mathsf{op}}$ over the natural isomorphism defined by δ

$$[\![E_1^*]\!] \overset{[\![\delta \triangleright \Gamma \vdash E_1 \cong E_2 : \mathcal{C}]\!]^{\mathsf{op}}}{\cong} [\![E_2^*]\!].$$

6.2.4 Lack of Coherence

As with judgements for functoriality, for any derivable isomorphism sequent there are several derivations. In this case it is not just a consequence of the presence of structural rules and duality, and in fact there is no coherence result. As the example below shows different derivations for the same isomorphism sequent could yield different natural isomorphisms.

The interpretation for the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\mathbf{Set}\vdash x:\mathbf{Set}}\,\textbf{identity}\qquad \dfrac{}{y:\mathbf{Set}\vdash y:\mathbf{Set}}\,\textbf{identity}}{x:\mathbf{Set}, y:\mathbf{Set}\vdash x\otimes y:\mathbf{Set}}\,\textbf{power}^*}{w:\mathbf{Set}\vdash w\otimes w:\mathbf{Set}}\,\textbf{contraction}}{w:\mathbf{Set}\vdash w\otimes w\cong w\otimes w:\mathbf{Set}}\,\textbf{reflexivity}$$

is the identity natural isomorphism. The copower over **Set** is just the cartesian product and then the interpretation corresponds to the componentwise identity. There is another derivation given by the definition of power. We first have the instances of copowers and powers definitions

$$\dfrac{\dfrac{}{x:\mathbf{Set}\vdash x:\mathbf{Set}}\,\textbf{identity}\qquad \dfrac{}{y:\mathbf{Set}\vdash y:\mathbf{Set}}\,\textbf{identity}}{x:\mathbf{Set}^{\mathsf{op}}, y:\mathbf{Set}^{\mathsf{op}}, z:\mathbf{Set}\vdash [x\otimes y, z]\cong \big[x,[y,z]\big]:\mathbf{Set}}\,\textbf{(def power)}^*$$

$$\dfrac{\dfrac{}{y:\mathbf{Set}\vdash y:\mathbf{Set}}\,\textbf{identity}\qquad \dfrac{}{w:\mathbf{Set}\vdash w:\mathbf{Set}}\,\textbf{identity}}{x:\mathbf{Set}^{\mathsf{op}}, y:\mathbf{Set}^{\mathsf{op}}, z:\mathbf{Set}\vdash \big[x,[y,z]\big]\cong \big[y,[x,w]\big]:\mathbf{Set}}\,\textbf{def power} \tag{6.2}$$

$$\dfrac{\dfrac{}{y:\mathbf{Set}\vdash y:\mathbf{Set}}\,\textbf{identity}\qquad \dfrac{}{x:\mathbf{Set}\vdash x:\mathbf{Set}}\,\textbf{identity}}{x:\mathbf{Set}^{\mathsf{op}}, y:\mathbf{Set}^{\mathsf{op}}, z:\mathbf{Set}\vdash \big[y,[x,z]\big]\cong [x\otimes y, z]:\mathbf{Set}}\,\textbf{(def power)}^*\textbf{+symmetry}$$

and then by transitivity

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ x:\mathbf{Set}^{\mathsf{op}}, y:\mathbf{Set}^{\mathsf{op}}, z:\mathbf{Set}\vdash [x\otimes y, z]\cong [y\otimes x, z]:\mathbf{Set}\end{array}}{x:\mathbf{Set}, y:\mathbf{Set}\vdash x\otimes y\cong y\otimes x:\mathbf{Set}}\,\textbf{representable}^*}{w:\mathbf{Set}\vdash w\otimes w\cong w\otimes w:\mathbf{Set}.}\,\textbf{contraction}$$

The interpretation for this derivation is the "twist" function acting as $(a,b)\mapsto(b,a)$. The core part of this interpretation is given by the semantics of (6.2). Then there two different derivations for

$$w:\mathbf{Set}\vdash w\otimes w\cong w\otimes w:\mathbf{Set}$$

which yield different natural isomorphisms.

6.3 Weighted Limits

When it comes to reasoning about presheaf categories it is more natural to approach universality through weighted limits. In principle weighted limits are no more than ordinary limits and they should be present already in the type theory. In this section we show that this is in fact the case.

Let us first extend the syntax with two binders for weighted limits and colimits

$$E ::= \ldots \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2) \mid \oint^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}}(E_2, E_1).$$

These binders are binary operations which capture one variable at each argument, *i.e.* they have arity $\langle 1, 1\rangle$ (where the lambda abstraction is $\langle 1\rangle$ and the integral is $\langle 2\rangle$, see [Plo90,FPT99]). Notice that here the order of the variables in $\oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_2, E_1)$ does matter.

The following rule defines the typing for weighted limit expressions:

wlimit

$$\frac{\Gamma_1, x{:}\mathcal{C} \vdash E_1{:}\mathbf{Set} \quad \Gamma_2, y{:}\mathcal{C} \vdash E_2{:}\mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2){:}\mathcal{D}.}$$

Remember from the discussion in § 3.5.1 that the weights are contravariant when thought as parameters, this is expressed by the dual context Γ_1^{op}. This induces the following definition for substitution

$$\oint_{y^{\mathcal{C}}, w^{\mathcal{C}}}(E_1, E_2)[E/x] = \oint_{y^{\mathcal{C}}, w^{\mathcal{C}}}(E_1[E^*/x], E_2[E/x])$$

for $x \neq y$ and $x \neq w$. This is accompanied by a suitable definition for the meta-operation $(\_)^*$ to cover the new expressions:

$$\left(\oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2)\right)^* = \oint^{x^{\mathcal{C}}, y^{\mathcal{C}^{\mathsf{op}}}}(E_1, E_2^*) \qquad \left(\oint^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}}(E_1, E_2)\right)^* = \oint_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}^{\mathsf{op}}}}(E_1, E_2^*).$$

Then the typing rule for weighted colimits is derivable from the one for limits and duality. Given the sequents

$$\Gamma_1, x{:}\mathbb{C}^{\mathsf{op}} \vdash E_1{:}\mathbf{Set} \quad \Gamma_2, y{:}\mathbb{C} \vdash E_2{:}\mathcal{D} \quad \mathbb{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{cocomplete}$$

we have the derivation

$$\cfrac{\cfrac{\Gamma_1, x{:}\mathbb{C}^{\mathsf{op}} \vdash E_1{:}\mathbf{Set} \quad \cfrac{\Gamma_2, y{:}\mathbb{C} \vdash E_2{:}\mathcal{D}.}{\Gamma_2^{\mathsf{op}}, y{:}\mathbb{C}^{\mathsf{op}} \vdash E_2^*{:}\mathcal{D}^{\mathsf{op}}}\ \mathbf{dual}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}} \vdash \oint_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}^{\mathsf{op}}}}(E_1, E_2^*){:}\mathcal{D}^{\mathsf{op}}}\ \mathbf{wlimit}}{\Gamma_1, \Gamma_2 \vdash \oint^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}}(E_1, E_2){:}\mathcal{D}.}\ \mathbf{dual}$$

That gives the derivable rule

wlimit*

$$\frac{\Gamma_1, x{:}\mathcal{C}^{\mathsf{op}} \vdash E_1{:}\mathbf{Set} \quad \Gamma_2, y{:}\mathcal{C} \vdash E_2{:}\mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{cocomplete}}{\Gamma_1, \Gamma_2 \vdash \oint^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}}(E_1, E_2){:}\mathcal{D}.}$$

Now we can give the definition of weighted limits as a representation:

def wlimit

$$\frac{\Gamma_1, x\,{:}\,\mathcal{C} \vdash E_1\,{:}\,\mathbf{Set} \quad \Gamma_2, y\,{:}\,\mathcal{C} \vdash E_2\,{:}\,\mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, w\,{:}\,\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2)) \cong [\mathcal{C}, \mathbf{Set}]\big(\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.\mathcal{D}(w, E_2)\big)\,{:}\,\mathbf{Set}}$$

where the left-hand side is justified by

$$\dfrac{\dfrac{}{w\,{:}\,\mathcal{D} \vdash w\,{:}\,\mathcal{D}}\ \textbf{identity} \qquad \dfrac{\Gamma_1, x\,{:}\,\mathcal{C} \vdash E_1\,{:}\,\mathbf{Set} \quad \Gamma_2, y\,{:}\,\mathcal{C} \vdash E_2\,{:}\,\mathcal{D}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2)\,{:}\,\mathcal{D}}\ \textbf{wlimit}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, w\,{:}\,\mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}}(E_1, E_2))\,{:}\,\mathbf{Set}}\ \textbf{hom+exchange}\text{'s}$$

and the right-hand side by

$$\dfrac{\dfrac{\Gamma_1, x\,{:}\,\mathcal{C} \vdash E_1\,{:}\,\mathbf{Set}}{\Gamma_1 \vdash \lambda x^{\mathcal{C}}.E_1\,{:}\,[\mathcal{C}, \mathbf{Set}]}\ \textbf{lambda} \qquad \dfrac{\dfrac{\dfrac{}{w\,{:}\,\mathcal{D} \vdash w\,{:}\,\mathcal{D}}\ \textbf{identity} \quad \Gamma_2, y\,{:}\,\mathcal{C} \vdash E_2\,{:}\,\mathcal{D}}{\Gamma_2, w\,{:}\,\mathcal{D}^{\mathsf{op}}, y\,{:}\,\mathcal{C} \vdash \mathcal{D}(w, E_2)\,{:}\,\mathbf{Set}}\ \textbf{hom+exchange}\text{'s}}{\Gamma_2, w\,{:}\,\mathcal{D}^{\mathsf{op}} \vdash \lambda y^{\mathcal{C}}.\mathcal{D}(w, E_2)\,{:}\,[\mathcal{C}, \mathbf{Set}]}\ \textbf{lambda}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, w\,{:}\,\mathcal{D}^{\mathsf{op}} \vdash [\mathcal{C}, \mathbf{Set}]\big(\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.\mathcal{D}(w, E_2)\big)\,{:}\,\mathbf{Set}.}\ \textbf{hom}$$

