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Bistructures, Bidomains and Linear Logic

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Abstract

Bistructures are a generalisation of event structures which allow a representation of spaces of functions at higher types in an order-extensional setting. The partial order of causal dependency is replaced by two orders, one associated with input and the other with output in the behaviour of functions. Bistructures form a categorical model of Girard's classical linear logic in which the involution of linear logic is modelled, roughly speaking, by a reversal of the roles of input and output. The comonad of the model has an associated co-Kleisli category which is closely related to that of Berry's bidomains (both have equivalent non-trivial full sub-cartesian closed categories).

1 Introduction

In this paper we link Winskel's bistructures [25], Girard's linear logic [10] and Berry's bidomains [25]. We show how bistructures provide a model of classical linear logic extending Girard's web model [10, 11]; we show too that a certain class of bistructures represent bidomains. We hope that the structures isolated here will help in the search for a direct, extensional and "mathematically natural" account of sequentiality and thereby of Milner's fully abstract model of PCF [20].

Girard has given an analysis of intuitionistic logic in terms of his more primitive linear logic. When we consider models, this is reflected in the fact that cartesian closed categories (categorical models of intuitionistic logic) arise as the co-Kleisli categories associated with categorical models of linear logic. In particular, linear logic yields refined analyses of the categories of domains used in denotational semantics. For instance, Berry and Curien's

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category of concrete data structures and sequential algorithms [5] may be obtained as the co-Kleisli category of a games model [6, 16]. The connection between games and sequentiality has in turn informed recent work on intensional models of PCF and their fully-abstract extensional collapse [1, 12].

After Berry isolated the mathematically natural notion of stability [3], it was soon realized that sequential functions are stable. While there is a cartesian closed category of stable functions, at higher orders the extensional ordering is not respected. It was therefore natural for Berry to introduce bidomains. These are *biorders*—that is, sets equipped with two partial orders. One is an intensional stable ordering, based on the method of computation; the other is an extensional ordering, inherited from Scott's domain theory. Models of this kind can be viewed as mathematically tractable "approximations" to the desired sequential structures.

Event structures are partial orders of events equipped with a conflict relation and obeying an axiom of finite causes. They were introduced in [21] as a model of concurrency, and turned out to have close connections with concrete domains [14] and hence sequentiality [5]; they are also a natural generalisation of Girard's webs. Winskel introduced bistructures (of events) in [25], representing a full sub-cartesian closed category of bidomains. They are biorders equipped with a binary consistency relation; the two orders are obtained by decomposing the event structure order into left and right (input and output) components.

The main idea of this paper is that the inherent symmetry of bistructures enables one to obtain a model of classical linear logic, generalising the web model. The model is obtained by modifying the original definition—retaining its axiom of finite causes, but with all axioms symmetric. The configurations of a bistructure can be equipped with both a stable and an extensional ordering, that is they are biorders; further, the morphisms of the category of bistructures yield linear functions of the biorders (in a certain sense). Unfortunately not all biorders obtained in this way are bidomains; further not all linear functions come from morphisms of bistructures.

However by considering the co-Kleisli category and then restricting the allowed bistructures, one obtains a category equivalent to a full sub-cartesian closed category of Berry's category of bidomains and which provides a model of PCF. It has to be admitted that the situation here is not entirely as one would like: perhaps the notions of bistructures and bidomains should be adjusted. Ideal would be to have a bidomain model of classical linear logic, with a co-Kleisli category equivalent to that of stable continuous functions, and containing a (full) submodel equivalent to one of bistructures; further, there should be a representation theorem, that the bidomains corresponding to bistructures are precisely those satisfying suitable axioms.

It may be that a natural extensional account of sequentiality can be given within a "bistructural" framework. One can imagine replacing the stable ordering by a structure for sequentiality. If one does not know the right axioms, one could instead look for suitable variants of bistructures of events.

However, Loader's undecidability result [19] for the finitary fragment of PCF shows that there is a major obstacle to finding a category of structured sets providing a fully abstract model of PCF. Such a category cannot be "finitary," in a certain sense.[2] It may nonetheless be possible to find suitable infinitary structure. The work in this paper suggests that one might do well to seek linear models whose co-Kleisli categories correspond to the sequential functions. There may even be enough symmetry that one has a model of classical linear logic.

