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Foundational and mathematical uses of higher types

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Dedicated to Solomon Feferman for his 70th Birthday

Abstract

In this paper we develop mathematically strong systems of analysis in higher types which, nevertheless, are proof-theoretically weak, i.e. conservative over elementary resp. primitive recursive arithmetic. These systems are based on non-collapsing hierarchies (Φ_n-WKL$_+$, Ψ_n-WKL$_+$) of principles which generalize (and for $n = 0$ coincide with) the so-called 'weak' König's lemma WKL (which has been studied extensively in the context of second order arithmetic) to logically more complex tree predicates. Whereas the second order context used in the program of reverse mathematics requires an encoding of higher analytical concepts like continuous functions $F : X \to Y$ between Polish spaces X, Y, the more flexible language of our systems allows to treat such objects directly. This is of relevance as the encoding of F used in reverse mathematics tacitly yields a constructively enriched notion of continuous functions which e.g. for $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ can be seen (in our higher order context)

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to be equivalent to the existence of a continuous modulus of pointwise continuity. For the direct representation of F the existence of such a modulus is independent even of full arithmetic in all finite types E-PA$^\omega$ plus quantifier-free choice, as we show using a priority construction due to L. Harrington. The usual WKL-based proofs of properties of F given in reverse mathematics make use of the enrichment provided by codes of F, and WKL does not seem to be sufficient to obtain similar results for the direct representation of F in our setting. However, it turns out that Ψ_1-WKL$_+$ **is** sufficient.
Our conservation results for (Φ_n-WKL$_+$, Ψ_n-WKL$_+$) are proved via a new elimination result for a strong non-standard principle of uniform Σ^0_1-boundedness which we introduced in 1996 and which implies the WKL-extensions studied in this paper.

1 Introduction

This paper addresses a central theme of proof theory expressed by the following question:

'What parts of ordinary mathematics (in particular of analysis) can be carried out in certain restricted formal systems?'

The relevance of this question is twofold:

1) **Foundational relevance:** suppose a formal system $\mathcal{T}_{\mathbf{PA}}$ allows to formalize a great amount of mathematics but can be shown (by restricted means) to be a conservative extension of first order Peano Arithmetic **PA**, then that part of mathematics has an arithmetical foundation (partial realization of H. Weyl's program, see S. Feferman's discussion in [8]).
   If we work in a system $\mathcal{T}_{\mathbf{PRA}}$ which can be shown (finitistically) even to be conservative over Primitive Recursive Arithmetic **PRA** and identify (following [35]) **PRA** with finitism, then the parts of mathematics which can be carried out in $\mathcal{T}_{\mathbf{PRA}}$ have a finitistic foundation (partial realization of D. Hilbert's program, see e.g. [33]).

2) **Mathematical relevance:** here the guiding question is

   'What more do we known if we have proved a theorem by restricted means than if we merely know that it is true?' (G. Kreisel)

   The aim is to get additional mathematical information out of the fact that a certain theorem S has been proved by certain restricted means. Such additional information may be the extractability of a realizing construction for an existential statement or of an algorithm or a numerical bound for a $\forall\exists$-theorem by unwinding the given proof.

Both motivations are of course closely related and research on them has mutually influenced each other: e.g. a proof of a Π^0_2-theorem carried out in a system which can be (effectively) reduced to **PRA** allows to extract at least a primitive recursive algorithm. In the other direction, e.g. our analysis of proofs in approximation theory (which used the principle of the attainment of the maximum of $f \in C[0,1]$, see [20]) led us to an elimination procedure of the weak König's lemma WKL over a variety of subsystems of arithmetic in all finite types thereby contributing to '1)' above (see [19]). Likewise our treatment of e.g. the Bolzano-Weierstraß principle in [26] via an elimination technique of Skolem functions yielded also new conservation results for comprehension principles ([27]).
However, there are also important differences due to the different points emphasized in 1) and 2):
Whereas there are hardly foundational (understood in the sense of Hilbert) reasons to study systems weaker than **PRA**, merely primitive recursive algorithms and bounds are in most cases much too complex to be of any mathematical value. So on the one hand further restrictions are needed to guarantee the extractability of mathematically more interesting data whereas on the other hand e.g. proofs of large classes of lemmas (having a certain logical form) can be shown not to contribute to the complexity or growth of algorithms or bounds extracted from proofs of theorems using these lemmas. Hence such lemmas can be treated simply as axioms (no matter how non-constructive their proofs might be) in the course of the analysis of a given proofs. Also, for successful unwindings the complexity of the proof transformations used is critical. It has turned out that methods using functionals of finite type like appropriate versions of Gödel's functional interpretation or modified realizability combined with tools like negative translation and/or the Friedman-Dragalin translation are most useful (in particular compared to techniques which try to avoid any passage through higher types, see [29]).

Whereas we have focused on '2)' in several publications (see [21],[20],[24] among others), this paper addresses '1)' to which S. Feferman has contributed so profoundly. We study mathematical strong, but nevertheless **PRA**-reducible, systems in all finite types emphasizing the need of third order variables already for a faithful formalization of continuous functions between Polish spaces.

Let us recall very briefly some of the history of research on '1)'. As Feferman pointed out in [7], 'Hermann Weyl initiated a program for the arithmetical foundation of

mathematics' in his book 'Das Kontinuum' ([39]. In this book, Weyl observed that large parts of analysis can be developed on the basis of **arithmetical** comprehension. This theme was further developed in the 50's by P. Lorenzen among others. In the late 70's Feferman [5] and G. Takeuti [36] independently designed formal systems based on arithmetical comprehension in the framework of higher order arithmetic which are conservative over **PA**. For this property it is important that the schema of induction is restricted to arithmetical formulas only.[1] Work on the program of so-called reverse mathematics by H. Friedman, S. Simpson and others has shown that almost all of the mathematics that can be developed based on arithmetical comprehension at all can also be carried out if induction is restricted in this way. This work uses a second order fragment $\mathbf{ACA}_0$ (formulated in the language of second order arithmetic) of the system from [5] (which is formulated in the language of functionals of all finite types). Via appropriate representations and codings of higher objects (like continuous functions between Polish spaces) a great deal of mathematics can be developed already in $\mathbf{ACA}_0$ (see [34] for a comprehensive treatment).
Feferman's system, however, allows a more **direct** treatment of such objects and their mathematics and also contains a strong uniform ('explicit') version of arithmetical comprehension via a non-constructive μ-operator. These features hold in an even stronger form for theories with flexible (variable) types which were developed successively by Feferman in his framework of explicit mathematics in [4],[6],[7] culminating in a formal system called **W** (where 'W' stands for 'Weyl') which was shown to be proof-theoretically reducible to and conservative over **PA** in [11]. The enormous mathematical power and flexibility of the system **W** led Feferman in [9] to the formulation of the thesis that all (or almost all) scientifically applicable mathematics can be developed in **W**.
In the late 70's, H. Friedman observed that large parts of the mathematics that can be carried out in $\mathbf{ACA}_0$ are already formalizable in a subsystem $\mathbf{WKL}_0$ which instead of the schema of arithmetical comprehension is based on the binary König's lemma (for quantifier-free trees) and Σ^0_1-induction only (see again [34] for a comprehensive treatment of ordinary mathematics in $\mathbf{WKL}_0$). This fact is of foundational relevance since $\mathbf{WKL}_0$ can be proof-theoretically reduced to and is Π^0_2-conservative over **PRA** (H. Friedman (1976, unpublished) and [32]; for a historical discussion which in particular points out various errors in the literature on WKL see [23] (p.69)).

[1] As was shown by Feferman in [5], the corresponding system with full induction is proof-theoretically stronger than **PA**. In [36], Takeuti considers in addition the variant where parameters (except arithmetical ones) are not permitted in the schema of arithmetical comprehension. In this case the resulting system is conservative over **PA** even in the presence of the full schema of induction.

In [19] we introduced an extension (in the spirit of Feferman's **PA**-conservative system from [5] mentioned above) of $\mathbf{WKL_0}$ to all finite types and proved among other things that this extension still can be proof-theoretically reduced to **PRA** and is Π^0_2-conservative over **PRA**.
Although this extension is already much more flexible than the system $\mathbf{WKL_0}$, the use of WKL still requires a complicated encoding of analytical objects. While working on '2)' mentioned above and investigating what parts of analysis produce only provable recursive function(al)s which can be bounded by polynomials (see [24] for a survey) we faced the problem that already the formulation of WKL involves coding devices of exponential growth. That is why we introduced a non-standard axiom F which together with some form of quantifier-free choice proves a strong principle of uniform boundedness Σ^0_1-UB which allows to give short proofs of the usual WKL-applications in analysis relative to very weak (polynomially bounded) systems (see [23],[25]) but does not contribute to the growth of provably recursive functionals. This axiom as well as the principle of uniform boundedness is 'non-standard' in the sense that it is not true in the full set-theoretic type structure. Nevertheless all of its analytic (i.e. second order) consequences are true. In [23] we also studied a restricted version F^- of F which yields a correspondingly restricted version of uniform boundedness which is sufficient for many applications (although a bit more complicated to use, see [25]) but which allows a very easy proof-theoretical elimination. In section 3 of this paper we show that in the presence of the axiom of extensionality and a form of quantifier-free choice, F actually is implied by F^- so that in this context (which we use throughout this paper) the F^--elimination applies to proofs based on F as well. The proof of this fact uses an argument due to Grilliot [14]. The result allows to construct a **PRA**-reducible finite type system $\mathcal{T}^*$ which is based on Σ^0_1-UB. The foundational relevance of this is due to fact that $\mathcal{T}^*$ allows to treat continuous functions between Polish spaces directly as certain type-2-functionals and to prove all the usual WKL-consequences known from reverse mathematics without the coding of such objects used reverse mathematics. We investigate that coding and show that it tacitly yields a constructively enriched representation of continuous functions. More precisely, we show that already for continuous functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, the representation used in reverse mathematics entails the existence of a (continuous) modulus of pointwise continuity functional, which for the direct formulation of such functions as type-2-functionals is not even provable in E-PA$^\omega$+QF-AC^{1,0} (arithmetic in all finite types with full induction and Gödel's T plus quantifier-free choice, see below for a precise definition). In the presence of arithmetical comprehension, the difference between both representations disappears, since the existence of such a modulus of