As expected the definition for weighted colimits is derived from the one for weighted limits together with duality. Given the judgements

$$\Gamma_1, x\,{:}\,\mathbb{C}^{\mathsf{op}} \vdash E_1\,{:}\,\mathbf{Set} \quad \Gamma_2, y\,{:}\,\mathbb{C} \vdash E_2\,{:}\,\mathcal{D} \quad \mathbb{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{cocomplete} \tag{6.3}$$

there is a derivation

$$\dfrac{\dfrac{\Gamma_1, x\,{:}\,\mathbb{C}^{\mathsf{op}} \vdash E_1\,{:}\,\mathbf{Set} \quad \Gamma_2, y\,{:}\,\mathbb{C} \vdash E_2\,{:}\,\mathcal{D}}{\Gamma_1, \Gamma_2 \vdash \oint_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}}(E_1, E_2)\,{:}\,\mathcal{D}}\ \textbf{wlimit}^* \qquad \dfrac{}{w\,{:}\,\mathcal{D} \vdash w\,{:}\,\mathcal{D}}\ \textbf{identity}}{\Gamma_1^{\mathsf{op}}, \Gamma_2^{\mathsf{op}}, w\,{:}\,\mathcal{D} \vdash \mathcal{D}\big(\oint^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}}(E_1, E_2), w\big) \cong \mathcal{D}^{\mathsf{op}}\big(w, \oint_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}^{\mathsf{op}}}}(E_1, E_2^*)\big)\,{:}\,\mathbf{Set}.}\ \textbf{opposite cat}$$

Then by transitivity we have

$$\begin{aligned} \mathcal{D}^{\mathsf{op}}\big(w, \oint_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}^{\mathsf{op}}}}(E_1, E_2^*)\big) &\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\lambda x^{\mathbb{C}^{\mathsf{op}}}.E_1, \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathcal{D}^{\mathsf{op}}(w, E_2^*)\big) && \textbf{def wlimit} \\ &\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\lambda x^{\mathbb{C}^{\mathsf{op}}}.E_1, \lambda y^{\mathbb{C}^{\mathsf{op}}}.\mathcal{D}(E_2, w)\big) && \textbf{opposite cat} + \textbf{congruence}. \end{aligned}$$

We can formalise the proof of Proposition 3.5.3 in the calculus, but more interesting from the rule **def singleton** we can show that a weighted limit is just a limit. Given a functoriality judgement

$$\Gamma, x\,{:}\,\mathbb{C} \vdash E\,{:}\,\mathcal{D}$$

where $\mathcal{D}$ is a complete category we can derive in the calculus the isomorphism

$$\Gamma \vdash \textstyle\int_{x^{\mathbb{C}}} E \cong \oint_{y^{\mathbb{C}}, x^{\mathbb{C}}}(1, E)\,{:}\,\mathcal{D} \tag{6.4}$$

where y is not present in Γ. The heart of this proof is given by the chain of isomorphisms

$$\begin{aligned} [\mathbb{C}, \mathcal{D}]\big(\lambda y.w, \lambda x.E\big) &\cong \textstyle\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} \mathcal{D}(w, E) && \textbf{nat form} \\ &\cong \textstyle\int_{y^{\mathbb{C}^{\mathsf{op}}}, x^{\mathbb{C}}} \big[1, \mathcal{D}(w, E)\big] && \textbf{def singleton} + \textbf{congruence} \\ &\cong [\mathbb{C}, \mathbf{Set}](\lambda y.1, \lambda x.\mathcal{D}(w, E)) && \textbf{nat form}. \end{aligned}$$

where y and w are not free in E.

6.4 Another Approach to Contravariance

The contravariance on the left-hand argument in hom-expressions has been modelled by dualising the type of the free variables. That forces a context-independent presentation for the rule

hom

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \qquad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}}$$

since we need to keep track of the free variables in E_1.

An alternative approach which avoids the modality $(\_)^{\mathsf{op}}$ acting on contexts is to require E_1 to be of type $\mathcal{C}^{\mathsf{op}}$

hom2

$$\frac{\Gamma \vdash E_1 : \mathcal{C}^{\mathsf{op}} \quad \Gamma \vdash E_2 : \mathcal{C}}{\Gamma \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}.}$$

This solution falls far from the usual practice in category theory, imagine for instance a calculus where the expression

$$x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C}^{\mathsf{op}}, z : \mathcal{C} \vdash \mathcal{C}(x \times y, z) \cong \mathcal{C}(x, z) \times \mathcal{C}(y, z) \tag{6.5}$$

holds (where $x \times y : \mathcal{C}^{\mathsf{op}}$).

There is a compromised solution which consists of introducing the rule **hom2** instead of **hom** but keeping an explicit track of the contravariance by adding a new constructor to the syntax:

$$E ::= \ldots | E^*.$$

Now the usual rule **dual** introduces the piece of syntax $(\_)^*$. The judgement (6.5) would still hold, but the more natural

$$x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C}^{\mathsf{op}}, z : \mathcal{C} \vdash \mathcal{C}((x \times y)^*, z) \cong \mathcal{C}(x, z) \times \mathcal{C}(y, z)$$

holds as well.

Propositions 5.1.2 and 5.1.4 are not longer meaningful, and their effect is achieved via explicit isomorphism rules, like for instance

$$\frac{\Gamma, x : \mathbb{C} \vdash E : \mathcal{D}}{\Gamma^{\mathsf{op}} \vdash (\lambda x^{\mathbb{C}}.E)^* \cong \lambda x^{\mathbb{C}^{\mathsf{op}}}.E^* : [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}]}$$

and so on for the other definition clauses in 5.1.1. Then from the congruence rules can be proved that the rule

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma \vdash E \cong (E^*)^* : \mathcal{D}}$$

is admissable. The definition for substitution is simplified since there is no special case to consider, and then we can as well show that

$$\frac{\Gamma, x : \mathcal{C} \vdash E_1 : \mathcal{D} \quad \Gamma \vdash E_2 : \mathcal{C}}{\Gamma \vdash (E_1[E_2/x])^* \cong (E_1^*[E_2^*]) : \mathcal{D}}$$

where $E_1[E_2/x]$ is the usual substitution.

The definition of DCW-normal derivation does not make any sense in this approach as duality is explicitly represented in the language. Now only the structural rules can lead to different derivations for the same functoriality judgement.

Most of these features are indeed disadvantages of the approach since the interpretation will distinguish more than necessary and force us to carry on with if not useless at least redundant isomorphisms. This is worsened by the fact that expressions grow in complexity and proofs need to deal with a syntactic bookkeeping of $(\_)^*$'s.

Chapter 7

Towards a Calculus for Enrichment

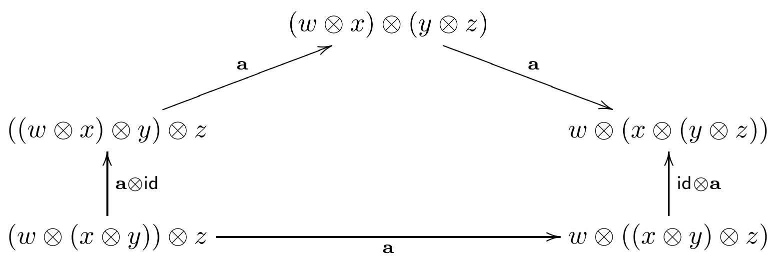
In enriched category theory hom-expressions are interpreted as objects in a base category $\mathcal{V}$. In this chapter we show how the 2-category of $\mathcal{V}$-categories gives a model for a substructural version of the calculus. Most of the constructions for locally small categories import to the enriched case smoothly but limits. In particular weighted limits are more general than ends and then take a prominent position. In the calculus this situation is reflected by the lack of structural rules for weakening and contraction. We introduce the basic concepts of enriched category theory by emphasising on the constructions modelled in the calculus. Most of the introductory material is extracted from *"Basic Concepts of Enriched Category Theory"* [Kel82], other references are [Dub70, Bor94b, BS00].

7.1 Symmetric Monoidal Closed Categories

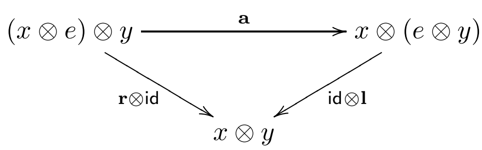
A *monoidal* category $\mathcal{V}$ is a locally small category $\mathcal{V}_0$ equipped with a bifunctor $- \otimes - : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, an object $e \in \mathcal{V}_0$ and natural isomorphisms

- $\langle (x \otimes y) \otimes z \overset{\mathbf{a}_{x,y,z}}{\cong} x \otimes (y \otimes z) \rangle_{x,y,z}$
- $\langle e \otimes x \overset{\mathbf{l}_x}{\cong} x \rangle_x$, and
- $\langle x \otimes e \overset{\mathbf{r}_x}{\cong} x \rangle_x$

such that the pentagon



commutes, called *associativity coherence*, and the triangle



commutes, called the *identity coherence*. As usual we omit the subscripts in the components of the natural transformations to simplify the notation.