In Sections 2 and 3 we give two approaches to bistructures; these represent two independent developments of the ideas of this paper [23, 7]. Section 2 starts from the world of webs and stable domain theory; Section 3 proceeds from that of event structures and continuous domain theory. We introduce bistructures in Section 4, and bistructure morphisms in Section 5. In Section 6 we show (Theorem 1) that bistructures provide a model of classical linear logic. In Section 7 we consider bidomains, establishing the connection with bistructures (Theorem 2). In Section 8 we discuss possible variations and connections with other work; in particular we consider strengthenings of bistructures incorporating Ehrhard's hypercoherences (see [8]) thereby accounting for strong stability within our approach.

In this paper, cpos are partial orders with a least element and lubs of all directed sets; continuous functions between cpos are those monotonic functions preserving all the directed lubs. For other domain-theoretic terminology see, for example, [28].

[2] The categories of partial orders and topological spaces are finitary, but structures involving reference to the natural numbers, *e.g.,* measure spaces, are not. The notion can be formalised; one requires that the structures, morphisms and product and function-space functors are given by formulas of higher-order logic referring only to the carrier sets, in such a way that the structured set corresponding to the Booleans has finite carrier, and the function space construction preserves finiteness of the carriers.

2 Motivation from stability

We recall the basics of Girard's stable model of classical linear logic [10, 11]. A *web* is a structure $(E, \smile)$, where:

- E is a set of *events* (or tokens), and
- $\smile$ is a binary irreflexive symmetric relation of *conflict* (called strict incoherence in [10]).

Throughout this paper we use Girard's notation: $\asymp$ is the reflexive closure of the irreflexive relation $\smile$, and $\stackrel{\frown}{\smile}$, the complement of $\smile$, is the reflexive closure of the irreflexive relation $\frown$. It is clear that specifying one relation determines all the others.

The *configurations* (called cliques in [10]) of $(E, \smile)$ are the subsets $x \subseteq E$ which are

- *consistent:* $\forall\, e, e' \in x \;\; e \stackrel{\frown}{\smile} e'$.

Ordered by inclusion, the configurations of E form a cpo $(\Gamma(E), \subseteq)$; as a collection of sets, $\Gamma(E)$ is a *coherence space* in the sense of [10, 11]. The webs form a category, taking the morphisms from E_0 to E_1 to be the stable functions from $\Gamma(E_0)$ to $\Gamma(E_1)$, *i.e.,* those continuous functions f such that whenever $e_1 \in f(x)$ there is a minimum finite $x_0 \subseteq x$ such that $e_1 \in f(x_0)$. In this setting, the stable functions coincide with the conditionally multiplicative functions, *i.e.,* the continuous functions that preserve binary compatible glbs (which are, categorically speaking, pullbacks).

The category is cartesian closed: the function space $E_0 \to E_1$ has as events the pairs (x, e_1) of a finite configuration of E_0 and an event of E_1, with incoherence defined by:

$$(x, e_1) \asymp (y, e_1') \Leftrightarrow (x \uparrow y) \text{ and } (e_1 \asymp e_1')$$

where $x \uparrow y$ means $\exists\, z \;\; x, y \subseteq z$. The configurations of $E_0 \to E_1$ are in 1-1 correspondence with the morphisms from E_0 to E_1, associating to each stable function f its *trace* $\mathrm{tr}(f)$, consisting of those pairs (x, e_1) such that $e_1 \in f(x)$ and $e_1 \notin f(y)$ if $y \subset x$. The inclusion of configurations determines an ordering on stable functions, refining the pointwise ordering and called the *stable ordering* [2].

The definition of $E_0 \to E_1$ is asymmetric in that configurations are paired with events, rather than events with events. This led Girard to two successive decompositions, each of which turned out to have deep logical significance.