pointwise continuity can be proved using arithmetical comprehension and QF-AC^{1,0}. However, for theories based on WKL instead this does not seem to be possible.
The main part of this paper (sections 5-7) analyzes that greater mathematical strength of the non-standard principle Σ^0_1-UB compared to WKL in terms of **standard** extensions of WKL. We develop a non-collapsing hierarchy Φ_n-WKL$_+$ of extensions of WKL. Basically, Φ_n-WKL$_+$ extends WKL from binary trees which are given by quantifier-free predicates to binary trees which are given by formulas belonging to a larger class Φ_n (see section 5 below for details). Φ_0-WKL$_+$ is equivalent to WKL, but for $n \geq 1$, Φ_n-WKL$_+$ is not even provable in E-PA$^\omega$+QF-AC^{1,0} + μ (here μ is Feferman's non-constructive μ-operator mentioned above). Nevertheless, Φ_n-WKL$_+$ is provable in $\mathcal{T}^*$ for all $n \in \mathbb{N}$ so that by the results mentioned before the whole hierarchy can be reduced proof-theoretically to **PRA**. Already Φ_2-WKL$_+$ (and even a variant Ψ_1-WKL$_+$ in between this principle and WKL) allows to carry out the usual WKL-applications now even for the direct representation of continuous functions instead of their constructively enriched encoding the WKL-based proofs in reverse mathematics rely on. The (**PRA**-reducible) system $\mathcal{T}$, which results from $\mathcal{T}^*$ be replacing Σ^0_1-UB by Φ_∞-WKL$_+ := \bigcup_{n\in\mathbb{N}}\{\Phi_n$-WKL$_+\}$, can be viewed as a **standard** approximation to $\mathcal{T}^*$.
One might also ask for an explicit version (with flexible types) of such systems based on (extensions of) WKL. However, things are quite delicate in this case. Already for the uniform ('explicit') version UWKL of WKL (analogously to the uniform version of arithmetical comprehension given by μ), the strength of the resulting system crucially depends on the amount of extensionality available (see [30]).

2 Description of the theories E-G$_n$A$^\omega$, E-PRA$^\omega$ and E-PA$^\omega$

The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write 0(00) instead of 0(0(0)). Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of

$\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$.

Our theories $\mathcal{T}$ used in this paper are based on many–sorted classical logic formulated in the language of functionals of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ which allow the definition of λ–abstraction.
The systems E-G$_n$A$^\omega$ (for all $n \geq 1$) are introduced in [23] to which we refer for details. E-G$_n$A$^\omega$ has as primitive relations $=_0, \leq_0$ for objects of type 0, the constant 0^0, functions $\min_0, \max_0, S^{00}$ (successor), $A_0, \ldots, A_n$, where A_i is the i–th branch of the Ackermann function (i.e. $A_0(x,y) = y', A_1(x,y) = x+y, A_2(x,y) = x \cdot y, A_3(x,y) = x^y, \ldots$), functionals of degree 2: $\Phi_1, \ldots, \Phi_n$, where $\Phi_1 fx = \max_0(f0, \ldots, fx)$ and Φ_i is the iteration of A_{i-1} on the f–values for $i \geq 2$, i.e. $\Phi_2 fx = \sum\limits_{i=0}^{x} fi, \Phi_3 fx = \prod\limits_{i=0}^{x} fi, \ldots$.
We also have a bounded search functional μ_b and bounded predicative recursion provided by recursor constants $\tilde{R}_\rho$ (where 'predicative' means that recursion is possible only at the type 0 as in the case of the (unbounded) Kleene-Feferman recursors $\widehat{R}_\rho$).
In this paper our systems always contain the axioms of extensionality

$$(E): \ \forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types ($x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1\ldots z_k =_0 yz_1 \ldots z_k)$ where $\rho = 0\rho_k \ldots \rho_1$).
In [23] we had in addition to the defining axioms for the constants of our theories all true sentences having the form $\forall x^\rho A_0(x)$, where A_0 is quantifier–free and $deg(\rho) \leq 2$, added as axioms.[2]
By 'true' we refer to the full set–theoretic model $\mathcal{S}^\omega$. In given proofs of course only very special universal axioms are used which can be proved in suitable extensions of our theories. Nevertheless one can include them all as axioms if one is only insterested in the applied aspect '2)' discussed above, since they (more precisely their proofs) do not contribute to the provable recursive function(al)s of the system. In particular this covers all instances of the schema of quantifier-free induction. In this paper, however, we include only the schema of quantifier-free choice to E-G$_n$A$^\omega$ instead of taking arbitrary universal axioms, since we are interested in proof-theoretical reductions.

E-PRA$^\omega$ results if we add the functional

$$\Phi_{it} 0yf =_0 y, \ \Phi_{it} x'yf =_0 f(x, \Phi_{it} xyf)$$

[2] The restriction $deg(\rho) \leq 2$ has a technical reason discussed in [23].

to $\text{E-G}_\infty\text{A}^\omega := \bigcup_{n\in\omega}\{\text{E-G}_n\text{A}^\omega\}$. The system E-PRA$^\omega$ is equivalent to Feferman's system $\text{E-}\widehat{\text{PA}}^\omega\!\restriction$ from [5] since Φ_{it} allows (relative to $\text{E-G}_\infty\text{A}^\omega$) to define the predicative recursor constants $\widehat{R}_\rho$ (see [23]).
E-PA$^\omega$ is the extension of E-PRA$^\omega$ obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of [13].

The schema of full choice is given by

$$\text{AC}^{\rho,\tau}\ :\ \forall x^\rho\exists y^\tau A(x,y)\to\exists Y^{\tau(\rho)}\forall x^\rho A(x,Yx),\quad \text{AC}\ :=\bigcup_{\rho,\tau\in\mathbf{T}}\{\,\text{AC}^{\rho,\tau}\}.$$

The schema of **quantifier-free choice** QF-AC$^{\rho,\tau}$ is defined as the restriction of AC$^{\rho,\tau}$ to quantifier-free formulas A_0.[3]

The theory $\mathcal{T}+\mu$ results from $\mathcal{T}$ if we add the non-constructive μ-operator μ^2 to $\mathcal{T}$ together with the characterizing axiom

$$\mu(f)=\begin{cases}\text{the least } x \text{ such that } f(x)=_0 0, \text{ if } \exists x^0(f(x)=_0 0)\\ 0, \text{ otherwise.}\end{cases}$$

Notation: For $\rho=0\rho_k\ldots\rho_1$, we define $1^\rho:=\lambda x_1^{\rho_1}\ldots x_k^{\rho_k}.1^0$, where $1^0:=S0$.

Definition 2.1 *1) Between functionals of type ρ we define the relation $\leq_\rho$:*

$$\begin{cases}x_1\leq_0 x_2:\equiv x_1\leq x_2,\\ x_1\leq_{\tau\rho}x_2:\equiv\forall y^\rho(x_1y\leq_\tau x_2y);\end{cases}$$

2) $$\begin{cases}\min_0(x_1^0,x_2^0):=\min(x_1,x_2),\\ \min_{\rho\tau}(x_1^{\rho\tau},x_2^{\rho\tau}):=\lambda y^\tau.\min_\rho(x_1y,x_2y).\end{cases}$$

In the following we will need the definition of the binary ('weak') König's lemma as given in [38]:

Definition 2.2 (Troelstra(74))
$\text{WKL}:\equiv\forall f^1(T(f)\wedge\forall x^0\exists n^0(lth\ n=_0 x\wedge fn=_0 0)\to\exists b\leq_1\lambda k.1\forall x^0(f(\overline{b}x)=_0 0))$, *where*
$Tf:\equiv\forall n^0,m^0(f(n*m)=_0 0\to fn=_0 0)\wedge\forall n^0,x^0(f(n*\langle x\rangle)=_0 0\to x\leq_0 1)$
(i.e. $T(f)$ asserts that f represents a 0,1–tree).

[3]Throughout this paper $A_0,B_0,C_0,\ldots$ denote quantifier-free formulas.

3 On two non-standard principles

In this section we in particular prove a new conservation result for the non-standard axiom F which was introduced first in [23][4] (and has been applied e.g. in [25]):

$$F :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0 \forall z \leq_1 yk(\Phi kz \leq_0 \Phi k(y_0 k)).$$

We call this axiom 'non-standard' since it does not hold in the full set-theoretic type structure $\mathcal{S}^\omega$. Nevertheless its use can be eliminated from certain proofs thereby yielding classically true results. This has been discussed extensively in [23] to which we refer for further information. In that paper we mainly made use of a weaker version F^- of F which allows a direct proof-theoretic elimination whereas the elimination of F was based on a model-theoretic argument. In this paper however we need the full version F. We show – using an argument known as Grilliot's trick in the context of recursion theory for the countable functionals (see [14])[5] – that in the fully extensional context of theories like E-PRA$^\omega$+QF-AC^{1,0}, F^- actually implies F. This allows to extend the proof-theoretic elimination of F^- to F thereby strengthening results in [23].