The bifunctor $\otimes$ is called the tensor product of the monoidal category. The celebrated "coherence theorem" on monoidal categories establishes that every diagram commutes. Here by diagram we mean the ones resulting from combining by means of $\otimes$ instances of $\mathbf{a}$, $\mathbf{l}$, $\mathbf{r}$, their inverses and identities (see [Mac98]). A simple example of monoidal category is given by categories with finite products: just take the product as inducing the tensor bifunctor and the terminal object as e; the pentagon and triangles follow from the universal property of product.

A *symmetric monoidal category* extends the list of natural transformations $\mathbf{a}$, $\mathbf{l}$ and $\mathbf{r}$ with a symmetry for the tensor

$$\langle x \otimes y \overset{\mathsf{s}_{x,y}}{\cong} y \otimes x \rangle_{x,y}$$

satisfying the axioms

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\mathsf{s}} & y \otimes x \\ & \searrow^{\mathsf{id}} & \downarrow{\scriptstyle \mathsf{s}} \\ & & x \otimes y \end{array}$$

and

$$\begin{array}{ccccc} (x \otimes y) \otimes w & \xrightarrow{\mathbf{a}} & x \otimes (y \otimes w) & \xrightarrow{\mathsf{s}} & (y \otimes w) \otimes x \\ \downarrow{\scriptstyle \mathsf{s}\otimes\mathsf{id}} & & & & \downarrow{\scriptstyle \mathbf{a}} \\ (y \otimes x) \otimes w & \xrightarrow[\mathbf{a}]{} & y \otimes (x \otimes w) & \xrightarrow[\mathsf{id}\otimes\mathsf{s}]{} & y \otimes (w \otimes x). \end{array}$$

A monoidal category $\mathcal{V}$ is said to be *closed* if each induced functor

$$- \otimes x : \mathcal{V}_0 \to \mathcal{V}_0$$

has a right adjoint

$$- \otimes x \dashv [x, -],$$

i.e. there is an isomorphism

$$\mathcal{V}_0(w \otimes x, y) \cong \mathcal{V}_0(w, [x, y]) \tag{7.1}$$

natural in w, y. As expected the counit of this adjunction is given by the *evaluation* morphisms

$$\mathit{eval} : [x, y] \otimes x \to y.$$

Remember from § 2.3 that for an arrow $f : w \to z$ the action $[x, f]$ corresponds under the adjunction to

$$[x, w] \otimes x \xrightarrow{\mathit{eval}} w \xrightarrow{f} z.$$

If for each $x \in \mathcal{V}_0$ the induced partial functor $x \otimes - : \mathcal{V}_0 \to \mathcal{V}_0$ has as well a right adjoint we say that $\mathcal{V}$ is *biclosed*. It follows that if $\mathcal{V}$ is symmetric and closed then it is biclosed. In this case by Proposition 2.3.2 the expression $[-, x]$ is functorial in x as well giving rise to a bifunctor

$$[=, -] : \mathcal{V}_0^{\mathsf{op}} \times \mathcal{V}_0 \to \mathcal{V}_0$$

where for $f : w \to z$ the arrow $[f, x]$ corresponds under the adjunction (7.1) to the arrow

$$[z, x] \otimes w \xrightarrow{\mathsf{id}\otimes f} [z, x] \otimes z \xrightarrow{\mathit{eval}} x.$$

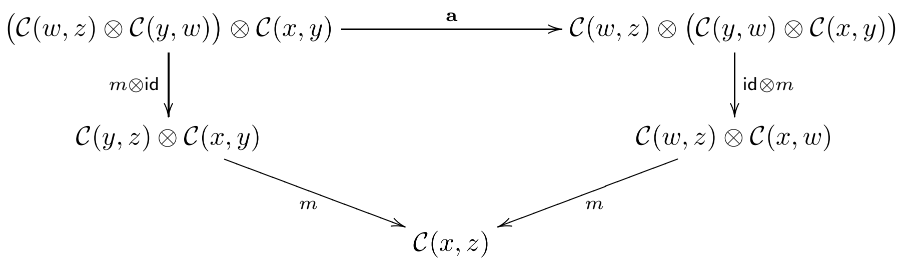
7.2 $\mathcal{V}$-Categories, $\mathcal{V}$-Functors and $\mathcal{V}$-Naturality

A monoidal category $\mathcal{V}$ is enough to replace the role of the category **Set** in defining the structure of the hom-expression. Then by adding more requirements on $\mathcal{V}$ we can recover most of the categorical constructions.

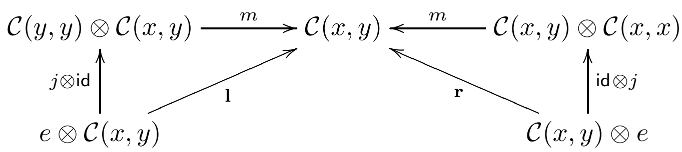
Let $\mathcal{V}$ be a monoidal category. A *$\mathcal{V}$-category* $\mathcal{C}$ is defined by

- a collection of objects (as usual we write $x \in \mathcal{C}$),
- an object $\mathcal{C}(x, y)$ in $\mathcal{V}_0$ for each pair $x, y \in \mathcal{C}$,
- a composition arrow $m : \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ for each $x, y, z \in \mathcal{C}$, and
- an identity "element" $j_x : e \to \mathcal{C}(x, x)$

such that for each x, y, w, z the diagram



commutes, called the *associativity axiom*, and the diagram



commutes, called the *identity axioms*. If the collection of objects of $\mathcal{C}$ is a set we say that $\mathcal{C}$ is a small $\mathcal{V}$-category.

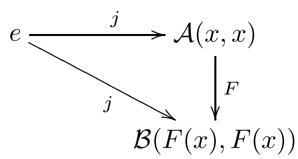
From $\mathcal{V}$-categories it is not difficult to see how to give suitable definitions for the notions of $\mathcal{V}$-functors and $\mathcal{V}$-naturality. Let us play this *enrichment game* by revising the basic categorical concepts in chapter 2 at the light of this more abstract notion of category.

Given two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ consists of

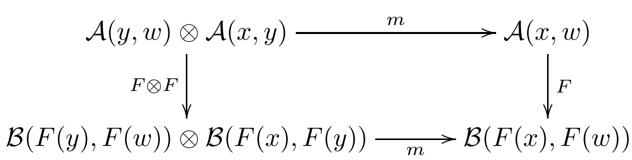
- a mapping F from the objects of $\mathcal{A}$ to the objects of $\mathcal{B}$, and
- for every pair $x, y \in \mathcal{A}$ there is an arrow

  $$F_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$$

  in $\mathcal{V}_0$ such that for each x, y, w the diagram



  commutes, *i.e.* F preserves identities; and the square



commutes, *i.e.* F preserves compositions.

For $\mathcal{V}$-functors $F, G : \mathcal{A} \to \mathcal{B}$ (this of course carries that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories), a $\mathcal{V}$-natural transformation $\alpha : F \Rightarrow G$ is defined by a family

$$\langle \alpha_x : e \to \mathcal{B}(F(x), G(x)) \rangle_x$$

of morphisms in $\mathcal{V}_0$ such that for each x, y

$$\begin{array}{ccccc} & \mathcal{A}(x,y) \otimes e & \xrightarrow{G\otimes\alpha} & \mathcal{B}(G(x),G(y)) \otimes \mathcal{B}(F(x),G(x)) & \\ {}^{\mathbf{r}^{-1}}\nearrow & & & & \searrow^{m} \\ \mathcal{A}(x,y) & & & & \mathcal{B}(F(x),G(y)) \\ {}_{\mathbf{l}^{-1}}\searrow & & & & \nearrow_{m} \\ & e \otimes \mathcal{A}(x,y) & \xrightarrow[\alpha\otimes F]{} & \mathcal{B}(F(y),G(y)) \otimes \mathcal{B}(F(x),F(y)) & \end{array}$$

commutes, *i.e.* the *naturality square* commutes.

The collection of identity elements gives as expected a natural transformation defined componentwise as

$$\langle j : e \to \mathcal{B}(F(x), F(x)) \rangle_{x \in \mathcal{A}}.$$

Compatible functors compose just by composing each of the arrows defining them, in the same spirit we can compose $\mathcal{V}$-natural transformations. Let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be $\mathcal{V}$-natural transformations. The composition $\beta \circ \alpha$ is defined at component x to be the arrow

$$e \cong (e \otimes e) \xrightarrow{\beta_x \otimes \alpha_x} \mathcal{B}(G(x), H(x)) \otimes \mathcal{B}(F(x), G(x)) \xrightarrow{m} \mathcal{B}(F(x), H(x))$$

in $\mathcal{V}_0$. It is routine though tedious to verify that these definitions satisfy the required conditions.

Thus the collections of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a 2-category that we called here $\mathcal{V}\mathbf{CAT}$. In particular if $\mathcal{V} = \mathbf{Set}$ the category of $\mathcal{V}$-categories is just $\mathbf{CAT}$. Another example comes by taking $\mathcal{V} = \mathbf{CAT}$ where we recover the categories of 2-categories. Henceforth we use the terms *category*, *functor* and *natural transformation* to mean the corresponding notion in the $\mathcal{V}$-enriched setting when there is not possibility of confusion.

