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On the uniform weak König's lemma

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Abstract

The so-called weak König's lemma WKL asserts the existence of an infinite path b in any infinite binary tree (given by a representing function f). Based on this principle one can formulate subsystems of higher-order arithmetic which allow to carry out very substantial parts of classical mathematics but are Π^0_2-conservative over primitive recursive arithmetic PRA (and even weaker fragments of arithmetic). In [10] we established such conservation results relative to finite type extensions PRA$^\omega$ of PRA (together with a quantifier-free axiom of choice schema). In this setting one can consider also a uniform version UWKL of WKL which asserts the existence of a functional Φ which selects uniformly in a given infinite binary tree f an infinite path Φf of that tree. This uniform version of WKL is of interest in the context of explicit mathematics as developed by S. Feferman. The elimination process in [10] actually can be used to eliminate even this uniform weak König's lemma provided that PRA$^\omega$ only has a quantifier-free rule of extensionality QF-ER instead of the full axioms (E) of extensionality for all finite types. In this paper we show that

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in the presence of (E), UWKL is much stronger than WKL: whereas WKL remains to be Π^0_2-conservative over PRA, PRA$^\omega + (E)$+UWKL contains (and is conservative over) full Peano arithmetic PA.

1 Introduction

The binary (so-called 'weak') König's lemma WKL plays an important role in the formulation of mathematically strong but proof-theoretically weak subsystems of analysis. In particular the fragment (WKL$_0$) of second-order arithmetic which is based on recursive comprehension (with set parameters), Σ^0_1-induction (with set parameters) and WKL occurs prominently in the context of reverse mathematics (see [16]). Although (WKL$_0$) allows to carry out a great deal of classical mathematics, it is Π^0_2-conservative over primitive recursive arithmetic PRA as was shown first by H. Friedman using a model-theoretic argument. In [15] a proof-theoretic argument is given for a variant of (WKL$_0$) which uses function variables instead of set variables. In [10] we established various conservation results for WKL relative to subsystems of arithmetic in all finite types. As a special case these results yield that

(1) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL is Π^0_2-conservative over PRA,

where E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL is a finite type extension of (WKL$_0$) (see below for a precise definition). The proof of this fact relies on a combination of Gödel's functional interpretation with elimination of extensionality (see [12]), negative translation and Howard's [8] majorization technique. The first step of the proof reduces the case with the full axiom of extensionality to a subsystem WE-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL which is based on a weaker quantifier-free rule of extensionality only (see below) which was introduced in Spector [17]. From this system, WKL is then eliminated. This elimination actually eliminates WKL via a strong uniform version of WKL, called UWKL below, which states the existence of a functional which selects uniformly in a given infinite binary tree an infinite path from that tree. This yields the following conservation result (which isn't stated explicitly in [10] but which can be obtained from the proofs in section 4 of that paper, see below)

(2) WE-PRA$^\omega$+QF-AC+UWKL is Π^0_2-conservative over PRA.[1]

For WE-PA$^\omega$ we get the following result (with the same convention on + as above)

(3) WE-PA$^\omega$+QF-AC+UWKL is conservative over PA.

(2) is of interest in the context of so-called explicit mathematics as developed by S. Feferman (starting with [3]) and further investigated also by A. Cantini, G. Jäger and T. Strahm among others, since the uniform weak König's lemma UWKL seems to be a very natural 'explicit' formulation of WKL. We have been asked about the status of UWKL in the presence of full extensionality. In this note we give a surprisingly simple answer to this question showing, in particular, that

(4)E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL contains (and is conservative over) PA

and

(5)E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL has the same strength as $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$,

where PA denotes full first-order Peano arithmetic.

Acknowledgement: This paper was prompted by discussions the author has had with Gerhard Jäger and Thomas Strahm who asked him about the status of the uniform weak König's lemma in a fully extensional context.

2 Preliminaries

The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

[1]In this weakly extensional context based on a quantifier-free rule of extensionality '+' must be understood in the sense that the axioms QF-AC and WKL must not be used in the proof of a premise of an application of the extensionality rule. See [10] (where we use a special symbol '$\oplus$' to emphasize this point) for details on this. Actually it is sufficient to impose this restriction on the use of the additional axioms for UWKL only.
The conservation results in [10] are much more general than the one we mentioned. This makes the proofs more involved than is needed for the special (Π^0_2-)case relevant here. A corresponding simplification of our argument has been worked out in [1].

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k\ldots\rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00))\ldots$ are so represented by $0, 1, 2, 3\ldots$ For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short deg(ρ)) is defined by $\deg(0) := 0$ and $\deg(\tau(\rho)) := \max(\deg(\tau),\deg(\rho)+1)$. For pure types the degree is just the number which represents this type.