- First, $E_0 \to E_1$ can be obtained as $(!E_0) \multimap E_1$, where, for any E, the web $!E$ (the *exponential* of E, pronounced "bang E") has as events the finite configurations of E (with $\smile\, = \uparrow$), and where, for any E_0, E_1, the web $E_0 \multimap E_1$, the *linear function space*, has as events pairs (e_0, e_1) of events of E_0 and events of E_1, with incoherence defined by:

$$(e_0, e_1) \asymp (e_0', e_1') \Leftrightarrow (e_0 \smile e_0') \text{ and } (e_1 \asymp e_1')$$

- Second, the remarkable symmetry between $\smile$ and $\asymp$ in the definition of $E_0 \multimap E_1$ leads to the decomposition $E_0 \multimap E_1 = (E_0^\perp) \wp E_1$, where, for any E, the web $E^\perp$, the *linear negation* of E, has the same events as E, but has as coherence the incoherence of E, and where, for any E_0, E_1, the web $E_0 \wp E_1$ (the *"par"* of E_0 and E_1) has as events the pairs (e_0, e_1) of an event of E_0 and an event of E_1, with incoherence defined by:

$$(e_0, e_1) \asymp (e_0', e_1') \Leftrightarrow (e_0 \asymp e_0') \text{ and } (e_1 \asymp e_1')$$

Returning to the consideration of stable functions, let us see how to describe the pointwise order between stable functions at the level of traces. In $E_0 \to E_1$ there arises a natural ordering between events (x, e_1) if we vary only the input x (whence the superscript ^{L}, for "left"):

$$(x, e_1) \leq^L (y, e_1') \Leftrightarrow (y \subseteq x \text{ and } e_1 = e_1')$$

Now define a partial order $\sqsubseteq$ on $\Gamma(E_0 \to E_1)$ by:

$$\phi \sqsubseteq \psi \Leftrightarrow \forall\, (x, e_1) \in \phi \;\; \exists\, y \subseteq x \;\; (y, e_1) \in \psi$$

or, equivalently:

$$\phi \sqsubseteq \psi \Leftrightarrow \forall\, e \in \phi \;\; \exists\, e' \in \psi \;\; e \leq^L e'$$

Then it is easy to see that for any two stable functions f, g:

$$(\forall\, x \;\; f(x) \subseteq g(x)) \Leftrightarrow \mathrm{tr}(f) \sqsubseteq \mathrm{tr}(g)$$

Since the stable ordering is a refinement of the pointwise ordering, it makes sense to ask whether there exists a sensible "complement" of the stable ordering. Indeed we shall see in Proposition 1 that we can always factor $\phi \sqsubseteq \psi$ uniquely as $\phi \sqsubseteq^L \chi \subseteq \psi$. Here $\phi \sqsubseteq^L \chi$ means that $\phi \sqsubseteq \chi$ and χ is minimal with respect to inclusion (*i.e.,* the stable ordering) among all χ' such that $\phi \sqsubseteq \chi'$; in other words, χ is "the part of ψ showing that $\phi \sqsubseteq \psi$" (notice that, given (x, e_1), the y in the definition of $\phi \sqsubseteq \psi$ is unique).

So far, our discussion has been implicitly carried at first-order types, where we have stable functions that can be ordered in two ways ($\subseteq$ and $\sqsubseteq$). If we next consider second-order types, or functionals, the explicit consideration of both the pointwise and the stable orderings at first-order types leads us to focus on functionals that are not only stable with respect to the stable ordering, but also monotonic with respect to the pointwise ordering. That is, we want to retain only those stable functionals H from $\Gamma(E_0 \to E_1)$ to $\Gamma(E_2)$ such that:

$$\forall\, \phi, \psi \;\; (\phi \sqsubseteq \psi \Rightarrow H(\phi) \subseteq H(\psi))$$

(where we now freely confuse functions with their traces), which, by the $\subseteq$-monotonicity of H and the definition of $\sqsubseteq^L$, can be rephrased as:

$$\forall\, \phi, \psi \;\; (\phi \sqsubseteq^L \psi \Rightarrow H(\phi) \subseteq H(\psi))$$

Now, specialising to finite ϕ and ψ, suppose that $(\phi, e_2) \in H$. Then we must have that $e_2 \in H(\psi)$, *i.e.,* there must exist (ψ_0, e_2) in H such that $\psi_0 \subseteq \psi$. Therefore we ask for the following condition, called the *securedness* condition:

$$\forall\, e \in H \;\; \forall\, e' \;\; (e' \leq^R e \Rightarrow \exists\, e'' \in H \;\; e' \leq^L e'')$$

where the order $\leq^R$ is defined by

$$(\psi, e_2') \leq^R (\phi, e_2) \Leftrightarrow (\phi \sqsubseteq^L \psi \text{ and } e_2' = e_2)$$

To summarise, by going from base types successively to first-order and then to second-order types, we have identified two orderings on events.