7.2.1 Opposite $\mathcal{V}$-Category

In order to be able to define opposite $\mathcal{V}$-categories we need $\mathcal{V}$ to be a symmetric monoidal category. The opposite category $\mathcal{A}^{\mathsf{op}}$ has the same objects as $\mathcal{A}$ and for each x, y the hom-object is defined as

$$\mathcal{A}^{\mathsf{op}}(x, y) = \mathcal{A}(y, x).$$

Remember that this refers to equality of objects in $\mathcal{V}_0$. The identity elements are those of $\mathcal{A}$. The symmetry of $\mathcal{V}$ is required for the definition of composition. The composition arrow of $\mathcal{A}^{\mathsf{op}}$ for objects x, y, w is

$$\mathcal{A}(w, y) \otimes \mathcal{A}(y, x) \xrightarrow{\mathbf{s}} \mathcal{A}(y, x) \otimes \mathcal{A}(w, y) \xrightarrow{m^{\mathcal{A}}} \mathcal{A}(w, x)$$

It is routine to verify that this definition gives indeed a $\mathcal{V}$-category.

In fact there is a 2-functor over $\mathcal{V}$**CAT** with the expected behaviour. Given a functor $F : \mathcal{A} \to \mathcal{B}$, the opposite functor $F^{\mathsf{op}} : \mathcal{A}^{\mathsf{op}} \to \mathcal{B}^{\mathsf{op}}$ acts over objects as F and over hom-objects as

$$\mathcal{A}^{\mathsf{op}}(x, y) = \mathcal{A}(y, x) \xrightarrow{\quad F \quad} \mathcal{B}(F(y), F(x)) = \mathcal{B}^{\mathsf{op}}(F(x), F(y)).$$

Given a natural transformation $\alpha : F \Rightarrow G$ the opposite $\alpha^{\mathsf{op}} : G^{\mathsf{op}} \Rightarrow F^{\mathsf{op}}$ is defined componentwise as

$$\begin{array}{ccc} e & \xrightarrow{\quad \alpha \quad} & \mathcal{B}(F(x), G(x)) \\ & \searrow^{\alpha^{\mathsf{op}}} & \Vert \\ & & \mathcal{B}^{\mathsf{op}}(G^{\mathsf{op}}(x), F^{\mathsf{op}}(x)). \end{array}$$

Now as expected we call a functor $F : \mathcal{A}^{\mathsf{op}} \to \mathcal{B}$ a contravariant functor. Still to be able to define hom-functors in the theory of $\mathcal{V}$-categories we need to support bifunctors.

7.2.2 $\mathcal{V}$-Bifunctors

As it is the case for product in **CAT**, the tensor product of $\mathcal{V}$ can be extended to a tensor product in $\mathcal{V}$**CAT**. For categories $\mathcal{A}$ and $\mathcal{B}$, the *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is the category where

- the collection of objects is the cartesian product of the collection of objects of $\mathcal{A}$ and $\mathcal{B}$,

- the hom-objects are defined as

$$\mathcal{A} \otimes \mathcal{B}((x, y), (x', y')) = \mathcal{A}(x, x') \otimes \mathcal{B}(y, y'),$$

- a composition arrow m is

$$\begin{array}{ccc} \big(\mathcal{A}(x', x'') \otimes \mathcal{B}(y', y'')\big) \otimes \big(\mathcal{A}(x, x') \otimes \mathcal{B}(y, y')\big) & \xrightarrow{\qquad m \qquad} & \mathcal{A}(x, x'') \otimes \mathcal{B}(y, y'') \\ \downarrow{\scriptstyle \textit{middle four interchanged}} & \nearrow_{m^{\mathcal{A}} \otimes m^{\mathcal{B}}} & \\ \big(\mathcal{A}(x', x'') \otimes \mathcal{A}(x, x')\big) \otimes \big(\mathcal{B}(y', y'') \otimes \mathcal{B}(y, y')\big) & & \end{array}$$

  where *middle four interchanged* is obtained from a combination of instances of **a** and **s**; and

- the identity element is the arrow

$$e \cong (e \otimes e) \xrightarrow{j \otimes j} \mathcal{A}(x, x) \otimes \mathcal{B}(y, y).$$

It is again routine though even more tedious to verify that this definition gives indeed a $\mathcal{V}$-category. Now functors with domains given by tensor categories, like $F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$, correspond to the bifunctors. As in the case of **CAT**, F induces the partial $\mathcal{V}$-functors $F^x : \mathcal{B} \to \mathcal{C}$ and $F_y : \mathcal{A} \to \mathcal{C}$. For our development is interesting to notice that indeed by following the definition we can show $(\mathcal{A} \otimes \mathcal{B})^{\mathsf{op}} = \mathcal{A}^{\mathsf{op}} \otimes \mathcal{B}^{\mathsf{op}}$.

7.2.3 The Underlying "Ordinary" Category

Let $\mathcal{V}$ be a monoidal category. A *global element* a of $x \in \mathcal{V}_0$ is an arrow $a : e \to x$. Now for a $\mathcal{V}$-category $\mathcal{A}$, the global elements of the object $\mathcal{A}(w,z)$ can be thought as the arrows from w to z in $\mathcal{A}$. Indeed as $\mathcal{V}$ is locally small the collection

$$\mathcal{V}_0\big(e, \mathcal{A}(w,z)\big)$$

is a set. This allows us to define the *underlying* ordinary category $\mathcal{A}_0$ where the objects are the ones from $\mathcal{A}$ and arrows are just the global elements. The composition of compatible "arrows" g and f is given by the global element

$$e \cong (e \otimes e) \xrightarrow{g \otimes f} \mathcal{A}(y,z) \otimes \mathcal{A}(x,y) \xrightarrow{m} \mathcal{A}(x,z).$$

In the same spirit we can define underlying functors and natural transformations. In fact those definitions give rise to an "underlying" 2-functor from $\mathcal{V}$**CAT** to **CAT**.

This constructions allows us to define the notion of isomorphism in a $\mathcal{V}$-category. Two object $a, b \in \mathcal{A}$ are isomorphic if there is an isomorphisms $a \cong b$ in $\mathcal{A}_0$.

7.3 Representable Functors and Universality

Thus the last step towards the notion of representability in the theory of enrichment comes with the verification that $\mathcal{V}$ itself can be seen as a $\mathcal{V}$-category. This requires to assume that $\mathcal{V}$ is a symmetric monoidal closed category.

Then there is a $\mathcal{V}$-category where

- the objects are those of $\mathcal{V}$,
- the hom-object for $x, y \in \mathcal{V}$ is

$$\mathcal{V}(x,y) = [x,y],$$

- the composition morphism

$$[y,z] \otimes [x,y] \xrightarrow{m} [x,z]$$

  corresponds under the adjunction (7.1) to the arrow

$$\big([y,z] \otimes [x,y]\big) \otimes x \xrightarrow{\mathbf{a}} [y,z] \otimes \big([x,y] \otimes x\big) \xrightarrow{\mathsf{id} \otimes eval} [y,z] \otimes y \xrightarrow{eval} z,$$

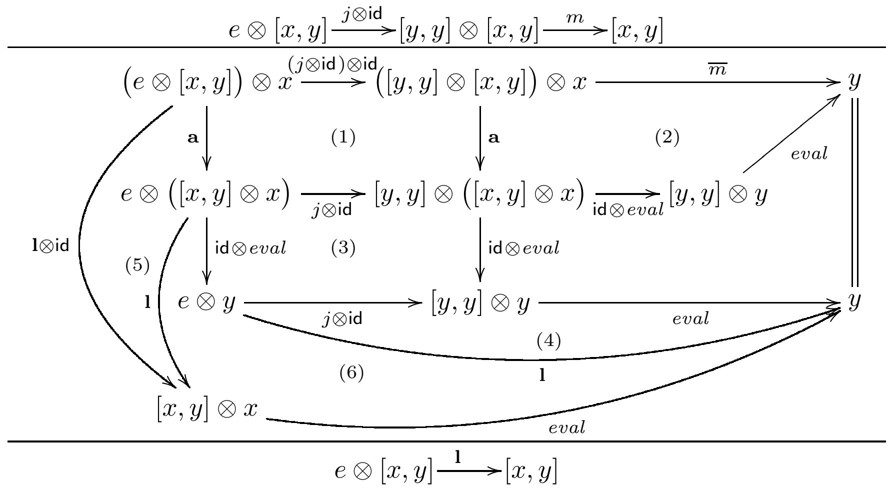
- the identity element

$$e \xrightarrow{j} [x,x]$$

  corresponds under (7.1) to

$$e \otimes x \xrightarrow{\mathbf{l}} x.$$

Of course we should verify by checking some big diagrams that this in fact gives a $\mathcal{V}$-category. As an example it follows the proof that j and m above satisfy one of the *identity axioms*:



where

(1) commutes by naturality of **a**,

(2) commutes by definition of m,

(3) commutes since $\otimes$ is a bifunctor,

(4) commutes since from the adjunction (7.2) and definition of j

$$eval \circ (j \otimes \mathsf{id}) = \overline{j} = \mathbf{l},$$

(5) commutes by coherence, and

(6) commutes by naturality of $\mathbf{l}$.

It is straightforward to verify that the underlying "ordinary" category of $\mathcal{V}$ as a $\mathcal{V}$-enriched one is $\mathcal{V}_0$. Then by definition every global element

$$f : e \to [x, y]$$

uniquely corresponds to an arrow $f : x \to y$.