We apply F via one of its consequences, the following principle of uniform Σ^0_1-boundedness:

Definition 3.1 ([23]) *The schema*[6] *of* **uniform Σ^0_1–boundedness** *is defined as*

$$\mathbf{\Sigma^0_1\text{–UB}} : \begin{cases} \forall y^{1(0)}(\forall k^0 \forall x \leq_1 yk \exists z^0 \ A(x,y,k,z) \\ \qquad \to \exists \chi^1 \forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k \ A(x,y,k,z)), \end{cases}$$

where $A \equiv \exists \underline{l} A_0(\underline{l})$ and $\underline{l}$ is a tuple of variables of type 0 and A_0 is a quantifier–free formula (which may contain parameters of arbitrary types).

Proposition 3.2 ([23]) *Let $\mathcal{T}$:=*E-G$_n$A$^\omega$ $(n \geq 2)$, E-PRA$^\omega$ *or* E-PA$^\omega$. *Then* $\mathcal{T}$+QF-AC^{1,0} + $F \vdash \Sigma^0_1$-UB.

Proposition 3.3 ([23]) E-G$_3$A$^\omega$ + Σ^0_1-UB $\vdash$ WKL.

[4] A special case of F was studied already in [21] and called also F in that paper but F_0 in [23].

[5] This argument recently has had a further proof-theoretic application in [30].

[6] Σ^0_1-UB can be written as a single axiom. However the schematic version is easier to apply.

Σ^0_1-UB implies the existence of a modulus of uniform continuity for each extensional $\Phi^{1(1)}$ on $\{z^1 : z \leq_1 y\}$ (where 'continuity' refers to the usual metric on the Baire space $\mathbb{N}^{\mathbb{N}}$):

Proposition 3.4 ([23])

$$\begin{aligned}&\text{E-G}_2\text{A}^\omega + \Sigma^0_1\text{-UB} \ \vdash \\ &\forall \Phi^{1(1)} \forall y^1 \exists \chi^1 \forall k^0 \forall z_1, z_2 \leq_1 y \big(\bigwedge_{i \leq_0 \chi k} (z_1 i =_0 z_2 i) \to \bigwedge_{j \leq_0 k} (\Phi z_1 j =_0 \Phi z_2 j)\big).\end{aligned}$$

Remark 3.5 *The argument above can actually be used to show that a sequence of functionals $\Phi^{1(1)}_i$ has a sequence of moduli of uniform continuity on a sequence of sets $\{z : z \leq_1 y_i\}$.*

As mentioned above, in [23] we mainly studied a weaker version

$$F^- :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0 \big(\bigwedge_{i <_0 n} (zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \Phi k(y_0 k)\big)$$

(where, for $z^{\rho 0}$, $(\overline{z,n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise) of F and gave a proof-theoretic elimination procedure for the use of F^- which – relative to so-called weakly extensional variants WE-G$_n$A$^\omega$+QF-AC of our systems E-G$_n$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} – applies for quite general classes of formulas. In the presence of the full extensionality axiom (E) we got corresponding results if the types involved were somewhat restricted. We now show that in the presence of (E), F is already implied by F^- and so that these results extend to F as well.

Proposition 3.6 E-G$_3$A$^\omega$+QF-AC^{1,0} + $F^- \vdash F$.

Proof: From [23] it follows that E-G$_3$A$^\omega$+QF-AC^{1,0}+F^- proves the following weakening of Σ^0_1-UB:

$$\Sigma^0_1\text{–UB}^- : \begin{cases} \forall y^{1(0)} (\forall k^0 \forall x \leq_1 yk \exists z^0\ A(x,y,k,z) \to \exists \chi^1 \forall k^0, x^1, n^0 \\ \qquad\qquad (\bigwedge_{i <_0 n} (xi \leq_0 yki) \to \exists z \leq_0 \chi k\ A((\overline{x,n}), y, k, z))), \end{cases}$$

with $A \equiv \exists l^0 A_0(l)$ as in Σ^0_1-UB. Σ^0_1-UB$^-$ combined with (E) in turn yields that

$$(1)\ \forall \Phi^{1(1)}, y^1 \exists \chi^1 \forall x, \tilde{x} \leq_1 y \forall k^0 \big(\bigwedge_{i=0}^{\chi k} (xi =_0 \tilde{x} i) \to \forall z^0 (\Phi(\overline{x,z})k =_0 \Phi(\overline{\tilde{x},z})k)\big).$$

So if $\Phi^{1(1)}$ satisfies the special case of pointwise continuity

$$(2)\ \forall x \forall k^0 \exists n \forall m \geq n(\Phi(\overline{x,m})k =_0 \Phi xk),$$

then we obtain

(3) Φ is uniformly continuous for $x \leq_1 y$ and has a modulus of uniform continuity χ.

It is easy to see, that (3) implies F (relative to E-G$_3$A$^\omega$). So it remains to show that

$$(4)\ \text{E-G}_3\text{A}^\omega\text{+QF-AC}^{1,0} + F^- \vdash (2).$$

Suppose that $\neg$(2), i.e. there exist $\Phi^{1(1)}, k^0, x^1$ such that

$$(5)\ \forall n^0 \exists m \geq n(\Phi(\overline{x,m})k \neq \Phi xk).$$

By QF-AC^{0,0} (which follows from QF-AC^{1,0}), (5) implies

$$(6)\ \exists f^1 \forall n(fn > n \wedge \Phi(\overline{x,fn})k \neq \Phi xk).$$

Hence for $x_i := \overline{x,fi}$ we have

$$(7)\ \forall i^0 \forall j \geq i(x_j(i) =_0 x(i))$$

and

$$(8)\ \forall i^0(\Phi(x_i,k) \neq \Phi xk).$$

Define $\Psi y^1 :=_0 \begin{cases} 1, \text{ if } \Phi yk \neq \Phi xk \\ 0, \text{ if } \Phi yk = \Phi xk. \end{cases}$

Then

$$(9)\ \forall i,j(\Psi x_i =_0 \Psi x_j \neq \Psi x).$$

Now one can apply an argument from [14], which can be formalized in E-G$_3$A$^\omega$ (see [30] for details on this and a further proof-theoretic application of that argument), to derive

$$(10)\ \exists \varphi^2 \forall f^1(\varphi f = 0 \leftrightarrow \exists x(fx = 0))$$

from (7) and (9). (10), however, contradicts F^- (relative to E-G$_3$A$^\omega$+QF-AC^{1,0}), since F^- implies that every Φ^2 is bounded on the set of all functions $\overline{x,n}$ with $x \leq_1 1, n \in \mathbb{N}$, whereas QF-AC^{1,0} together with (10) yields the existence of a functional μ such that

$$\forall f^1(\exists x^0(fx = 0) \to f(\mu(f)) = 0),$$

which obviously is unbounded on this set. $\Box$

Theorem 3.7 *Let* $\forall f^1, x^0 \exists y^0 A_0(f,x,y)$ *be a sentence of the language of* $\mathcal{T}$ *where* $\mathcal{T} :=$E-G$_n$A$^\omega$ $(n \geq 3)$, E-PRA$^\omega$ *or* E-PA$^\omega$. *Then the following rule holds*

$$\begin{cases} \mathcal{T}+\text{ QF-AC}^{1,0}+\text{QF-AC}^{0,1}+F \vdash \forall f^1, x^0 \exists y^0 A_0(f,x,y) \\ \Rightarrow \text{ one can extract a closed term } \Psi^{001} \text{ of } \mathcal{T} \text{ such that} \\ \mathcal{T} \vdash \forall f^1, x^0 A_0(f,x,\Psi f x). \end{cases}$$

Proof: The theorem follows from proposition 3.6 together with theorem 4.21 from [23]. □

4 Continuous functions: direct representations versus codes

A functional $\Phi^{1(1)}$ is continuous at x^1 if

$$\forall k^0 \exists n^0 \forall y^1 (\bigwedge_{i=0}^{n} (xi =_0 yi) \to \bigwedge_{j=0}^{k} (\Phi xj =_0 \Phi yj)).$$

Φ is continuous if it is continuous at every x.
Using a suitable so-called standard represenation of complete seperable metric ('Polish') spaces X (which in turn relies on a representation of real numbers as Cauchy sequences of rational numbers with fixed rate of convergence), elements of X can be represented by number-theoretic functions x^1 and, moreover, every such function can be considered as a representative of a uniquely determined element of X (see [2] and [20] for details). On these representatives we have a pseudo metric d_X. The elements of X can be identified with the equivalence classes w.r.t. $x =_Y x :\equiv (d_X(x,y) =_{\mathbb{R}} 0)$.
Functions $G : X \to Y$ between Polish spaces therefore are just functionals $\Phi_G^{1(1)}$ which repect $=_X, =_Y$, i.e.

$$\forall x^1, y^1 (x =_X y \to \Phi_G x =_Y \Phi_G y).$$

Φ_G represents a continuous function $G : X \to Y$ if

$$\forall x^1 \forall k^0 \exists n^0 \forall y^1 (d_X(x,y) \leq_{\mathbb{R}} \frac{1}{n+1} \to d_Y(\Phi_G x, \Phi_G y) \leq_{\mathbb{R}} \frac{1}{k+1}).$$

This definition is just the usual ε-δ–definition of continuous functions. One could also consider to define continuity as sequential continuity. In the presence of QF-AC^{0,1}

(which is included in all the systems we consider in this paper) both definitions are equivalent as we will show now.
As usual $G : X \to Y$ is called **sequentially continuous** in x iff

$$\forall x^{1(0)}_{(\cdot)}(\lim_{n\to\infty} x_n =_X x \to \lim_{n\to\infty} \Phi_G(x_n) =_Y \Phi_G(x)),$$

where $(\lim_{n\to\infty} x_n =_X x) :\equiv \forall k^0 \exists n^0 \forall m \geq_0 n(d_X(x_m, x) \leq \frac{1}{k+1})$.