- The $\leq^L$ ordering allows us to describe the extensional ordering between traces.
- The securedness condition, which involves both orderings $\leq^L$ and $\leq^R$, allows us to capture the preservation of this extensional ordering by functionals.

This suggests that we consider structures $(E, \leq^L, \leq^R, \frown)$, where $(E, \frown)$ is a web, with the aim of building a cartesian closed category of biordered domains (cf. the introduction), and, as it turns out, a model of classical linear logic.

3 Motivation from continuity

In event structures (which predate Girard's webs), a causal dependency relation inspired from Petri net theory is considered in addition to the conflict relation [21]. In full, an event structure is a structure $(E, \leq, \smile)$ where [3] :

- E is a set of *events*,
- $\leq$ is a partial order of *causal dependency*, and
- $\smile$ is a binary, irreflexive, symmetric relation of *conflict*.

The *configurations* (or *states*) of such an event structure are those subsets $x \subseteq E$ which are:

- *consistent*: $\forall\, e, e' \in x \;\; e \frown e'$, and
- *left closed*: $\forall\, e, e' \in E \;\; e' \leq e \in x \Rightarrow e' \in x$.

Ordered by inclusion, the configurations form a coherent prime algebraic domain $(\Gamma(E), \subseteq)$ [21]; such domains are precisely the infinitely distributive, coherent Scott domains [27]. An instance of the causal dependency ordering $e' \leq e$ when e and e' are distinct, is understood as meaning that the event e causally depends on the event e', in that the event e can only occur after e' has occurred. Given this understanding it is reasonable to impose a finiteness axiom, expressing that an event has finite causes:

$\{e' \mid e' \leq e\}$ is finite, for all events e.

The event structures satisfying this axiom yield the dI-domains [2] which are coherent, and therefore lead to a cartesian closed category of stably ordered stable functions. (See [26] where an alternative description of event structures

[3]In [21], an axiom relating causal dependency and conflict is imposed; however it is inessential in that it does not affect the class of domains represented.

using an enabling relation instead of an ordering on events is used to give a simple description of the function space construction.)

But event structures can also be used to describe a continuous model of intuitionistic linear logic, equivalent to the category of coherent prime algebraic domains, with completely additive functions (*i.e.,* functions preserving arbitrary lubs—just called "additive" below). We take as objects event structures (but without the axiom of finite causes: this is the price to pay), and as morphisms configurations of a "function space" of event structures. Let $E_i = (E_i, \leq_i, \smile_i)$, $i = 0, 1$, be event structures. Define:

$$
\begin{array}{rl}
& E_0 \multimap E_1 = (E_0 \times E_1, \leq, \smile) \\
\text{where} & (e_0, e_1) \leq (e'_0, e'_1) \Leftrightarrow e'_0 \leq_0 e_0 \text{ and } e_1 \leq_1 e'_1, \\
\text{and} & (e_0, e_1) \smile (e'_0, e'_1) \Leftrightarrow e_0 \stackrel{\frown}{\smile}_0 e'_0 \text{ and } e_1 \smile_1 e'_1.
\end{array}
$$

The configurations of $E_0 \multimap E_1$ are in 1-1 correspondence with the additive functions from $\Gamma(E_0)$ to $\Gamma(E_1)$—additive functions are determined by their action on complete primes[4] which correspond to events. The configuration associated with an additive function f is its *graph*, consisting of those pairs (e_0, e_1) such that $e_1 \in f(\{e'_0 \mid e'_0 \leq e_0\})$.