For a $\mathcal{V}$-category $\mathcal{A}$ and an object $x \in \mathcal{A}$ we can now define the *representable* $\mathcal{V}$-functor $\mathcal{A}(x, -) : \mathcal{A} \to \mathcal{V}$ mapping $y \in \mathcal{A}$ into the hom-object $\mathcal{A}(x, y)$ and acting on hom-objects as defined by (7.1)

$$\frac{\mathcal{A}(y,w) \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}(x,y), \mathcal{A}(x,w)]}{\mathcal{A}(y,w)\otimes\mathcal{A}(x,y) \xrightarrow{m} \mathcal{A}(x,w)\,,}$$

i.e. by post-composition. In a similar way we can define the contravariant representable functor $\mathcal{A}(-, z) : \mathcal{A}^{\mathsf{op}} \to \mathcal{V}$. Indeed $\mathcal{A}(x, -)$ and $\mathcal{A}(-, z)$ are the partial functors of the hom-functor

$$\mathcal{A}(=, -) : \mathcal{A}^{\mathsf{op}} \otimes \mathcal{A} \to \mathcal{V}.$$

This bifunctor acts on hom-objects as

$$\frac{(\mathcal{A}^{\mathsf{op}}\otimes\mathcal{A})\big((y,w),(y',w')\big) \xrightarrow{\mathcal{A}(=,-)} [\mathcal{A}(y,w), \mathcal{A}(y',w')]}{\big(\mathcal{A}(y',y)\otimes\mathcal{A}(w,w')\big)\otimes\mathcal{A}(y,w) \xrightarrow{(\mathsf{id}\otimes m)\circ\mathbf{a}} \mathcal{A}(y',y)\otimes\mathcal{A}(y,w') \xrightarrow{m\circ\mathsf{s}} \mathcal{A}(y',w')\,.}$$

It is cleat that in the case $\mathcal{A} = \mathcal{V}$ the hom-functor coincide with $[=,-]$ and then have

$$[x,[y,z]] \cong [x \otimes y, z] \cong [y \otimes x, z] \cong [y,[x,z]]. \tag{7.2}$$

As in the ordinary case a functor $F\colon\mathcal{C}^{\mathsf{op}} \to \mathcal{V}$ is representable if there exists an *isomorphism*

$$\mathcal{C}(-,c) \overset{\theta}{\cong} F$$

for some $c \in \mathcal{C}$. However, we find here the first divergence from the theory of locally small categories in that there is no characterisation of $\mathcal{V}$-representability as the one given by Theorem 2.1.4. There is otherwise a parametrised representability result.

Proposition 7.3.1 Let $F\colon\mathcal{A}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ be a functor such that for each $a \in \mathcal{A}$ the induced functor $F^a\colon\mathcal{B} \to \mathcal{V}$ has a representation

$$\mathcal{B}(G[a],-) \overset{\theta^a}{\cong} F^a.$$

Then for each $x, y \in \mathcal{A}$ there is a unique arrow

$$\mathcal{A}(x,y) \longrightarrow \mathcal{B}(G[x],G[y])$$

in $\mathcal{V}$ which makes G a functor and then θ is natural in a as well.

The proof of this results is in [Kel82]. This is crucial for the semantics of the calculus since it allows us to model the interpretation of typing judgements as asserting $\mathcal{V}$-functoriality.

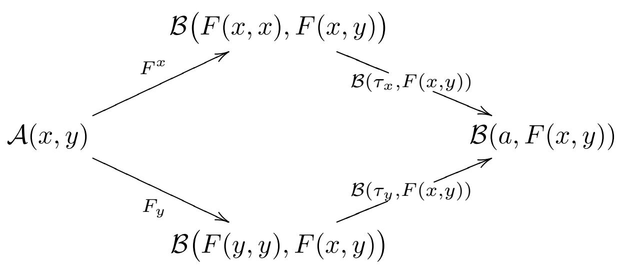
7.3.1 Functor $\mathcal{V}$-Categories and the Enriched Yoneda Lemma

We also expect to recover the naturality formula and the Yoneda lemma to successfully model the calculus in $\mathcal{V}$**CAT**. This leads first to the notion of extraordinary $\mathcal{V}$-naturality, or more informally the enriched wedges.

Let $F\colon\mathcal{A}^{\mathsf{op}} \otimes \mathcal{A} \to \mathcal{B}$ be a bifunctor. A family of morphisms in $\mathcal{V}_0$

$$\langle \tau_x\colon e \to \mathcal{B}(a, F(x,x))\rangle_{x\in\mathcal{A}}$$

is an extraordinary $\mathcal{V}$-natural transformation if for each $x, y \in \mathcal{A}$ the diagram



commutes. The arrow $\mathcal{B}(\tau_x, F(x,y))$ is an abbreviation for the composition

$$\mathcal{B}(F(x,x),F(x,y)) \otimes e \xrightarrow{\mathsf{id}\otimes\tau_x} \mathcal{B}(F(x,x),F(x,y)) \otimes \mathcal{B}(a,F(x,x)) \xrightarrow{m} \mathcal{B}(a,F(x,y))$$

There is of course a symmetric definition for families $\langle \sigma_x\colon F(x,x) \to b\rangle_x$.

For the special case $\mathcal{B} = \mathcal{V}$ the condition expressed by the diagram above can be rephrased as the commutativity of

$$\begin{array}{ccccc} & & F(x,x) & & \\ & \overset{\tau_x}{\nearrow} & & \overset{f_x}{\searrow} & \\ a & & & & [\mathcal{A}(x,y), F(x,y)] \\ & \underset{\tau_y}{\searrow} & & \underset{f_y}{\nearrow} & \\ & & F(y,y). & & \end{array}$$

This is a consequence of the isomorphism (7.2) where f_x is the arrow which corresponds under the adjunction (7.1) to

$$F(x,x) \otimes \mathcal{A}(x,y) \overset{\mathbf{s}}{\cong} \mathcal{A}(x,y) \otimes F(x,x) \xrightarrow{\overline{F^x}} F(x,y).$$

A universal such a family is of course called the *end* of F and can be expressed as the equaliser in $\mathcal{V}_0$

$$\textstyle\int_x F(x,x) \xrightarrow{\tau} \prod_{x \in \mathcal{A}} F(x,x) \underset{\prod_{y \in \mathcal{A}} f_y}{\overset{\prod_{x \in \mathcal{A}} f_x}{\rightrightarrows}} \prod_{x,y \in \mathcal{A}} \big[\mathcal{A}(x,y), F(x,y)\big].$$

Given two categories $\mathcal{A}$ and $\mathcal{B}$, then the "ordinary" category of $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$ can be provided with the structure of a $\mathcal{V}$-category. The objects of the functor $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$ are the $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$. For functors $F, G : \mathcal{A} \to \mathcal{B}$ the corresponding hom-object is defined to be the end

$$[\mathcal{A}, \mathcal{B}](F, G) = \textstyle\int_x \mathcal{B}(F(x), G(x)).$$

From the concrete definition of end above and through a lengthly calculation we can verify that a global element

$$e \xrightarrow{\alpha} [\mathcal{A}, \mathcal{B}](F, G)$$

in fact corresponds to a $\mathcal{V}$-natural transformation from F to G.

Notice that in order to ensure the corresponding equaliser exists we require $\mathcal{V}$ to be complete. As completeness here refers to *"all small limits"* the functor category construction requires the domain to be small. Thus the naturality formula holds by definition in the theory of enriched categories. We can now state the Yoneda lemma for the enriched case:

Proposition 7.3.2 (Enriched Yoneda Lemma) Given a $F : \mathcal{A} \to \mathcal{V}$ and $a \in \mathcal{A}$ the morphism

$$\mathcal{A}(a,x) \xrightarrow{F} [F(a), F(x)]$$

corresponds under the adjunction (7.1) and symmetry to the morphism

$$F(a) \xrightarrow{\rho_x} [\mathcal{A}(a,x), F(x)].$$

Then the pair $(F(a), \rho)$ is the end of $[\mathcal{A}(a, =), F(-)] : \mathcal{A}^{\mathsf{op}} \otimes \mathcal{A} \to \mathcal{V}$.

The table below summarises the relation between the requirements on $\mathcal{V}$ and the categorical constructions obtained in $\mathcal{V}$**CAT**

$\mathcal{V}$	$\mathcal{V}$**CAT**
monoidal	categories, functors and natural transformation
symmetric monoidal	opposite categories and bifunctors
symmetric monoidal closed	hom-functors and representables
complete symmetric monoidal closed	functor categories

Observe that we have not mentioned yet what is the notion of universality for the most general case. Ends so far are only defined for $\mathcal{V}$-valued functors, which in turn correspond to universal elements in the ordinary sense.

To define the right notion of limit in the enriched case is more subtle than the generalisations in the previous sections. Here the enrichment game ends abruptly justifying in first place the theory of enriched categories.

7.3.2 Tensors and Cotensors

We can start by generalising the notions of power and copower which in this setting are called *cotensor* and *tensor* respectively. We just take the usual definition as a representation. For instance given $a \in \mathcal{A}$ and $x \in \mathcal{V}$ the cotensor is defined by

$$\mathcal{A}(-,[x,a]) \cong \big[x, \mathcal{A}(-,a)\big].$$

If $\mathcal{A}$ admits all cotensor products we say that it is *cotensored*. $\mathcal{V}$ for instance is clearly a cotensored category.