The system E-PRA$^\omega$ is formulated in the language of functionals of all finite types and contains $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$-combinators for all types (which allow to define λ-abstraction) and all primitive recursive functionals in the sense of Kleene (i.e. primitive recursion is available only on the type 0). More formally, E-PRA$^\omega$ results from Feferman's system $\widehat{\mathrm{PA}}^\omega\!\restriction$ in [4] if we add the axioms of extensionality

$$(E): \ \forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types (where for $\rho = 0\rho_k\ldots\rho_1$, $x =_\rho y$ is defined as $\forall z_1^{\rho_1},\ldots,z_k^{\rho_k}(xz_1\ldots z_k =_0 yz_1\ldots z_k)$).[2] We only include equality $=_0$ between numbers as a primitive predicate.

E-PA$^\omega$ is the extension of E-PRA$^\omega$ obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of Gödel [6] and coincides with Troelstra's [18] system (E-HA$^\omega$)^c.
The 'weakly extensional'[3] versions WE-PRA$^\omega$ and WE-PA$^\omega$ of these systems result if we replace the extensionality axioms (E) by a quantifier-free rule of extensionality (due to Spector [17])

$$\text{QF-ER: } \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]},$$

where A_0 is quantifier-free, $s^\rho, t^\rho, r[x^\rho]^\tau$ are arbitrary terms of the system and $\rho, \tau \in$ are arbitrary types.
Note that QF-ER allows to derive the extensionality axiom for type 0 but already the extensionality axiom for type-1-arguments, i.e.

$$\forall z^2 \forall x^1, y^1(x =_1 y \to zx =_0 zy)$$

[2]We deviate slightly from our notation in [11]. The system denoted by E-PRA$^\omega$ in the present paper results from the corresponding system in [11] if we replace the universal axioms 9) in the definition of the latter by the schema of quantifier-free induction.

[3]This terminology is due to [18].

is underivable in WE-PA$^\omega$ (see [8]).
The schema of quantifier-free choice is given by

$$\text{QF-AC}^{\rho,\tau}\ :\ \forall x^\rho \exists y^\tau A_0(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A_0(x, Yx),\quad \text{QF-AC} \ :=\ \bigcup_{\rho,\tau\in\mathbf{T}} \{\,\text{QF-AC}^{\rho,\tau}\},$$

where A_0 is a quantifier-free formula.

In the following we use the formal definition of the binary ('weak') König's lemma as given in [19] (here $*, \overline{b}x, lth(n)$ refer to the primitive recursive coding of finite sequences from [18]):

Definition 2.1 (Troelstra(74))
WKL:$\equiv \forall f^1(T(f) \wedge \forall x^0 \exists n^0(lth(n) = x \wedge fn = 0) \to \exists b^1 \forall x^0(f(\overline{b}x) = 0))$, *where*
$Tf :\equiv \forall n^0, m^0(f(n*m) =_0 0 \to fn =_0 0) \wedge \forall n^0, x^0(f(n*\langle x\rangle) =_0 0 \to x \leq_0 1)$
(i.e. $T(f)$ asserts that f represents a 0,1–tree).

Notation 2.2 $T^\infty(f) :\equiv T(f) \wedge \forall x^0 \exists n^0(lth(n) = x \wedge fn = 0)$, *i.e.* $T^\infty(f)$ *expresses that f represents an infinite binary tree. So*
WKL $\equiv \forall f^1(T^\infty(f) \to \exists b^1 \forall x^0(f(\overline{b}x) = 0))$.

Definition 2.3 *The uniform weak König's lemma* UWKL *is defined as*

$$\text{UWKL} :\equiv \exists \Phi^{1(1)} \forall f^1(T^\infty(f) \to \forall x^0(f((\overline{\Phi f})x) = 0)).$$

3 Results

For the weakly extensional systems WE-PRA$^\omega$ and WE-PA$^\omega$ we have the following conservation results for UWKL:

Theorem 3.1 *1)* WE-PRA$^\omega$+QF-AC+UWKL *is* Π^0_2*-conservative over* PRA.

2) WE-PA$^\omega$+QF-AC+UWKL *is conservative over* PA.
(Here, again, + must be understood in the sense of footnote 1).