Proposition 4.1 *The theory* E-G$_3$A$^\omega$+QF-AC^{0,1} *proves*

$\forall G : X \to Y \forall x \in X($*G is sequentially continuous at* $x \leftrightarrow G$ *is* ε–δ–*continuous at* $x)$.

Proof: '$\leftarrow$': Obvious!
'$\to$': Suppose that G is not ε–δ–continuous at x, i.e.

$$(*)\ \exists k^0 \forall n^0 \exists y^1 (\underbrace{d_X(x,y) <_{\mathbb{R}} \frac{1}{n+1} \wedge d_Y(\Phi_G(x), \Phi_G(y)) >_{\mathbb{R}} \frac{1}{k+1}}_{\equiv: A \in \Sigma^0_1}).$$

By coding pairs of natural numbers and numbers into functions one can express $\exists y^1 A$ in the form $\exists y^1 A_0$. Hence QF-AC^{0,1} applied to $(*)$ yields

$$\exists k^0, \xi^{1(0)} \forall n^0 (d_X(x, \xi n) <_{\mathbb{R}} \frac{1}{n+1} \wedge d_Y(\Phi_G(x), \Phi_G(\xi n)) >_{\mathbb{R}} \frac{1}{k+1}),$$

i.e. $(\xi n)_{n\in\mathbb{N}}$ represents a sequence of elements of X which converges to x. But $\neg \lim_{n\to\infty} \Phi_G(\xi n) =_{\mathbb{R}} \Phi_G(x)$ and thus G is not sequentially continuous at the point represented by x. $\Box$

Remark 4.2 The use of QF-AC^{0,1} in the proof of '$\to$' in the proposition above is unavoidable already for $X = Y = \mathbb{R}$ since in this case the implication is known to be unprovable even in Zermelo–Fraenkel set theory ZF, see [16],[15] and [12].

We now discuss the indirect representation of continuous functions $G : X \to Y$ between Polish spaces X, Y via codes g as used in the context of reverse mathematics (see definition II.6.1 in [34]). Since reverse mathematics takes place in the language of second-order arithmetic (instead of a language with higher types), the direct representation of such continuous function which is available in our systems is not possible. We will show that provably in E-G$_3$A$^\omega$+QF-AC^{1,0}, for every such code g there exists a direct representation in our sense of the function coded by g, but that the reverse direction in general is not even provable in E-PA$^\omega$+QF-AC. The latter phenomenon is due to the fact that the indirect representation of continuous functions G via codes g tacitly yields a constructive enrichment of the direct representation of G by a modulus of pointwise continuity. To be more specific, let us consider the special case X = Baire space, $Y = \mathbb{N}$ (with the usual metrics). Then the existence of a code g for a continuous functional Φ^2 is (relative to E-G$_3$A$^\omega$+QF-AC^{1,0}) equivalent to the existence of a continuous modulus of pointwise continuity functional Ψ^2 for Φ^2 which in turn is equivalent to the existence of an associate of Φ in the sense of the Kleene/Kreisel countable functionals.

Definition 4.3 *1) α^1 is a neighborhood function if*

> *(a) $\forall \beta^1 \exists n^0 (\alpha(\overline{\beta}n) > 0)$ and*
>
> *(b) $\forall m, n(m \sqsubseteq n \wedge \alpha(m) > 0 \to \alpha(m) = \alpha(n))$, where '$m \sqsubseteq n$' expresses the (elementary recursive) predicate that the sequence encoded by m is an initial segment of the one encoded by n.*

2) α^1 is an associate of Φ^2 if

> *(a) $\forall \beta^1 \exists n^0 (\alpha(\overline{\beta}n) > 0)$ and*
>
> *(b) $\forall \beta, n(n$ least s.t. $\alpha(\overline{\beta}n) > 0 \to \alpha(\overline{\beta}n) = \Phi\beta + 1)$.*

Without loss of generality we may assume that an associate of Φ^2 is a neighborhood function, since otherwise we define

$$\tilde{\alpha}(n) := \begin{cases} \alpha(m), \text{ where } m \text{ shortest initial segment of } n \text{ s.t. } \alpha(m) > 0, \text{ if existing} \\ 0, \text{ otherwise.} \end{cases}$$

Proposition 4.4 E-G$_3$A$^\omega$+QF-AC^{1,0} *proves (uniformly in Φ^2) that the following properties are pairwise equivalent:*

1) $\exists f(f$ is an r.m.-code of $\Phi)$,[7]

2) $\exists \alpha^1(\alpha$ is an associate of $\Phi)$,

[7]By 'r.m-code' we here refer to definition II.6.1 in [34] specialized to $\widehat{A} := \mathbb{N}^{\mathbb{N}}$ and $\widehat{B} := \mathbb{N}$. We identify the set Φ in that definition with its characteristic function f.

3) $\exists \omega^2_\Phi(\omega_\Phi$ is a continuous modulus of pointwise continuity for $\Phi)$.

Proof: '1) $\to$ 3)': Let f be a r.m.-code of Φ^2. Since Φ is total, we have[8]

$$\forall \beta^1 \exists a^0, r^0, b^0, s^0 (d(\beta, \lambda i.(a)_i) <_{\mathbb{R}} 2^{-r} \wedge (a,r) f(b,s) \wedge 2^{-s} <_{\mathbb{Q}} 1)$$

and hence

$$\forall \beta^1 \exists a^0, r^0, b^0, s^0, l^0 (\underbrace{d(\beta, \lambda i.(a)_i) + 2^{-l} <_{\mathbb{R}} 2^{-r} \wedge (a,r) f(b,s) \wedge 2^{-s} <_{\mathbb{Q}} 1}_{\equiv: \exists v^0 A_0(f,\beta,a,r,b,s,l,v)}),$$

where A_0 is quantifier-free. By quantifier-free induction and QF-AC^{1,0} we obtain a functional X^2 such that

$$\forall \beta (X\beta \text{ minimal s.t. } A_0(f, \beta, \nu^6_1(X\beta), \ldots, \nu^6_6(X\beta))).$$

It is clear that X is continuous[9] and that $\Phi\beta = \nu^6_3(X\beta)$. With X, also

$$\omega_\Phi \beta :=_{\mathbb{Q}} 2^{-\nu^6_5(X\beta)}$$

is continuous. One easily verifies that ω_Φ is a modulus of pointwise continuity for Φ.
'3) $\to$ 2)': Let ω_Φ be a continuous modulus of pointwise continuity for Φ^2. Then

$$(1)\ \forall \beta, \gamma (\overline{\beta}(\omega_\Phi \beta) =_0 \overline{\gamma}(\omega_\Phi \beta) \to \Phi\beta =_0 \Phi\gamma)$$

and

$$(2)\ \forall \beta \exists n^0 (\omega_\Phi(\overline{\beta, n}) \le n)$$

(where $\overline{\beta, n}$ is the continuation of $\overline{\beta}n$ with 0).
Define

$$\alpha(n) := \begin{cases} \Phi(\lambda i.(n)_i) + 1, & \text{if } \omega_\Phi(\lambda i.(n)_i) \le lth(n) \\ 0, & \text{otherwise.} \end{cases}$$

(2) yields

$$\forall \beta \exists k (\alpha(\overline{\beta}k) > 0).$$

[8] As in reverse mathematics we represent real numbers as Cauchy sequences with fixed rate of convergence. As a consequence of this, $<_{\mathbb{R}} \in \Sigma^0_1$.

[9] Here we use the fact that $A_0(f, \beta, a, r, b, s, l, v)$ can be written as $t_{A_0}(f, \beta, a, r, b, s, l, v) =_0 0$ for a suitable closed term t_{A_0} of E-G$_3$A$^\omega$ and that every closed term t^2 of E-G$_3$A$^\omega$ is provably pointwise continuous.

Assume that $\alpha(\overline{\beta}k) > 0$, then – by (1) and the definition of α – $\omega_\Phi(\overline{\beta,k}) \le k \wedge \Phi(\overline{\beta,k}) = \Phi\beta$ and therefore $\alpha(\overline{\beta}k) = \Phi\beta + 1$.
'2) $\to$ 1)': Let α be an associate for Φ. By the remark above we may assume that α is a neighborhood function. Define an r.m.-code f for Φ by

$$(a,r)f(b,s) :\equiv \alpha((\overline{\lambda i.(a)_i})r) > 0 \wedge |(\alpha(\overline{\lambda i.(a)_i}r) - 1) - b| < 2^{-s}.$$

This is a quantifier-free (and hence Σ^0_1-)predicate (which we identify with its characteristic function). It is straightforward to verify that f satisfies the properties of an r.m.-code and that f is a code for Φ. We omit the tedious details. □

Remark 4.5 *For the equivalence between 2) and 3), see also* [2] (p.143, E.8).

Theorem 4.6 E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} *does not prove that every continuous functional* Φ^2 *has an r.m.-code (i.e. that* Φ *is continuous in the sense of reverse mathematics).*

Proof: In [31](6.4) a type-structure $A = \langle A_k\rangle_{k\in\mathbb{N}}$ over ω is constructed with the following properties:
(i) $E_2{\restriction}A_1 \notin A_2$, where $E_2(f^1) = 0 \leftrightarrow \exists x(fx = 0)$;
(ii) A is closed under computation in the sense of Kleene's schemata S1-S9.
(iii) there exists a $\Phi \in A_2$ such that Φ has no associate in A_1. By (ii), A is a model of the restriction of E-PA$^\omega$+QF-AC^{1,0} to the fragment with pure types only. Modulo the well-known reduction to pure types (see [37](1.8.5-1.8.8)), E-PA$^\omega$+QF-AC^{1,0} therefore has a model in which there exists a functional Φ^2 which has no associate and therefore – by the previous proposition – no r.m.-code f. Nevertheless, all functionals Φ^2 of type 2 are continuous: one could use here an argument due to [14] to show that the existence of a non-continuous functional in A_2 would contradict (i). However, it requires some care to verify that this argument (which usually is formulated for the full type-structure) relativises to A. We therefore use directly the construction of A which is based on a certain type-2 functional $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ (constructed by L. Harrington using a complicated priority construction, see [31](4.21)) which has the following properties

(i) F is continuous (and therefore has an associate in $\mathbb{N}^\mathbb{N}$),

(ii) $F{\restriction}$REC is not computable (in the sense of S1-S9) and therefore has no recursive associate,

(iii) 1-sc(F) =REC.