Similarly for $a \in \mathcal{A}$ and $x \in \mathcal{V}$ the tensor is defined by the representation

$$\mathcal{A}(x \otimes c, -) \cong [x, \mathcal{A}(c,-)].$$

If $\mathcal{A}$ has all tensor products we say that it is a *tensored* category. For the special case $\mathcal{A} = \mathcal{V}$ this definition recovers the tensor product in $\mathcal{V}_0$. We expect these definitions to occur as special cases of the more general definition of limit.

7.3.3 A Notion of Limit

In **CAT** a limit for a diagram $D : \mathbb{I} \to \mathcal{C}$ is given by a representation of the **Set**-valued functor

$$[\mathbb{I}, \mathcal{C}](\Delta_-, D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

where Δ_- is the diagonal functor. For $c \in \mathcal{C}$ the expression $[\mathbb{I}, \mathcal{C}](\Delta c, D)$ denotes the set of "cones" from c to D (see 2.2). The constant functor Δc is define as the composition

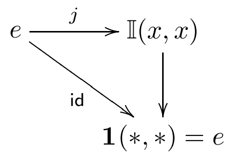
$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\Delta c} & \mathcal{C} \\ & {\scriptstyle !}\searrow \quad \nearrow{\scriptstyle c} & \\ & \mathbf{1} & \end{array} \tag{7.3}$$

where $\mathbf{1}$ is the singleton category and c is the functor selecting $c \in \mathcal{C}$.

In the enriched setting there is a counterpart of the singleton category. The $\mathcal{V}$-category $\mathbf{1}$ has one object $*$ and $\mathbf{1}(*,*) = e$ where $j = \mathsf{id}$. This allows us to define for each object $c \in \mathcal{C}$ a $\mathcal{V}$-functor $c : \mathbf{1} \to \mathcal{C}$ mapping $*$ to c and acting on the unique hom-object as

$$\mathbf{1}(*,*) = e \xrightarrow{j} \mathcal{C}(c,c).$$

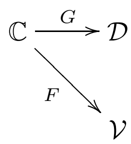
Perhaps surprisingly there is no canonical $\mathcal{V}$-functor from $\mathbb{I}$ to $\mathbf{1}$. The obvious mapping $x \mapsto *$ can not be matched by a suitable action on hom-objects. For instance if there is an arrow from $\mathbb{I}(x,x)$ to $\mathbf{1}(*,*) = e$ there is no argument which ensures that the *identity* axiom for $\mathcal{V}$-functors holds, *i.e.* that the diagram



commutes.

There is however a canonical functor from $\mathbb{I}$ to 1, the category with one element where $1(*,*)$ is $\top$ the terminal object in $\mathcal{V}$. Notice that the terminal object exists since we assume $\mathcal{V}$ to be complete. It is enough to have a section-retraction from $\top$ to e to be able to define the enriched version of the constant functor. However, this is not sufficient to express cotensors as instances of limits (for an example refer to [Kel82]). In any case for the most general case the notion of limit defined in terms of cones is not satisfiable.

Borceux and Kelly [BK75] first introduced the weighted limits as attempt to find the suitable definition of limits for $\mathcal{V}$-categories. For functors



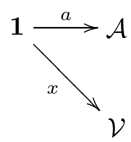
the limit of G weighted by F is given by the enriched representation

$$\mathcal{D}(x, \{F, G\}) \cong [\mathbb{C}, \mathcal{V}](F, \mathcal{D}(x, G(\_))).$$

For examples of $\mathcal{V}$ where usual conical limits and ends are defined in $\mathcal{V}$**CAT** we know they can be expressed as weighted limits as well. Cotensor products are instances of weighted limits as required: given $x \in \mathcal{V}$ and $a \in \mathcal{A}$

$$[x, a] = \{x, a\}$$

where a and x in the right-hand side are just the corresponding $\mathcal{V}$-functors



which exists in $\mathcal{V}$**CAT**. Of course the dual construction for weighted colimits gives the tensors products.

All the properties in § 3.5.1 about weighted limits can be rephrased for the enriched case. In particular the enriched version of Proposition 3.5.3 determines the definition of completeness for the most abstract case: a $\mathcal{V}$-category is complete is if it is cotensored and has all ends.

7.4 An Enriched Model for the Calculus

Most of the rules for the calculus can be interpreted in the category $\mathcal{V}$**CAT** but the structural rules **weakening** and **contraction**. The absence of these rules reflect in the calculus the inability of defining limits as universal cones in the theory of enrichment.

7.4.1 Contexts and Judgements

The basic type **Set** is now replaced by $\mathcal{V}$ representing a complete symmetric monoidal closed category $\mathcal{V}$. At the level of expressions the singleton 1 is now written as e.

Types in the calculus denote $\mathcal{V}$-categories and expressions $\mathcal{V}$-functors. As tensors substitute products we need to abandon the notion of cartesian context and then some of the related structural rules. A context

$$\Gamma = x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n$$

is interpreted by the tensor product of categories

$$[\![\Gamma]\!] = \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n.$$

In this setting the empty context corresponds to the $\mathcal{V}$-category $\mathbf{1}$.

This notion of context implies that the rule for **weakening** and **contraction** cannot be interpreted since there are not projection or diagonal functors associated to the tensor. The symmetry on $\mathcal{V}$, however, extends to a symmetry over the tensor product of categories providing an interpretation for the rule **exchange**.

A derivation

$$\begin{array}{c} \vdots\ \delta \\ \Gamma = \mathcal{C}_1, \ldots \mathcal{C}_n \vdash E : \mathcal{C} \end{array}$$

denotes a $\mathcal{V}$-functor

$$[\![\delta \rhd \Gamma \vdash E : \mathcal{C}]\!] : \mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n \to \mathcal{C}.$$

Although the rules for weakening and contraction are not longer present still the rule for duality produces different derivations for the same final sequent. The coherence result, however, is stronger since different derivation for the same final sequent result in the same functor. Remember that the only source of isomorphisms in the proof of coherence comes from the interaction of **weakening** and **contraction** with the rule for ends.

An isomorphism judgement

$$\begin{array}{c} \vdots\ \delta \\ \Gamma \vdash E_1 \cong E_2 \end{array}$$

denotes the existence of $\mathcal{V}$-natural isomorphisms between the functors $[\![E_1]\!]$ and $[\![E_2]\!]$. Observe that some of the equations derivable in the calculus correspond to the natural transformations in the definition of $\mathcal{V}$. For instance the judgements

$$x : \mathcal{V} \vdash x \otimes e \cong x : \mathcal{V} \quad \text{and} \quad x : \mathcal{V} \vdash e \otimes x \cong x : \mathcal{V}$$

are derivable in the calculus and denote the natural isomorphisms $\mathbf{l}$ and $\mathbf{r}$ respectively.

The syntactic identification over types

$$(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} = \mathcal{C} \quad (\mathcal{A} \otimes \mathcal{B})^{\mathsf{op}} = \mathcal{A}^{\mathsf{op}} \otimes \mathcal{B}^{\mathsf{op}} \quad [\mathbb{C}, \mathcal{D}]^{\mathsf{op}} = [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}] \quad (\mathbf{1}^{\mathsf{op}})^{\mathsf{op}} = \mathbf{1}$$

are by definition still valid in the enriched setting. Finally the identification of a hom-expression over $\mathcal{V}$ with the cotensor product

$$\mathcal{V}(E_1, E_2) = [E_1, E_2]$$

corresponds to the definition of $\mathcal{V}_0$ as a $\mathcal{V}$-category.

7.4.2 Structural Rules, Limits and Dinaturalities

Limits are not equivalent to weighted limits in the enrichment case. This is reflected in the calculus by the impossibility to derive the isomorphism (3.9). Concretely as there is no rule for weakening then we can not derive the constant functor in the calculus. The attempt

$$\cfrac{\cfrac{}{w\!:\!\mathcal{D}, y\!:\!\mathbb{C} \vdash w\!:\!\mathcal{D}}}{w\!:\!\mathcal{D} \vdash \lambda y^{\mathbb{C}}.w\!:\![\mathbb{C},\mathcal{D}]}\ \textbf{lambda}$$

cannot proceed into an axiom.

On the other hand the lack of a rule for **contraction** leaves us without a derivation for the diagonal functor. As a consequence of the *context-independent* presentation of the rules for a derivation

$$\begin{array}{c}\vdots\\ \Gamma \vdash \mathcal{C}(E_1, E_2)\!:\!\mathcal{V}\end{array}$$

we can always split Γ in two disjoint contexts Γ_1 and Γ_2 such that the sequents

$$\Gamma_1^{\mathsf{op}} \vdash E_1\!:\!\mathcal{C} \quad \text{and} \quad \Gamma_2 \vdash E_2\!:\!\mathcal{C}$$

are derivable.

Another interesting fact of $\mathcal{V}$-enriched categories is that there is no notion of dinatural transformations. Although there is no explicit syntax for dinaturalities in the calculus they can be expressed as an end formula via the dinaturality formula (3.5). The corresponding expression, however, can only be derived with the use of **contraction**. For instance given the functoriality judgements

$$\Gamma_1, x_1\!:\!\mathbb{C}^{\mathsf{op}}, y_1\!:\!\mathbb{C} \vdash E_1\!:\!\mathcal{D} \quad \text{and} \quad \Gamma_2, x_2\!:\!\mathbb{C}^{\mathsf{op}}, y_2\!:\!\mathbb{C} \vdash E_2\!:\!\mathcal{D}$$

we aim to derive the sequent

$$\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \textstyle\int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} \mathcal{D}\big(E_1[y/x_1, x/y_1], E_2[x/x_2, y/y_2]\big)\!:\!\mathcal{V}.$$

This can be only obtained by the application of **end** over the sequent

$$\Gamma_1^{\mathsf{op}}, \Gamma_2, x\!:\!\mathbb{C}^{\mathsf{op}}, y\!:\!\mathbb{C} \vdash \mathcal{D}\big(E_1[y/x_1, x/y_1], E_2[x/x_2, y/y_2]\big)\!:\!\mathcal{V}.$$

Now the only possible way to identify the variables x_1 with y_2 and y_1 with x_2 is through **contraction**.