Proof: 1) In [10] (4.2-4.7), we constructed a primitive recursive functional $f^1, g^1 \mapsto \zeta fg := (\widehat{f_g})$ such that

$$(1)\ \text{WE-PRA}^\omega \vdash \forall f, g\, T^\infty(\zeta fg)$$

and

$$(2)\ \text{WE-PRA}^\omega \vdash \forall f (T^\infty(f) \to \exists g (f =_1 \zeta f g)).$$

By the proof of theorem 4.8 in [10] (and the fact that WE-PRA$^\omega$ is Π^0_2-conservative over PRA), it follows that

$$\text{WE-PRA}^\omega\text{+QF-AC+UWKL}^* \text{ is } \Pi^0_2\text{-conservative over PRA},$$

where

$$\text{UWKL}^* := \exists B \forall f, g, x ((\zeta f g)((\overline{B f g})x) =_0 0).$$

It remains to show that

$$\text{WE-PRA}^\omega \vdash \text{UWKL}^* \to \text{UWKL}.$$

The proof of (2) in [10](4.7) shows that g can be primitive recursively defined in f as

$$\tilde{f}(x) := \begin{cases} \min n \leq \overline{1^1}x[lth(n) = x \wedge f(n) = 0], \text{ if such an } n \text{ exists} \\ 0^0, \text{ otherwise.} \end{cases}$$

Thus for $\xi f := \zeta(f, \tilde{f})$

$$(2)'\ \text{WE-PRA}^\omega \vdash \forall f (T^\infty(f) \to f =_1 \xi f).$$

Define $\Phi f := B(f, \tilde{f})$ for B satisfying UWKL*. Then

$$\forall x ((\xi f)((\overline{\Phi f})x) =_0 0)$$

and so for f such that $T^\infty(f)$ (which implies $f =_1 \xi f$)

$$\forall x (f((\overline{\Phi f})x) =_0 0),$$

i.e. Φ satisfies UWKL.
2) As in 1) we obtain from the proof of 4.8 in [10] that

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL is } \forall \underline{x}^{\underline{\rho}} \exists \underline{y}^0 A_0(\underline{x}, \underline{y})\text{-conservative over WE-PA}^\omega,$$

where $\underline{x}^{\underline{\rho}}$ is a tuple of variables of type levels ≤ 1, A_0 is quantifier-free and contains only $\underline{x}, \underline{y}$ as free variables. Now let A be a sentence in the language of PA which can be assumed to be in prenex normal form and assume that

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL} \vdash A.$$

Then a fortiori

$$\text{WE-PA}^\omega\text{+QF-AC+UWKL} \vdash A^H,$$

where A^H is the Herbrand normal form of A. By the conservation result just mentioned we get

$$\text{WE-PA}^\omega \vdash A^H$$

and therefore by [9](theorem 4.1)

$$\text{PA} \vdash A.$$

Remark 3.2 *The passage from the provability of A^H to that of A used in the proof of 2) above does not apply to* WE-PRA$^\omega$ *and* PRA *(see* [9] *for a counterexample). Indeed, already* WE-PRA$^\omega$+QF-AC^{0,0} *is not* Π^0_3*-conservative over* PRA*: the former theory proves the schema of* Σ^0_1*-collection* Σ^0_1-CP*, but it is known that there are instances of* Σ^0_1-CP *(which always can be prenexed as* Π^0_3*-sentences)*[4] *which are unprovable in* PRA *(see* [14]*).*

We now show that the picture changes completely if we consider the systems E-PRA$^\omega$ amd E-PA$^\omega$ with full extensionality instead of WE-PRA$^\omega$, WE-PA$^\omega$. This phenomenon is due to the following

Proposition 3.3
E-PRA$^\omega$ $\vdash$ UWKL $\leftrightarrow \exists\varphi^2\forall f^1(\varphi f =_0 0 \leftrightarrow \exists x^0(fx =_0 0))$.

Proof: 1) '$\rightarrow$': We first show that any Φ satisfying UWKL is – provably in E-PRA$^\omega$ – (effectively) discontinuous[5], i.e.

$$\text{E-PRA}^\omega \vdash \begin{cases} \forall\Phi^{1(1)}(\forall f^1(T^\infty(f) \rightarrow \forall x^0(f((\overline{\Phi f})x) =_0 0)) \rightarrow \\ \qquad \exists g^{1(0)}_{(\cdot)}, g^1(T^\infty(g) \wedge \forall i T^\infty(g_i) \wedge \forall i \forall j \geq i(g_j(i) =_0 g(i)) \\ \qquad\qquad \wedge \forall i, j(\Phi(g_i, 0) = \Phi(g_j, 0) \neq \Phi(g, 0)))) \end{cases}$$

and, moreover, $g_{(\cdot)}, g$ can be computed uniformly in Φ by closed terms of E-PRA$^\omega$. Define g primitive recursively such that

$$g(k) = \begin{cases} 0, \text{ if } \forall m < lth(k)((k)_m = 0) \vee \forall m < lth(k)((k)_m = 1) \\ 1, \text{otherwise.} \end{cases}$$

[4]Here PRA is understood not as a quantifier-free theory but with full first-order predicate logic.
[5]The term 'effectively discontinuous' is due to [7] on which we rely in the second part of our proof.