A_1 :=REC, $A_{k+1} := \{\Phi : A_k \to \mathbb{N} : \Phi$ computable in $F{\restriction}$REC.$\}$
It is clear that every $\Phi \in A_2$ is continuous.
As a further consequence of this, QF-AC^{0,1} reduces in A to QF-AC^{0,0} since $\forall x^0 \exists f^1 A_0(x,f) \to \forall x^0 \exists y^0 A_0(\lambda i.(y)_i)$. So $A \models$ QF-AC^{0,1}. $\Box$

The fact that the representation of continuous functions in reverse mathematics via codes goes together with a constructive enrichment is also used heavily in many proofs of basic properties of continuous functions in the system WKL$_0$, while WKL does not seem to be sufficient to prove the same results for our direct representation. We discuss this for simplicity again for the case of continuous functions $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. As we have seen above, reverse mathematics treats Φ via an associate α^1. This representation allows to prove the uniform continuity of Φ on the Cantor space of all 0-1-functions by WKL. Define a binary tree by

$$f(n) := \begin{cases} 1, \text{ if } \forall i < lth(n)((n)_i \leq 1) \wedge \alpha(n) > 0 \\ 0, \text{ otherwise.} \end{cases}$$

Since we may assume that α is a neighborhood function, f satisfies $T(f)$. The contraposition of WKL applied to f yields

$$\forall \beta \leq_1 1 \exists x^0 (\alpha(\overline{\beta}x) > 0 \to \exists x \forall \beta \leq_1 1(\alpha(\overline{\beta}x) > 0),$$

i.e. $\Phi\beta = \alpha(\overline{\beta} \min n[\alpha(\overline{\beta}n) > 0]) - 1$ is uniformly continuous on $\{\beta : \beta \leq 1\}$.
Together with QF-AC^{0,0}, WKL even implies the existence of a modulus of uniform continuity function for continuous functionals $\Phi^{1(1)}$ on $\{\beta : \beta \leq 1\}$ (if given by an associate or – equivalently – by an r.m.-code). This is due to the fact that WKL yields

$$\exists k^0 \forall x^0 \exists \beta \leq_1 1(\alpha(\langle k \rangle * \overline{\beta}x) = 0) \to \exists k \exists \beta \leq 1 \forall x(\alpha(\langle k \rangle * \overline{\beta}x) = 0)$$

and so with QF-AC^{0,0} (and the fact that '$\exists \beta \leq_1 1(\alpha(\langle k \rangle * \overline{\beta}x) = 0)$' can be written as a quantifier-free formula) using contraposition

$$(+)\ \forall k \forall \beta \leq_1 1 \exists x(\alpha(\langle k \rangle * \overline{\beta}x) > 0) \to \exists \omega^1 \forall k \forall \beta \leq 1(\alpha(\langle k \rangle * \overline{\beta}(\omega k)) > 0).$$

Thus ω is a modulus of uniform continuity for the functional $\Phi^{1(1)}$ encoded by α.
This argument can be adopted to real functions coded as in reverse mathematics

and is reponsible for the fact that in that context one can prove e.g. that every continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous and has a modulus of uniform continuity.

In our direct type-2-treatment of continuous functions $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ as functionals Φ^2 satisfying

$$\forall f^1 \exists n^0 \forall g^1 (\overline{f}n = \overline{g}n \to \Phi f = \Phi g),$$

the binary tree to which we have to apply König's lemma in order to prove the uniform continuity of Φ on $\{f : f \leq_1 1\}$ is given by

$$Tree(n) :\equiv \exists g, h \leq_1 1 (\bigwedge_{i<lth(n)} (g(i) = (n)_i = h(i)) \wedge \Phi g \neq \Phi h)$$

which no longer is quantifier-free and apparently does not possess a characteristic function (in E-PA$^\omega$+QF-AC^{1,0}) which would be necessary to apply WKL. So we need an extension of WKL to trees of the form $Tree$ above. To show the existence of a modulus of continuity function for a continuous $\Phi^{1(1)}$ on $\{f : f \leq_1 1\}$, not even this extension is enough since QF-AC no longer suffices to prove the version of this extension corresponding to (+) above.
On the other hand – as we saw in proposition 3.4 – the non-standard principle Σ^0_1-UB easily proves the existence of such a modulus function for **arbitrary** functionals $\Phi^{1(1)}$ (and also of functions $G : [0,1]^d \to \mathbb{R}$ represented directly as type-2 functionals; see [23],[25]).
In the next section we study extensions Φ_n-WKL$_+$ and Ψ_n-WKL$_+$ of WKL to trees given by Φ_n- (resp. Ψ_n-)formulas, where, roughly, a formula is in Φ_n (Ψ_n) if it has n alternating bounded function quantifiers – starting with a universal (resp. existential) one – in front of a Π^0_1-formula.[10] For $n = 0$, these principles are equivalent to the usual WKL, but from $n \geq 1$ (resp. $n \geq 2$) on they form a proper hierarchy (even relative to E-PA$^\omega$+QF-AC$^{1,0} + \mu$, where μ is Feferman's non-constructive μ-operator corresponding to the E_2-functional). Adopting the argument above, one can show that Ψ_1-WKL$_+$ suffices to prove the existence of a modulus of uniform continuity for continuous functionals Φ^2 on $\{f^1 : f \leq 1\}$ but also (using the representation of $[0,1]^d, \mathbb{R}$ from [26]) for continuous functions $f : [0,1]^d \to \mathbb{R}$ (and – via suitable standard representations – for other Polish spaces K,Y instead of $[0,1]^d, \mathbb{R}$ with K

[10] That we allow a universal number quantifier underneath the bounded function quantifiers is usefull for the treatment of continuous functions $G : K \to X$ for spaces like $K = [0,1]^d, X = \mathbb{R}$ instead of $2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$.

compact) in their direct type-2 representation.
For all $n \in \mathbb{N}$, the principles Φ_n-WKL$_+$ and Ψ_n-WKL$_+$ (which – in contrast to Σ^0_1-UB – **are** true in the full set-theoretic model) follow from Σ^0_1-UB (relative to E-G$_3$A$^\omega$+QF-AC^{1,0}). So by theorem 3.7 (and proposition 3.2), **proof-theoretically** these extensions of WKL are not stronger than WKL which allows to define PRA-reducible systems of analysis whose **mathematical** strength goes beyond that of the system WKL$_0$ used in reverse mathematics and which in particular allow to treat continuous functions directly without a constructively enriched representation.

We close this section with an open problem whose solution which we conjecture to be true would relativise the foundational significance of WKL for a partial realization of Hilbert's program (see [33]): It seems unlikely in view of the comments above, that WKL (used in a finite type extension like E-PRA$^\omega$+QF-AC^{1,0} of the base system RCA$_0$ used in reverse mathematics) suffices to prove e.g. the existence of a modulus of uniform continuity for continuous functions $F : [0,1] \to \mathbb{R}$ or $F : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ when those are represented directly as type-2 objects (and not via r.m.-codes). However we have not been able to show its unprovability. This problem has connections to apparently rather non-trivial questions in the context of recursion theory for continuous functionals. We now formulate a conjecture which would imply this unprovability:[11]

Conjecture: There exists a type-structure $A = \langle A_n \rangle_{n\in\mathbb{N}}$ such that

1) A is closed under μ-recursion;
2) $A_0 = \omega$;
3) A_1 is a model of WKL;
4) every $\Phi \in A_2$ is continuous (in the usual sense);
5) there exists a $\Phi(f,n) \in A_2$ such that the restriction of Φ to $x \in \omega$ and $f \in A_1$ with $f \leq 1$ does not have an associate in A_1.

Corollary 4.7 (to the conjecture) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL *does not prove that every continuous* $\Phi^{1(1)}$ *has a modulus of uniform continuity when restricted to* $2^{\mathbb{N}}$*. Thus the prominent role of* WKL *in the context of analysis for continuous functions as carried out in reverse mathematics crucially depends on the particular –*

[11] We are indebted to Professor Dag Normann for correspondence about this problem and which led us to formulate it as a conjecture.

constructively enriched – representation of continuous functions via codes (enforced by the restricted language of second-order arithmetic used in reverse mathematics).

Proof: By 1)-3), A is a model of (the pure-type fragment of) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL. If the (restriction of the) functional Φ from 5) had such a modulus in A_1, then one could construct an associate for this restriction in A_1. $\square$

Remark 4.8 *A stronger version of this conjecture results if 1) is replaced by '1*) A is closed under S1-S9 computation'. This strong version implies that even* E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL *does not prove the existence of a modulus of uniform continuity for continuous functions* $F : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

5 Generalization of WKL to more complex trees: Φ_∞-WKL$_+$

Definition 5.1 *1)* $A \in \Phi_n$ *if*

$$A \equiv \forall f_1 \leq_1 s_1[\underline{a}]\exists f_2 \leq_1 s_2[\underline{a}] \ldots \forall^{(d)} f_n \leq_1 s_n[\underline{a}]\forall x^0 A_0(\underline{a}, f_1, \ldots, f_n, x),$$

where A_0 *is quantifier-free and* $\underline{a}$ *contains all free variables of* A *and* s_i *(which may have arbitrary types). The* f_i *must not occur in* $\underline{a}$.