Chapter 8

Extensions and Applications

We presented a typed language for category theory which formalises the notion of universal property. The logic is based on the presentation of universality as given by representations for suitable **Set**-valued functors. End for functors of multiple variance play a central role by providing the algebraic manipulation of expressions required. The calculus supports a syntactic treatment of duality which enforces a novel definition of substitution of raw terms. The model for the calculus is given by **CAT**, the category of locally small categories and functors. The types of the language represent categories and expressions are functors. In this interpretation the typing rules correspond to operations on functors.

The calculus is encompassed with a theory for equations. The equational theory is based on that one for lambda calculus with new rules for ends, the Yoneda lemma and Fubini theorem among others. An equation-in-context in the theory asserts the existence of a natural isomorphisms between two functors.

In the previous chapter we elaborated on the possibility of interpreting the calculus in a more abstract setting: the 2-category of enriched $\mathcal{V}$-categories $\mathcal{V}$**CAT**. As tensors replace products in the semantics for contexts there is no possible interpretation for **weakening** and **contraction**. These rules are key to prove the equivalence between limits, ends and weighted limits in the ordinary case. Indeed in the enriched setting the most general notion for universality is weighted limits where the other notions are special cases.

We discuss below some possible extensions and further applications of the calculus.

Profunctors

One of the original aims of our research was to give a formal framework to the categorical language which arose from the work done by Kelly, Borceux, Dubuc, Street, Day and others in enriched category theory. The main motivation is the use of functor categories and in particular presheaves in computer science. In recent years we have seen the development of domain-theoretic models for concurrency based on profunctors [Cat99, CW]. The manipulation of end formulæ advocated in this dissertation is central in this approach. Concrete models for concurrency as for instance labelled transitions systems can be interpreted more abstractly as presheaves over path categories. In this setting a path indicates a simple thread of execution. Thus functors between presheaves represent operations on models. The interest is focused on expressions denoting cocontinuous functors since they preserve the notion of equivalence.

A future direction which would add value to the calculus as a tool for semantics consists of characterising the *(co)continuous* functors in the syntax. For instance, a direct consequence of Fubini is that coend expressions are cocontinuous in their variables whereas the tensor gives a simple counterexample since

$$x:\mathbf{Set} \vdash x \otimes x:\mathbf{Set}$$

is not cocontinuous in x. At a first glance a first measure to capture cocontinuity should constraint the use of **contraction** to avoid expressions as the one above.

Addressing the Lack of Coherence

In order to be able to define a coherent equational theory we need to consider a more serious approach to types. In the presentation of the language and rules we left the issues concerning types deliberately aside. A proposal suggested by Glynn Winskel consists of adding to the logic a new form of judgement

$$\Gamma \vdash E_1 \overset{!}{\cong} E_2 : \mathcal{C}$$

read as there is a *unique* natural isomorphisms between the functors interpreted by E_1 and E_2. Observe as a simple result that

$$\Gamma \vdash E_1 \overset{!}{\cong} E_1 : \mathcal{C}$$

implies that for all E_2

$$\Gamma \vdash E_1 \cong E_2 : \mathcal{C} \qquad \Rightarrow \qquad \Gamma \vdash E_1 \overset{!}{\cong} E_2 : \mathcal{C}.$$

Since the natural isomorphism represented by the left-hand side must be unique.

Richer Types

As an attempt to obtain unique isomorphism judgements we extend the syntax for types with type variables. So far we use metavariables at the level of types which basically represented ground types. Now the syntax for types looks like

$$\mathcal{C}_1, \mathcal{C}_2, \ldots ::= A, B, \ldots \mid \mathcal{C}_1^{\mathsf{op}} \mid \ldots$$

where $A, B, \ldots$ are variable types for *small* categories.

To keep track of the type variables in a judgement we extend the syntax with contexts for them:

$$\underbrace{A_1, \ldots, A_m}_{\Delta}; \underbrace{x_1 : \mathcal{C}_1, \ldots, x_n : \mathcal{C}_n}_{\Gamma} \vdash E : \mathcal{C}.$$

In this judgement all the free term-variables in E are in Γ and all the type variables in the type expressions $\mathcal{C}_1, \ldots, \mathcal{C}_n, \mathcal{C}$ are in Δ. These kind of sequents denote a "dinatural" family of functors. Indeed the context for term variables Γ and the type expression $\mathcal{C}$ define functors

$$[\![\mathcal{C}]\!], [\![\Gamma]\!] : (\mathbf{Cat}^{\mathsf{op}})^m \times \mathbf{Cat}^m \to \mathbf{CAT}.$$

Thus for every instance for the variables in Δ the interpretation gives a functor.

Exponentials in the type expressions require contravariant arguments. Consider for instance the judgement

$$A; f : [A, \mathcal{D}], x : A \vdash f(x) : \mathcal{D}$$

where $\mathcal{D}$ stands for given locally small category. The context gives rise to the functor

$$\mathbf{Cat}^{\mathsf{op}} \times \mathbf{Cat} \xrightarrow{[=, \mathcal{D}] \times -} \mathbf{CAT}.$$

while the right-hand side expressions gives the constant functor over $\mathcal{D}$. The expression $f(x)$ is obviously interpreted as the evaluation functor which gives a dinatural family.

In this setting an isomorphism judgement

$$\Delta;\Gamma \vdash E_1 \cong E_2 : \mathcal{D}$$

should be interpreted as a "transformation" between the corresponding natural families. We borrow here some terminology from 2-categories and called those transformations of *"di-modifications"*. Glynn Winskel has shown that the lambda calculus fragment of the calculus can be interpreted in these higher-order semantics. Unfortunately already the hom-expressions fail to give a dinatural family.

Mechanisation in a Theorem Prover

The calculus provides the basis for a mechanisation of category theory in a theorem prover. Our approach is based on an algebraic characterisation of universality which removes the usual set-theoretic issues.

A draft of a possible implementation in the theorem prover Isabelle [Pau94] is described in [CW01]. The different syntactic components of the calculus are distributed in several user-defined theories which extend the initial theory `Pure` of the system. Basically, `Pure` gives the built-in higher-order logic [Pau88] on top of which we define the object-logic for categories. The judgements of the calculus correspond as usual to theorems in the meta-logic.

Appendix A

The Calculus

A.1 Syntax

Types

$$\mathcal{C}, \mathcal{D}, \ldots ::= \mathbf{Set} \mid \mathcal{C}^{\mathsf{op}} \mid [\mathcal{C}, \mathcal{D}].$$

Expressions

$$\begin{aligned} E_1, E_2, \ldots ::= \; & x \mid 1 \mid \lambda x^{\mathcal{C}}.E \mid E_1 E_2 \mid \mathcal{C}(E_1, E_2) \mid [E_1, E_2] \mid E_1 \otimes E_2 \mid \\ & \int_{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E \mid \int^{x^{\mathbb{C}^{\mathsf{op}}}, y^{\mathbb{C}}} E \end{aligned}$$

Syntactic Identities

$$\begin{aligned} (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} &= \mathcal{C} \\ [\mathbb{C}, \mathcal{D}]^{\mathsf{op}} &= [\mathbb{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}}] \\ (\mathcal{C}_1 \times \ldots \times \mathcal{C}_n)^{\mathsf{op}} &= \mathcal{C}_1^{\mathsf{op}} \times \ldots \times \mathcal{C}_n^{\mathsf{op}} \\ \mathbf{Set}(E_1, E_2) &= [E_1, E_2] \end{aligned}$$

A.2 Rules for Functoriality

A.2.1 Axioms

singleton

$$\frac{}{\vdash 1 : \mathbf{Set}}$$

identity

$$\frac{}{x : \mathcal{C} \vdash x : \mathcal{C}}$$

A.2.2 Structural Rules

weakening

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n, x_{n+1} : \mathcal{C}_{n+1} \vdash E : \mathcal{C}}$$

exchange

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_i : \mathcal{C}_i, x_{i+1} : \mathcal{C}_{i+1}, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_{i+1} : \mathcal{C}_{i+1}, x_i : \mathcal{C}_i, \cdots, x_n : \mathcal{C}_n \vdash E : \mathcal{C}}$$

contraction

$$\frac{\Gamma, x : \mathcal{C}, y : \mathcal{C} \vdash E : \mathcal{D}}{\Gamma, z : \mathcal{C} \vdash E[z/x, z/y] : \mathcal{D}}$$

A.2.3 Logical Rules

hom

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \qquad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) : \mathbf{Set}}$$

power

$$\frac{\Gamma_1 \vdash E_1 : \mathbf{Set} \qquad \Gamma_2 \vdash E_2 : \mathcal{C} \qquad \mathcal{C} \ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] : \mathcal{C}}$$

lambda

$$\frac{\Gamma, x : \mathcal{C} \vdash E : \mathcal{D} \qquad \mathcal{C} \ \mathsf{small}}{\Gamma \vdash \lambda x^{\mathcal{C}}.E : [\mathcal{C}, \mathcal{D}]}$$

application

$$\frac{\Gamma_1 \vdash E_1 : [\mathcal{C}, \mathcal{D}] \quad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 : \mathcal{D}}$$

end

$$\frac{\Gamma, x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash E : \mathcal{D} \quad \mathcal{C} \ \mathsf{small} \quad \mathcal{D} \ \mathsf{complete}}{\Gamma \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E : \mathcal{D}}$$

dual

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma^{\mathsf{op}} \vdash E^* : \mathcal{D}^{\mathsf{op}}}$$

where $(\_)^*$ is defined in 5.1.1.