It is clear that (provably in E-PRA$^\omega$) $T^\infty(g)$. Now let $\Phi^{1(1)}$ be such that

$$\forall f^1(T^\infty(f) \to \forall x(f((\overline{\Phi f})x) =_0 0)).$$

<u>Case 1</u>: $\Phi(g,0)=0$. Define a primitive recursive function $\lambda i,k.g_i(k)$ such that

$$g_i(k) = \begin{cases} 0, \text{ if } [lth(k) \le i \wedge \forall m < lth(k)((k)_m = 0)] \vee [\forall m < lth(k)((k)_m = 1)] \\ 1, \text{otherwise.} \end{cases}$$

Again we easily verify within E-PRA$^\omega$ that $\forall i T^\infty(g_i)$. From the construction of g_i and g it is clear that

$$\forall k \forall l \ge lth(k)(g_l(k) = g(k)).$$

Since our coding has the property that $lth(k) \le k$, we get

$$\forall k \forall l \ge k(g_l(k) = g(k)).$$

Since $\lambda x.1$ is the only infinite path of the binary tree represented by g_i, it follows that

$$\forall i(\Phi(g_i,0) = 1).$$

<u>Case 2</u>: $\Phi(g,0)=1$. The proof is analogous to case 1 with

$$g_i(k) := \begin{cases} 0, \text{ if } [lth(k) \le i \wedge \forall m < lth(k)((k)_m = 1)] \vee [\forall m < lth(k)((k)_m = 0)] \\ 1, \text{otherwise.} \end{cases}$$

This finishes the proof of the discontinuity of Φ. We now show – using an argument from [7] known as 'Grilliot's trick' [6] – that the functional φ^2 defined by $(+) \forall f^1(\varphi f =_0 0 \leftrightarrow \exists x(fx =_0 0))$ can be defined primitive recursively in Φ such that $(+)$ holds provably in E-PRA$^\omega$:
We can construct a closed term $t^{1(1)}$ of E-PRA$^\omega$ such that (provably in E-PRA$^\omega$) we have

$$thi = \begin{cases} g_j(i), \text{ for the least } j < i \text{ such that } h(j) > 0, \text{ if existent} \\ g_i(i), \text{ otherwise.} \end{cases}$$

[6] This argument plays an important role in the context of the Kleene/Kreisel countable functionals, see [13] whose formulation of it we adopt here.

Together with $\forall i \forall j \geq i (g_j(i) = g_i(i))$ this yields

$$\exists j(h(j) > 0) \to th =_1 g_j \text{ for the least such } j$$

and together with $\forall i(g_i(i) = g(i))$

$$\forall j(h(j) = 0) \to th =_1 g.$$

Hence using the extensionality axiom for type-2-functionals we get

$$\forall j(h(j) = 0) \leftrightarrow \Phi(th, 0) =_0 \Phi(g, 0).$$

So $\varphi := \lambda h^1.\overline{sg} \circ |\Phi(t(\overline{sg} \circ h), 0) - \Phi(g, 0)|$ where $\overline{sg}(x) := 0$ for $x \neq 0$ and $= 1$ otherwise, does the job.
We now combine the two constructions of φ corresponding to the two cases above into a single functional which defines φ primitive recursively in Φ: Let χ be a closed term such that

$$\text{E-PRA}^\omega \vdash \forall x^0((x =_0 0 \to \chi x =_{1(1)} t) \wedge (x \neq 0 \to \chi x =_{1(1)} \tilde{t})),$$

where t is defined as above with g_i from case 1 whereas $\tilde{t}$ is defined analogously but with g_i as in case 2. Then define $\varphi := \lambda h^1.\overline{sg} \circ |\Phi((\chi(\Phi(g, 0))(\overline{sg} \circ h), 0) - \Phi(g, 0)|$.
2) '$\leftarrow$': Primitive recursively in φ one can easily compute a functional Φ which even selects the leftmost infinite branch of an infinite binary tree.