2) $A \in \Psi_n$ *if*

$$A \equiv \exists f_1 \leq_1 s_1[\underline{a}]\forall f_2 \leq_1 s_2[\underline{a}] \ldots \exists^{(d)} f_n \leq_1 s_n[\underline{a}]\forall x^0 A_0(\underline{a}, f_1, \ldots, f_n, x),$$

where A_0 *and* s_i *as above.*

3) The classes Φ_n^- *and* Ψ_n^- *result if we restrict ourselves to parameters* $\underline{a}$ *of type level* ≤ 1 *in* A_0 *and* s_i.

Remark 5.2 *One could also allow further universal number quantifiers* $\forall x^0$ *(but no existential quantifiers) to occur in between the bounded function quantifiers in the definition of* Φ_n. *The results of this paper easily extend to this slightly generalized case. However, for applications to continuous functions on Polish spaces one apparently does not need this. So we restrict ourselves to the definition of* Φ_n *as stated above in order to improve the readability of the proofs.*

Remark 5.3 *In the extensional context of our theories $\mathcal{T}$ we can code pairs of bounded function quantifiers of the same sort together:*

$$\forall f_1 \leq_1 s_1 \forall f_2 \leq_1 s_2 A(f_1, f_2) \leftrightarrow \forall f \leq_1 j(s_1, s_2)\, A(\min_1(j_1 f, s_1), \min_1(j_2 f, s_2))$$

for some monotone function pairing as used e.g. in [23]. *Analogously for $\exists f \leq_1 s$.*

Definition 5.4 *The generalization of* WKL *to Φ_n-trees is given by*

$$\Phi_n\text{-WKL}: \ \forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n\, A(\overline{f}\tilde{n}) \rightarrow \exists f \leq_1 1 \forall n^0 A(\overline{f}n),$$

where $A(k^0) \in \Phi_n$ (with arbitrary further parameters of arbitrary types). Ψ_n-WKL *is defined analogously.* Φ_∞-WKL:$= \bigcup_{n\in\omega}\{\Phi_n\text{-WKL}\}$.

Remark 5.5 *1)* Φ_n-WKL (Ψ_n-WKL) *can be written as a single axiom for each fixed n.*

2) Instead of the special bounding function $\lambda x.1$ in Φ_n-WKL *we may also have a function variable g^1. All proofs in this paper remain valid. For notational simplicity and because of the fact that this more general version actually can be derived from the special one, we formulate only the latter in this paper.*

The next proposition shows that in the absence of parameters of types ≥ 2 (and so in particular in a second-order context) there is no point in considering Φ_n-WKL instead of WKL.[12] For its proof we need the following

Lemma 5.6 *Let $A_0(\underline{a}, g^1, y^0)$ be a quantifier-free formula of $\mathcal{T}$:=*E-G$_n$A$^\omega$ *($n \geq 3$),* E-PRA$^\omega$ *or* E-PA$^\omega$ *containing (in addition to g, y) only parameters $\underline{a}$ of type levels ≤ 1 and let s be a term of $\mathcal{T}$ containing at most $\underline{a}$ as free variables. Then one can construct a Π^0_1-formula $B(\underline{a})$ of $\mathcal{T}$ (containing only $\underline{a}$ free) such that*

$$\mathcal{T} + \text{WKL} \vdash \forall \underline{a}(B(\underline{a}) \leftrightarrow \exists g \leq_1 s[\underline{a}] \forall y^0 A_0(\underline{a}, g, y)).$$

Proof: For $\mathcal{T}$ =E-PRA$^\omega$ and $\mathcal{T}$ = E-PA$^\omega$ this follows from (the proofs of) proposition 4.14 and corollary 4.15 in [19]. The use of the modulus $\tilde{t}xyk$ of pointwise continuity in y used in the proof of proposition 4.14 in [19] can easily be replaced by a modulus $\widehat{t}xk$ of uniform continuity on $\{y : y \leq_1 sx\}$. For closed $t \in$E-G$_n$A$^\omega$ such a modulus $\widehat{t}$ can be constructed in E-G$_n$A$^\omega$ by the method of [18] since the majorization argument used there is available in E-G$_n$A$^\omega$ as was shown in [23]. □

[12] This is in sharp contrast to the case where arbitrary parameters are allowed as we will see below.

Proposition 5.7 *Let $m,n \geq 0$. Over $\mathcal{T}$:=E-G$_k$A$^\omega$ ($k \geq 3$), E-PRA$^\omega$ or E-PA$^\omega$ the following principles are equivalent:*
(i) WKL, (ii) Φ_0-WKL, (iii) Ψ_0-WKL, (iv) Φ_m^--WKL, (v) Ψ_n^--WKL.

Proof: We first show the following
Claim: Let $A(\underline{a})$ be a Φ_n^- (or Ψ_n^-) formula containing only parameters $\underline{a}$ of type degree ≤ 1. Then one can construct a Π_1^0-formula $B(\underline{a})$ such that

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow B(\underline{a}).$$

Proof of the claim: We proceed by meta-induction on n:
$n = 0$: In this case $A \in \Pi_1^0$ and so $B := A$ suffices.
$n \to n+1$: Case 1: $A \in \Phi_{n+1}$. Then $A(\underline{a}) \equiv \forall f \leq_1 s[\underline{a}]\, \tilde{A}(\underline{a}, f)$, where $\tilde{A} \in \Psi_n$. By the induction hypothesis there exists a formula $\tilde{B}(\underline{a}, f) \equiv \forall y^0 \tilde{B}_0(\underline{a}, f, y) \in \Pi_1^0$ with

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \forall f \leq_1 s[\underline{a}] \forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

Let $t_{\tilde{B}_0}$ be a closed term of $\mathcal{T}$ such that

$$\mathcal{T} \vdash \forall \underline{a}, f, y (t_{\tilde{B}_0}(\underline{a}, f, y) =_0 0 \leftrightarrow \tilde{B}_0(\underline{a}, f, y).$$

From results in [18] (using for the case of E-G$_k$A$^\omega$ also [23]) it follows that one can construct a closed term $\widehat{t}_{\tilde{B}_0}$ of $\mathcal{T}$ such that $\widehat{t}_{\tilde{B}_0}(\underline{a}, y)$ is (provably in $\mathcal{T}$) a modulus of uniform continuity for $\lambda f. t_{\tilde{B}_0}(\underline{a}, f, y)$ on $\{f : f \leq_1 s[\underline{a}]\}$. Using this modulus, $\forall f \leq_1 s[\underline{a}]\, \tilde{B}_0(\underline{a}, f, y)$ can be written as a quantifier-free formula and hence $\forall f \leq_1 s[\underline{a}] \forall y\, \tilde{B}_0(\underline{a}, f, y)$ as a Π_1^0-formula $\widehat{B}(\underline{a})$. So

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \widehat{B}(\underline{a}).$$

Case 2: $A(\underline{a}) \in \Psi_{n+1}$. Then $A(\underline{a}) \equiv \exists f \leq_1 s[\underline{a}]\, \tilde{A}(\underline{a}, f)$ with $\tilde{A}(\underline{a}, f) \in \Phi_n$. By I.H. there exists a formula $\tilde{B}(\underline{a}, f) \equiv \forall y^0 \tilde{B}_0(\underline{a}, f, y) \in \Pi_1^0$ with

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \exists f \leq_1 s[\underline{a}] \forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

By the lemma, there exists a Π_1^0-formula $\widehat{B}(\underline{a})$ such that

$$\mathcal{T} + \text{WKL} \vdash \widehat{B}(\underline{a}) \leftrightarrow \exists f \leq_1 s[\underline{a}] \forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

So again

$$\mathcal{T} + \mathrm{WKL} \vdash A(\underline{a}) \leftrightarrow \widehat{B}(\underline{a})$$

with $\widehat{B} \in \Pi^0_1$. This finishes the proof of the claim.
The claim implies that

$$\mathcal{T} + \mathrm{WKL} \vdash \Phi^-_m\text{-WKL} \leftrightarrow \Psi^-_n\text{-WKL}$$

for all $m, n \geq 0$. Since trivially Φ^-_0-WKL $\leftrightarrow$ Φ_0-WKL, it therefore remains to show that

$$\mathcal{T} \vdash \Phi_0\text{-WKL} \ \leftrightarrow \ \Psi_0\text{-WKL} \ \leftrightarrow \ \mathrm{WKL}.$$

Φ_0-WKL$\equiv$ Ψ_0-WKL holds by definition. We have to show WKL$\leftrightarrow$ Φ_0-WKL:
The right-hand side obviously implies the left-hand side since Φ_0-WKL allows the tree-predicate to be given even by a Π^0_1-formula whereas in WKL $T(f)$ is quantifier-free. So it remains to show that WKL$\rightarrow$ Φ_0-WKL: Assume

$$(+)\ \forall n^0 \exists g \leq_1 1 \forall \tilde{n} \leq n \forall z^0 A_0(\overline{g}\tilde{n}, z).$$

Define f such that

$$(++)\ f(x) =_0 0 \leftrightarrow \forall i < lth(x)((x)_i \leq 1) \wedge \forall \tilde{x} \sqsubseteq x \forall z \leq lth(x) A_0(\tilde{x}, z),$$

where '$\tilde{x} \sqsubseteq x$' means that $\tilde{x}$ is the code of an initial segment of the sequence coded by x (note that the right-hand side of (++) can be written as a quantifier-free formula in $\mathcal{T}$).
f satisfies $T(f)$ and – by (+) – represents an infinite binary tree, i.e.

$$\forall n \exists g \leq_1 1\, (f(\overline{g}n) = 0).$$

Hence WKL yields

$$\exists g \leq_1 1 \forall n\, (f(\overline{g}n) = 0),$$

which implies

$$\exists g \leq_1 1 \forall n \forall m \leq n \forall z \leq n\, A_0(\overline{g}m, z),$$

and therefore

$$\exists g \leq_1 1 \forall n \forall z\, A_0(\overline{g}n, z).$$

$\square$

In the presence of higher type parameters, however, we get non-collapsing hierarchies of principles Φ_n-WKL and Ψ_n-WKL as we will show now.