A.3 Products

A.3.1 Syntax

Types

$$\mathcal{C} ::= \ldots \mid \mathbf{1} \mid \mathcal{C}_1 \times \mathcal{C}_2$$

Expressions

$$E ::= \ldots \mid \langle\ \rangle \mid \langle E_1, E_2 \rangle \mid \mathit{fst}^{\mathcal{C} \times \mathcal{D}}(E) \mid \mathit{snd}^{\mathcal{C} \times \mathcal{D}}(E)$$

Syntactic Identity

$$\mathbf{1}^{\mathsf{op}} = \mathbf{1}$$

A.3.2 Rules

unit

$$\overline{\vdash \langle\,\rangle : \mathbf{1}}$$

pair

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \quad \Gamma_2 \vdash E_2 : \mathcal{D}}{\Gamma_1, \Gamma_2 \vdash \langle E_1, E_2 \rangle : \mathcal{C} \times \mathcal{D}}$$

first

$$\frac{\Gamma \vdash E : \mathcal{C} \times \mathcal{D}}{\Gamma \vdash \mathit{fst}^{\mathcal{C}\times\mathcal{D}}(E) : \mathcal{C}}$$

second

$$\frac{\Gamma \vdash E : \mathcal{C} \times \mathcal{D}}{\Gamma \vdash \mathit{snd}^{\mathcal{C}\times\mathcal{D}}(E) : \mathcal{D}}$$

A.4 Rules for Natural Isomorphisms

A.4.1 Structural Rules

weakening

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_n : \mathcal{C}_n, x_{n+1} : \mathcal{C}_{n+1} \vdash E_1 \cong E_2 : \mathcal{C}}$$

exchange

$$\frac{x_1 : \mathcal{C}_1, \cdots, x_i : \mathcal{C}_i, x_{i+1} : \mathcal{C}_{i+1}, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}{x_1 : \mathcal{C}_1, \cdots, x_{i+1} : \mathcal{C}_{i+1}, x_i : \mathcal{C}_i, \cdots, x_n : \mathcal{C}_n \vdash E_1 \cong E_2 : \mathcal{C}}$$

contraction

$$\frac{\Gamma, x : \mathcal{C}, y : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D}}{\Gamma, z : \mathcal{C} \vdash E_1[z/x, z/y] \cong E_2[z/x, z/y] : \mathcal{D}}$$

A.4.2 Conversion Rules

β-conversion

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 : \mathcal{D} \quad \Gamma_2 \vdash E_2 : \mathcal{C} \quad \mathcal{C}\ \mathsf{small}}{\Gamma_1, \Gamma_2 \vdash (\lambda x^{\mathcal{C}}.E_1)E_2 \cong E_1[E_2/x] : \mathcal{D}}$$

η-conversion

$$\frac{\Gamma \vdash E : [\mathcal{C}, \mathcal{D}]}{\Gamma \vdash \lambda x^{\mathcal{C}}.Ex \cong E : [\mathcal{C}, \mathcal{D}]} \quad x \text{ is not free in } E$$

A.4.3 Equivalence Rules

reflexivity

$$\frac{\Gamma \vdash E : \mathcal{D}}{\Gamma \vdash E \cong E : \mathcal{D}}$$

symmetry

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{D}}{\Gamma \vdash E_2 \cong E_1 : \mathcal{D}}$$

transitivity

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{D} \quad \Gamma \vdash E_2 \cong E_3 : \mathcal{D}}{\Gamma \vdash E_1 \cong E_3 : \mathcal{D}}$$

A.4.4 Congruence Rules

hom cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : \mathcal{C} \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) \cong \mathcal{C}(E_1', E_2') : \mathbf{Set}}$$

power cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : \mathbf{Set} \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C} \quad \mathcal{C} \text{ complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [E_1, E_2] \cong [E_1', E_2'] : \mathcal{C}}$$

ξ-conversion lambda

$$\frac{\Gamma, x : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D} \quad \mathcal{C} \text{ small}}{\Gamma \vdash \lambda x^{\mathcal{C}}.E_1 \cong \lambda x^{\mathcal{C}}.E_2 : [\mathcal{C}, \mathcal{D}]}$$

application cong

$$\frac{\Gamma_1 \vdash E_1 \cong E_1' : [\mathcal{C}, \mathcal{D}] \quad \Gamma_2 \vdash E_2 \cong E_2' : \mathcal{C}}{\Gamma_1, \Gamma_2 \vdash E_1 E_2 \cong E_1' E_2' : \mathcal{D}}$$

ξ-conversion end

$$\frac{\Gamma, x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash E_1 \cong E_2 : \mathcal{D} \quad \mathcal{C} \text{ small} \quad \mathcal{D} \text{ complete}}{\Gamma \vdash \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E_1 \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E_2 : \mathcal{D}}$$

A.4.5 Categorical Rules

nat formula

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 : \mathcal{D} \quad \Gamma_2, y : \mathcal{C} \vdash E_2 : \mathcal{D} \quad \mathcal{C} \text{ small}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash [\mathcal{C}, \mathcal{D}](\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.E_2) \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(E_1, E_2) : \mathbf{Set}}$$

Fubini

$$\frac{\Gamma, x_1 : \mathcal{C}^{\mathsf{op}}, x_2 : \mathcal{C}, y_1 : \mathcal{D}^{\mathsf{op}}, y_2 : \mathcal{D} \vdash E : \mathcal{E} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{small} \quad \mathcal{E}\ \mathsf{complete}}{\Gamma \vdash \int_{x_1^{\mathcal{C}^{\mathsf{op}}}, x_2^{\mathcal{C}}} \int_{y_1^{\mathcal{D}^{\mathsf{op}}}, y_2^{\mathcal{D}}} E \cong \int_{y_1^{\mathcal{D}^{\mathsf{op}}}, y_2^{\mathcal{D}}} \int_{x_1^{\mathcal{C}^{\mathsf{op}}}, x_2^{\mathcal{C}}} E : \mathcal{E}}$$

def end

$$\frac{\Gamma, x : \mathcal{C}^{\mathsf{op}}, y : \mathcal{C} \vdash E : \mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma, w : \mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} E) \cong \int_{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \mathcal{D}(w, E) : \mathbf{Set}}$$

def power

$$\frac{\Gamma_1 \vdash E_1 : \mathbf{Set} \quad \Gamma_2 \vdash E_2 : \mathcal{D} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, x : \mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(x, [E_1, E_2]) \cong [E_1, \mathcal{D}(x, E_2)] : \mathbf{Set}}$$

def singleton

$$\frac{\Gamma \vdash E : \mathcal{D} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma \vdash [1, E] \cong E : \mathcal{D}}$$

Yoneda

$$\frac{\Gamma, x : \mathcal{C}^{\mathsf{op}} \vdash E : \mathbf{Set} \quad \mathcal{C}\ \mathsf{small}}{\Gamma, x : \mathcal{C}^{\mathsf{op}} \vdash E \cong \int_{w^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} \big[\mathcal{C}(y, x), E[w/x]\big] : \mathbf{Set}}$$

opposite cat

$$\frac{\Gamma_1 \vdash E_1 : \mathcal{C} \quad \Gamma_2 \vdash E_2 : \mathcal{C}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \mathcal{C}(E_1, E_2) \cong \mathcal{C}^{\mathsf{op}}(E_2^*, E_1^*) : \mathbf{Set}}$$

dual iso

$$\frac{\Gamma \vdash E_1 \cong E_2 : \mathcal{C}}{\Gamma^{\mathsf{op}} \vdash E_1^* \cong E_2^* : \mathcal{C}^{\mathsf{op}}}$$

A.5 Rules for Weighted Limits

Syntax

$$E ::= \ldots \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}} (E_1, E_2) \mid \oint^{x^{\mathcal{C}^{\mathsf{op}}}, y^{\mathcal{C}}} (E_2, E_1)$$

Typing Rule

wlimit

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 : \mathbf{Set} \quad \Gamma_2, y : \mathcal{C} \vdash E_2 : \mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2 \vdash \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}} (E_1, E_2) : \mathcal{D}}$$

Definition

def wlimit

$$\frac{\Gamma_1, x : \mathcal{C} \vdash E_1 : \mathbf{Set} \quad \Gamma_2, y : \mathcal{C} \vdash E_2 : \mathcal{D} \quad \mathcal{C}\ \mathsf{small} \quad \mathcal{D}\ \mathsf{complete}}{\Gamma_1^{\mathsf{op}}, \Gamma_2, w : \mathcal{D}^{\mathsf{op}} \vdash \mathcal{D}(w, \oint_{x^{\mathcal{C}}, y^{\mathcal{C}}} (E_1, E_2)) \cong [\mathcal{C}, \mathbf{Set}]\big(\lambda x^{\mathcal{C}}.E_1, \lambda y^{\mathcal{C}}.\mathcal{D}(w, E_2)\big) : \mathbf{Set}}$$

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