Corollary to the proof of proposition 3.3: One can construct closed terms t_1, t_2 of E-PRA$^\omega$ such that

$$\text{E-PRA}^\omega \vdash \begin{cases} \forall \Phi^{1(1)}(\forall f^1(T^\infty(f) \to \forall x^0(f((\overline{\Phi f})x) = 0)) \to \\ \qquad \forall f^1((t_1\Phi)f =_0 0 \leftrightarrow \exists x(fx = 0))) \end{cases}$$

and

$$\text{WE-PRA}^\omega \vdash \begin{cases} \forall \varphi^2(\forall f^1(\varphi f = 0 \leftrightarrow \exists x(fx = 0)) \to \\ \qquad \forall f^1(T^\infty(f) \to \forall x^0(f((\overline{t_2\varphi f})x) = 0))). \end{cases}$$

Corollary 3.4
E-PRA$^\omega$ + QF-AC^{1,0} $\vdash$ UWKL $\leftrightarrow \exists \mu^2 \forall f^1(\exists x^0(fx = 0) \to f(\mu f) = 0)$.

Proof: The existence of μ obviously implies the existence of φ in proposition 3.3 and hence of Φ. For the other direction we only have to observe that the existence of φ implies the existence of μ be applying QF-AC^{1,0} to

$$\forall f \exists x(\varphi(f) = 0 \to fx = 0).$$

Remark 3.5 In contrast to the corollary to the proof of proposition 3.3 above there exists no closed term t in E-PRA$^\omega$ which computes μ in Φ, i.e.

$$\mathcal{S}^\omega \not\models \forall \Phi^{1(1)}(\forall f^1(T^\infty(f) \to \forall n^0(f(\overline{\Phi f})n = 0)) \to \forall f^1(\exists x(fx = 0) \to f(t\Phi f) = 0)$$

for every closed term t (of appropriate type) of E-PRA$^\omega$, since – by [8] – every such term has a majorant t^*, Φ is majorized by $\lambda f^1, x^0.1$ and so μ would have a majorant $\lambda f^M.t^*(1^{1(1)}, f^M)$ (where $f^M(x) := \max(f0, \ldots, fx)$), which contradicts the easy observation that μ has not even a majorant in $\mathcal{S}^\omega$ (here $\mathcal{S}^\omega$ denotes the full set-theoretic type structure).

Theorem 3.6 *1)* E-PRA$^\omega$+UWKL *contains Peano arithmetic* PA*.*

2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL *is conservative over* PA*.*

3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+UWKL *proves the consistency of* PA *and has the same proof-theoretic strength as (and is Π^1_2-conservative over) the second order system* $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$*.*

Proof: 1) Using φ from proposition 3.3 one easily gets characteristic functions for all arithmetical formulas $A(\underline{x})$. By applying the quantifier-free induction axiom of E-PRA$^\omega$ to them, one obtains every arithmetical instance of induction.
2) This follows from corollary 3.4 and the conservation of E-PRA$^\omega$+QF-AC^{1,0}+QF-AC$^{0,1} + \mu$ over PA which is due [4] (note that the usual elimination of extensionality procedure – which applies to the existence of μ but not to UWKL – yields a reduction of E-PRA$^\omega$+QF-AC^{1,0}+QF-AC$^{0,1} + \mu$ to its variant where the extensionality axioms for types > 0 are dropped, see [12] for details on this).
3) follows from [4],[5] using again corollary 3.4 above and elimination of extensionality.

Remark 3.7 *1) The functionals φ and μ from proposition 3.3 and corollary 3.4 provide uniform versions (in the same sense in which* UWKL *is a uniform version of* WKL*) of*

$$(1)\ \Pi^0_1\text{-CA}: \ \forall f \exists g \forall x^0(g(x) =_0 0 \leftrightarrow \exists y^0(f(x,y) =_0 0))$$

respectively of

$$(2)\ \Pi^0_1\text{-}\widehat{\mathrm{CA}}:\ \forall f \exists g \forall x^0, z^0 (f(x,gx) =_0 0 \vee f(x,z) \neq 0),$$

but yet φ, μ *are not stronger than* $(1), (2)$ *relative to* E-PRA$^\omega$ *(but only relative to* E-PA$^\omega$*) as Feferman's results cited in the proof above show. The reason for this is, that* E-PRA$^\omega$ *is too weak to iterate* φ *or* μ *uniformly since this would require a primitive recursion of type level 1. In contrast to this fact,* UWKL *is stronger than* WKL *already relative to* E-PRA$^\omega$*.*

2) One might ask whether UWKL *gets weaker if we allow* $\Phi^{1(1)}$ *to be a partial functional which is required to be defined only on those functions* f *which represent an infinite binary tree. However the construction* ξ *(used in the proof of theorem 3.1) such that*

$$(1)\ \forall f^1 T^\infty(\xi f)$$

and

$$(2)\ \forall f^1 (T^\infty(f) \to \xi f =_1 f)$$

shows that any such partial Φ *could be easily extended to a total one.*

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