

Trembling hand perfection is NP-hard (note)

Peter Bro Miltersen

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Abstract

It is NP-hard to decide if a given pure strategy Nash equilibrium of a given three-player game in strategic form with integer payoffs is trembling hand perfect.

1 Introduction

Trembling hand perfection [4] is a well-established refinement of Nash equilibrium. We prove:

Theorem 1. *It is NP-hard to decide if a given pure strategy Nash equilibrium of a given three-player game in strategic form is trembling hand perfect.*

In particular, unless $P=NP$, there is no polynomial time algorithm for deciding if a given equilibrium of a given three-player game in strategic form is trembling hand perfect. Arguably, this can be interpreted as a deficiency of the trembling hand solution concept.

Note that in contrast to the above hardness result, one may efficiently determine if a given equilibrium of a *two*-player game is trembling hand perfect. Indeed, for the two-player case, an equilibrium is trembling hand perfect if and only if it is undominated. This can be checked by linear programming in polynomial time.

The proof below can be rather easily modified to show that it is NP-hard to decide if a given equilibrium of a three-player game is *proper*. For properness, we do not know if the two-player case is easy or not.

Finally, we remark that we do not know if it is *in* NP to decide if a given equilibrium is perfect (or proper). It seems that an obvious nondeterministic algorithm would be to guess a *lexicographic belief structure* and appeal to the characterizations of Blume *et al* [1] and Govindan and Klumpp [3] of trembling hand perfection in terms of these. However, we do not know if a lexicographic belief structure witnessing perfection (or properness) can be represented as a polynomial length string over a finite alphabet.

2 Proof

Our proof is a reduction from the problem of approximately computing minmax values of 3-player games with 0-1 payoffs, a problem that was recently shown to be NP-hard by Borgs *et al* [2]. In particular, it follows from Borgs *et al.* that the following *promise problem* MINMAX is NP-hard:

MINMAX:

1. YES-instances: Pairs (G, r) for which the minmax value for Player 1 in the 3-player game G is strictly smaller than the rational number r .
2. NO-instances: Pairs (G, r) for which the minmax value for Player 1 in G is strictly greater than r .

In fact, by multiplying the payoffs of the game with the denominator of r , we can without loss of generality assume that r is an integer. We now reduce MINMAX to deciding trembling hand perfection.

Let G be a three-player game in strategic form and let r be an integer. We define G' be the game where the strategy space of each player is as in G , except that it is extended by a single pure strategy, \perp . The payoffs of G' are defined as follow. The payoff to Players 2 and 3 are 0 for all strategy combinations. The payoff to Player 1 is r for all strategy combinations where at least one player plays \perp . For those strategy combinations where no player plays \perp , the payoff to player 1 is the same as it would have been in the game G . Obviously, $\mu = (\perp, \perp, \perp)$ is a Nash equilibrium of G' .

We claim that if the minmax value for Player 1 in G is strictly smaller than r , then μ is a trembling hand perfect equilibrium of G' . Indeed, let (τ_2, τ_3) be a minmax strategy profile of Players 2 and 3 in G . Let τ be any profile of G' where Players 2 and 3 play (τ_2, τ_3) . Also, let u be the strategy profile of G' where each player mixes all pure strategies uniformly. Now define

$$\sigma_k = \left(1 - \frac{1}{k} - \frac{1}{k^2}\right)\mu + \frac{1}{k}\tau + \frac{1}{k^2}u$$

We have that σ_k is a fully mixed strategy profile of G' converging to μ as $k \rightarrow \infty$. Also, for sufficiently large k , the strategies of μ are best replies to σ_k . This follows from the fact that Players 2 and 3 are indifferent about the outcome and the fact that Player 1 gets payoff r by playing \perp while he gets a payoff strictly smaller than r for large values of k by playing any other strategy. We conclude that μ is trembling hand perfect, as desired.

On the other hand, we claim that if the minmax value for Player 1 in G is strictly greater than r , then μ is *not* a trembling hand perfect equilibrium of G' . Indeed, let $(\sigma_{k,1}, \sigma_{k,2}, \sigma_{k,3})_k$ be any sequence of fully mixed strategy profiles converging to (\perp, \perp, \perp) . Since $\sigma_{k,2}$ and $\sigma_{k,3}$ do not put all their probability mass on \perp , Player 1 has a reply to $(\sigma_{k,2}, \sigma_{k,3})$ with an expected payoff strictly greater than r . Therefore, \perp is not a best reply of Player 1 to $(\sigma_{k,2}, \sigma_{k,3})$ and we conclude that (\perp, \perp, \perp) is not trembling hand perfect.

That is, we have reduced the promise problem MINMAX to deciding trembling hand perfection and are done.

References

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