Definition 5.8 *We define the classes of formulas* $\Pi^{1,b}_n$ *and* $\Psi^{1,b}_n$ *simultaneously by induction on* n*:*

(i) $A \in \Pi^{1,b}_0 = \Sigma^{1,b}_0$*, if* A *is quantifier-free;*

(ii) if $A(f) \in \Pi^{1,b}_n$*, then* $\exists f \leq_1 1\, A(f) \in \Sigma^{1,b}_{n+1}$*;*

(iii) if $A(f) \in \Sigma^{1,b}_n$*, then* $\forall f \leq_1 1\, A(f) \in \Pi^{1,b}_{n+1}$*.*

A *may contain arbitrary parameters (of arbitrary types).*

Remark 5.9 $\Pi^{1,b}_n \subseteq \Phi_n$ *and* $\Sigma^{1,b}_n \subseteq \Psi_n$*.*

Definition 5.10 *1) The schema of* $\Pi^{1,b}_n$*-comprehension is given by*

$$\Pi^{1,b}_n\text{-CA}:\ \exists g^1 \forall x^0 (gx = 0 \leftrightarrow A(x)),$$

where $A(x) \in \Pi^{1,b}_n$ *and may contain arbitrary parameters (of arbitrary types) in addition to* x*.* $\Sigma^{1,b}_n$-CA *is defined analogously but with* $\Sigma^{1,b}_n$ *instead of* $\Pi^{1,b}_n$*.*

2) The schema of $\Pi^{1,b}_n$*-choice for numbers is given by*

$$\Pi^{1,b}_n\text{-AC}^{0,0}:\ \forall x^0 \exists y \leq_0 1\, A(x,y) \rightarrow \exists g \leq_1 1 \forall x\, A(x, gx),$$

where $A(x,y) \in \Pi^{1,b}_n$ *and may contain arbitrary parameters.*

Proposition 5.11 *Let* $\mathcal{T}$:=E-PA$^\omega$*. Then*

$$\mathcal{T} + \Phi_{n+1}\text{-WKL} \vdash \Pi^{1,b}_n\text{-CA}$$

(Likewise for Ψ_{n+1}-WKL*).*

Proof: We use the following tree-predicate from [38]:

$$\tilde{A}(k) :\equiv \begin{cases} (k)_{lth(k) \dot{-} 1} \leq 1 \wedge ((k)_{lth(k) \dot{-} 1} = 0 \rightarrow A(lth(k) \dot{-} 1)) \wedge \\ \qquad ((k)_{lth(k) \dot{-} 1} = 1 \rightarrow \neg A(lth(k) \dot{-} 1))), \text{ if } lth(k) > 0 \\ true, \text{ otherwise.} \end{cases}$$

For $A \in \Pi^{1,b}_n$, $\tilde{A}(k)$ can be written as a Φ_{n+1}-formula (using remark 5.3). By induction on n we can prove in E-PA$^\omega$ that

$$\forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n \tilde{A}(\overline{f}\tilde{n}).$$

Φ_{n+1}-WKL therefore yields the characteristic function for $A(n)$. □

Proposition 5.12 E-PA$^\omega$ + $\Pi^{1,b}_n$-CA+μ *contains (modulo a canonical embedding which doesn't change the first order part) the second order system* (Π^1_n-CA) *known from reverse mathematics.*[13]

Proof: Systems formulated in the language of second-order arithmetic with set variables like (Π^1_n-CA) can be embedded in (suitable) systems formulated in the language of functionals of all finite types by representing sets X by their characteristic functions χ_X and replacing formulas '$t \in X$' by '$\chi_X(t) =_0 0$'. In doing so and using the fact that the presence of μ allows to absorb an arbitrary arithmetical quantifier-prefix in front of a quantifier-free formula with arbitrary parameters uniformly in these parameters, the comprehension schema of (Π^1_n-CA) reduces to $\Pi^{1,b}_n$-CA above. $\Box$

The two propositions above show that the systems E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + $\mu + \Phi_n$-WKL (and similar with Ψ_n-WKL) form a non-collapsing hierarchy which as n increases eventually exhausts full second-order arithmetic.

Together with the result due to Feferman that E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + μ can be reduced proof-theoretically to $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$[14] and hence is proof-theoretically much weaker than (Π^1_1-CA), it in particular follows that for $n \geq 2$, Φ_n-WKL and Ψ_n-WKL are underivable in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + μ. The next proposition improves this further:

Proposition 5.13 E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + $\mu \nvdash \Phi_1$-WKL.

Proof: One easily verifies that E-PA$^\omega$ + Φ_1-WKL proves $\Pi^{1,b}_1$-AC^{0,0} which in the presence of μ yields the so-called Σ^1_1-separation principle (see [34]), hence (again by [34]) the subsystem ATR of second order arithmetic, whose proof-theoretic strength is much higher than that of $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$, is contained in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + $\mu + \Phi_1$-WKL. $\Box$

Remark 5.14 *A more detailed analysis of the proof-theoretic strength of the systems* E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + $\mu + \Phi_n$-WKL *which would allow to determine the precise relationship between* Φ_n-WKL *and* Ψ_m-WKL *has to be postponed for a subsequent paper.*

As we have seen already above, Ψ_1-WKL suffices to prove the uniform continuity of continuous functions $\Phi : [0,1]^d \to \mathbb{R}$ (and more general: for continuous functions

[13] In the notation of [34], (Π^1_n-CA) is the system Π^1_n-CA$_0$+full induction.

[14] This follows from [5] together with elimination of extensionality (see also [1]).

from compact metric spaces into Polish spaces). However, in order to show the existence of a modulus of uniform continuity function we apparently need a slightly stronger form Ψ_1-WKL$_+$:

Definition 5.15 *Let $A(a^0, k^0) \in \Phi_n$ (with arbitrary parameters).*

$$\Phi_n\text{-WKL}_+ : \forall h^1 \exists a^0 \exists f \leq_1 1 \forall \tilde{n} \leq h(a)\, A(a, \overline{f}\tilde{n}) \to \exists a \exists f \leq_1 1 \forall n^0 A(a, \overline{f}n)$$

(Ψ_n-WKL$_+$ is defined analogously with $A \in \Psi_n$.)

Remark 5.16 *In $\mathcal{T}$:=E-G$_k$A$^\omega$ ($k \geq 3$), E-PRA$^\omega$ or E-PA$^\omega$, trivially Φ_n-WKL$_+ \to$ Φ_n-WKL.*

For $n = 0$, Φ_0-WKL (and hence WKL) together with QF-AC^{0,0} implies already Φ_0-WKL$_+$:

Proposition 5.17 *Let $\mathcal{T}$:=E-G$_k$A$^\omega$ ($k \geq 3$), E-PRA$^\omega$ or E-PA$^\omega$. Then $\mathcal{T}$+QF-AC$^{0,0} \vdash \Phi_0$-WKL$\leftrightarrow \Phi_0$-WKL$_+$.*

Proof: The direction '$\leftarrow$' is trivial.
'$\to$': For $A(a, k) \in \Phi_0$ ($= \Pi^0_1$), Φ_0-WKL implies

$$\exists a^0 \forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n\, A(a, \overline{f}\tilde{n}) \to \exists a \exists f \leq_1 1 \forall n^0 A(a, \overline{f}n).$$

'$\exists f \leq_1 1$' in '$\exists f \leq_1 1 \forall \tilde{n} \leq n\, A(a, \overline{f}\tilde{n})$' can be replaced by a bounded number quantifier. Together with the fact that the Σ^0_1-collection principle is derivable in $\mathcal{T}$+QF-AC^{0,0}, this implies that '$\exists f \leq_1 1 \forall \tilde{n} \leq n\, A(a, \overline{f}\tilde{n})$' can be written as a Π^0_1-formula. Hence (again using QF-AC^{0,0}), $\mathcal{T}$+QF-AC^{0,0} proves

$$\exists a^0 \forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n\, A \leftrightarrow \forall h^1 \exists a^0 \exists f \leq_1 1 \forall \tilde{n} \leq h(a)\, A,$$

which concludes the proof. □

The proposition above is the reason for the phenomenon that in the context of reverse mathematics (where the more constructive definition of continuous functions used makes it possible to replace the use of Ψ_1-WKL by WKL=Φ_0-WKL) WKL suffices even to show the existence of a modulus of uniform continuity. For our direct representation of continuous functions, however, we have to use Ψ_1-WKL$_+$ which does not seem to be implied by Ψ_1-WKL and QF-AC^{0,0}.
In the next two sections we will show that, nevertheless, proof-theoretically Φ_∞-WKL$_+$ (=Ψ_∞-WKL$_+$) is not stronger than WKL.

6 The computational strength of Φ_∞-WKL$_+$

Proposition 6.1 *Let* $\mathcal{T}$:=E-G$_k$A$^\omega$ $(k \geq 3)$, E-PRA$^\omega$ *or* E-PA$^\omega$. *Then*

$$\mathcal{T} + \text{QF-AC}^{1,0} + F^- \vdash \Phi_\infty\text{-WKL}_+.$$

Proof: Because of proposition 3.6 it suffices to show that

$$\mathcal{T} + \text{QF-AC}^{1,0} + F \vdash \Phi_\infty\text{-WKL}_+.$$

The idea of the proof is to use proposition 3.4 (together with propositions 3.2 and 3.3) to show similarly to the argument in the proof of proposition 5.7 that every $A \in \Phi_n$ (or $\in \Psi_n$) can be written as a Π^0_1-formula B. Whereas in the proof of proposition 5.7 we could use the fact that for every term $t^2[\underline{a}]$ of $\mathcal{T}$ containing only variables $\underline{a}$ of type ≤ 1 one can construct a modulus of uniform continuity on $\{x : x \leq_1 b\}$ (uniformly in $\underline{a}$ and b), we have to use proposition 3.4 in the presence of arbitrary parameters. The latter provides such a modulus of uniform continuity only uniformly in number parameters but not uniformly in function parameters f unless the latter are themselves restricted to a compact set $\{f : f \leq_1 b\}$ (in which case a modulus that is independent of f does exist). However this is just the case in the situation at hand since all function variables $f_1, \ldots, f_n$ of $A \in \Phi_n$ which are not parameters are bounded. So all we need is

$$(*) \begin{cases} \forall \Phi, \underline{a} \exists \alpha^1 \forall x^0, z^0 (\lambda \underline{f}.(\Phi x z \underline{f} \underline{a})^0 \text{ is uniformly continuous for all} \\ f_1 \leq_1 s_1[x, \underline{a}], \ldots, f_n \leq_1 s_n[x, \underline{a}] \text{ with modulus } \alpha x z), \end{cases}$$

where $\underline{a}$ are all the remaining free variables of s_i (which may have arbitrary types).[15] $(*)$ is implied by

$$(**) \begin{cases} \forall \Phi, \underline{a}, \underline{b}^{1(0)} \exists \alpha^1 \forall x^0, z^0 (\lambda \underline{f}.(\Phi x z \underline{f} \underline{a})^0 \text{ is uniformly continuous for all} \\ f_1 \leq_1 b_1 x, \ldots, f_n \leq_1 b_n x \text{ with modulus } \alpha x z). \end{cases}$$

But this follows in $\mathcal{T} + \Sigma^0_1$-UB (and therefore in $\mathcal{T}$+QF-AC^{1,0} + F by proposition 3.2) similarly to the proof of proposition 3.4. Since by proposition 3.3 also WKL

[15] Here 'z' is the variable from the Π^0_1-kernel of A (which of course can be merged together with x).

is available in this theory, we can argue as in the proof of the claim in the proof of proposition 5.7 and show that for $A(x) \in \Phi_n$ (with arbitrary additional parameters)

$$\mathcal{T} + \Sigma^0_1\text{-UB} \vdash \exists \Phi \forall x^0 (A(x) \leftrightarrow \forall z^0 (\Phi x z =_0 0)).$$

Hence for all $n \in \mathbb{N}$

$$(***) \mathcal{T} + \Sigma^0_1\text{-UB} \vdash \Phi_0\text{-WKL}_+ \to \Phi_n\text{-WKL}_+$$

and therefore (using propositions 3.3,5.7 and 5.17)

$$\mathcal{T} + \text{QF-AC}^{0,0} + \Sigma^0_1\text{-UB} \vdash \Phi_n\text{-WKL}_+$$

and therefore by proposition 3.2

$$\mathcal{T} + \text{QF-AC}^{1,0} + F \vdash \Phi_n\text{-WKL}_+,$$

which concludes the proof. $\Box$
Corollary to the proof of proposition 6.1:

$$\mathcal{T} + \text{QF-AC}^{0,0} + \Sigma^0_1\text{-UB} \vdash \Phi_\infty\text{-WKL}_+$$

7 PRA-reducible theories

Theorem 7.1 *1)* E-G$_3$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+Σ^0_1-UB *is* Π^0_2*-conservative over* EA*,*

2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Σ^0_1-UB *is* Π^0_2*-conservative over* PRA*,*

3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Σ^0_1-UB *is conservative over* PA*.*

Proof: We first prove 3): Let A be a sentence of PA which is provable in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+Σ^0_1-UB and hence (using proposition 3.2) in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + F. Then the Herbrand normal form $A^H \equiv \forall \underline{f} \exists \underline{y} A_0(\underline{f}, \underline{y})$ of A is provable there a-fortiori. Hence by theorem 3.7

$$\text{E-PA}^\omega \vdash \forall \underline{f}\, A_0(\underline{f}, \underline{\Psi}(\underline{f}))$$

for suitable closed terms $\underline{\Psi}$ of E-PA$^\omega$. Thus

$$\text{E-PA}^\omega \vdash A^H.$$

By [17](thm.4.1) we can conclude that[16]

$$\mathrm{PA} \vdash A.$$

1) and 2): For Π^0_2-sentences A the argument above applies equally to E-G$_3$A$^\omega$ (resp. E-PRA$^\omega$) yielding E-G$_3$A$^\omega \vdash A$ (resp. E-PRA$^\omega \vdash A$). The conclusion now follows from the fact that E-G$_3$A$^\omega$ (resp. E-PRA$^\omega$) is Π^0_2-conservative over EA (resp. PRA). □

Theorem 7.2

1) E-G$_3$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Φ_∞-WKL$_+$ *is* Π^0_2*-conservative over* EA*,*

2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Φ_∞-WKL$_+$ *is* Π^0_2*-conservative over* PRA*,*

3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Φ_∞-WKL$_+$ *is conservative over* PA

Proof: The theorem follows from theorem 7.1, proposition 6.1 and 3.2. □

Remark 7.3 *The purely proof-theoretic proofs of theorems 7.1 and 7.2 also yield corresponding proof-theoretic reductions.*

Summary about PRA-reducibility:

In this paper we in particular have constructed two new mathematically strong PRA-reducible and Π^0_2-conservative extensions of PRA. One of these systems

$$\mathcal{T}^* := \text{E-PRA}^\omega\text{+QF-AC}^{1,0}\text{+QF-AC}^{0,1} + \Sigma^0_1\text{-UB}$$

is a non-standard system in the sense that the full set-theoretic type structure $\mathcal{S}^\omega$ is not a model of $\mathcal{T}^*$.
Analysing the greater mathematical strength of $\mathcal{T}^*$ (w.r.t. to derivable consequences which **are** true in $\mathcal{S}^\omega$) in terms of generalizations of WKL to logically more complex binary trees, we developed the subsystem

$$\mathcal{T} := \text{E-PRA}^\omega\text{+QF-AC}^{1,0}\text{+QF-AC}^{0,1} + \Phi_\infty\text{-WKL}_+$$

which has $\mathcal{S}^\omega$ as a model.
In particular, $\mathcal{T}$ allows to carry out substantial parts of classical analysis in a much more direct way than the second order system WKL$_0$ or even E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL.

[16]Warning: this argument does not apply to the subsystems E-PRA$^\omega$, PRA; see [17] for a counterexample to this.

Concluding remarks:

1) There is also a different route to design PRA-reducible systems which is based on E-$G_\infty A^\omega$ instead of E-PRA$^\omega$. Although E-$G_\infty A^\omega$ contains all primitive recursive functions and primitive recursive functionals of every Grzegorczyk level n, it does not contain all ordinary Kleene-primitive recursive functionals of type 2, in particular it does not contain Φ_{it}. As a consequence of this, E-$G_\infty A^\omega$+QF-AC^{0,0} does not prove the schema of Σ^0_1-induction. As we have shown in [26],[27] and [28], one can add to E-$G_\infty A^\omega$+QF-AC^{1,0}+ QF-AC^{0,1} function parameter-free schematic forms of e.g. Π^0_1-comprehension, the Bolzano-Weierstraß principle for sequences in $[0,1]^d$, the Arzela-Ascoli lemma etc. and still obtain a PRA-reducible system (whereas the addition of any of these principles to E-PRA$^\omega$ would make the Ackermann function provably total). This result was obtained via a certain Σ^0_2-generalization of the principle Σ^0_1-UB$^-$ mentioned in the proof of proposition 3.6. Using the results of this paper we can even allow a corresponding generalization of the principle Σ^0_1-UB instead. As a consequence of this and the fact that Φ_∞-WKL$_+$ follows from Σ^0_1-UB already relative to E-$G_\infty A^\omega$, we may add Φ_∞-WKL$_+$ to the principles listed above without losing PRA-conservation. This results in a mathematically fairly strong system (note that E-$G_\infty A^\omega$+QF-AC^{0,0} allows to interpret the weak base system RCA$^*_0$ from reverse mathematics and see remark X.4.3 in [34]) which is incompatible with the systems studied in this paper. A detailed treatment of this theme, however, has to be postponed for another paper.

2) The results of this paper and [30] suggest to propose the following extension of the program of reverse mathematics to finite types: Replace the base system RCA$_0$ by its finite type extension RCA$^\omega_0$:= E-PRA$^\omega$+QF-AC^{1,0}. This system can be shown to be conservative over (an inessential variant with function variables instead of set variable of) RCA$_0$. So for second order statements A, B (i.e. the type of statements which can be discussed in the framework of currently existing reverse mathematics) **nothing is lost** if we prove an equivalence between A and B relative to RCA$^\omega_0$ instead of RCA$_0$. However, the richer language allows to consider new statements (in their direct formulation) which can not even be expressed in RCA$_0$ and to apply reverse mathematics to them as well. As first example, we can recast a result from [30] as a result in reverse mathematics in this extended sense:

   'Relative to RCA$^\omega_0$, the uniform weak König's lemma UWKL and the existence of Feferman's μ-operator are equivalent'.

Likewise, the equivalence between μ and strong uniform versions of analytical theorems like the attainment of the maximum of $f \in C[0,1]$ can be obtained.

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