The inclusion ordering on configurations reflects the pointwise ordering on functions; in particular, the function events (e_0, e_1) correspond to the prime additive one-step functions (see [31]); and the order $\leq$ to the pointwise order between them.

A morphism $E_0 \to E_1$ is defined to be a configuration of $E_0 \multimap E_1$. As such it is a relation between the events of E_0 and E_1. Composition in the category is that of relations. The category is a model of intuitionistic linear logic, as defined in [24, 4]. For instance, its tensor is given in a coordinatewise fashion. For event structures $E_i = (E_i, \leq_i, \smile_i)$, for $i = 0, 1$, define:

$$
\begin{array}{rl}
& E_0 \otimes E_1 = (E_0 \times E_1, \leq, \smile) \\
\text{where} & (e_0, e_1) \leq (e'_0, e'_1) \Leftrightarrow e_0 \leq_0 e'_0 \text{ and } e_1 \leq_1 e'_1, \\
\text{and} & (e_0, e_1) \stackrel{\frown}{\smile} (e'_0, e'_1) \Leftrightarrow e_0 \stackrel{\frown}{\smile}_0 e'_0 \text{ and } e_1 \stackrel{\frown}{\smile}_1 e'_1.
\end{array}
$$

[4]A complete prime of a Scott domain $(D, \sqsubseteq)$ is an element p for which whenever X is bounded above and $p \sqsubseteq \bigsqcup X$ then $p \sqsubseteq x$ for some x in X. Complete primes are *a fortiori* compact, where the definition of compact is obtained by replacing "X is bounded above" by "X is directed".

Monoidal-closure follows from the isomorphism

$$(E_0 \otimes E_1 \multimap E_2) \cong (E_0 \multimap (E_1 \multimap E_2))$$

natural in E_0 and E_2. Product and coproduct are obtained by disjoint juxtaposition of event structures, extending conflict across the two event sets in the case of coproduct. The comonad operation is:

$$!E = (\Gamma(E)^0, \subseteq, \smile)$$

for an event structure E, with events the *compact* configurations $\Gamma(E)^0$, and where $\smile$ stands for incompatibility with respect to inclusion. The continuous functions $\Gamma(E_0) \to \Gamma(E_1)$, between configurations of event structures E_0, E_1, are in 1-1 correspondence with the configurations of $!E_0 \multimap E_1$.

Notice that this does not yield a model of classical linear logic. The reader should compare the asymmetric definition of conflict in $E_0 \multimap E_1$ given above to capture continuity with the symmetric definition of incoherence in the stable framework (cf. Section 2).

Moreover, in this model of intuitionistic linear logic, all hope of considering the order $\leq$ as causal dependency is lost. The difficulty stems from the definition of the order $\leq$ for $(E_0 \multimap E_1)$. Its events are ordered by:

$$(e_0, e_1) \leq (e_0', e_1') \;\Leftrightarrow\; e_0' \leq_0 e_0 \;\text{ and }\; e_1 \leq_1 e_1'$$

The reversal in the $\leq_0$ order can lead to $\leq$ violating the axiom of finite causes, even though $\leq_0$ and $\leq_1$ do not: an infinite, ascending chain of events in E_0 can give rise to an infinite, *descending* chain in $E_0 \multimap E_1$. Of course, there is no reason why the extensional ordering on functions should be a relation of causal dependency, so it was not to be expected that its restriction to step functions should be finitary.

However, if we factor $\leq$ into two orderings, one associated with input (on the left) and one with output (on the right), we can expose two finitary orderings. Define

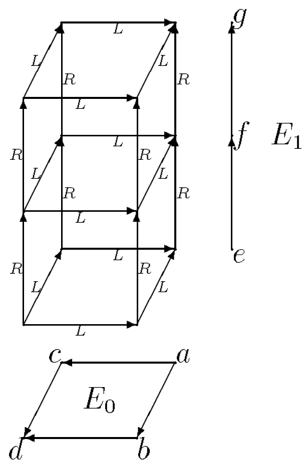
$$(e_0, e_1) \leq^L (e_0', e_1') \Leftrightarrow e_0' \leq_0 e_0 \;\text{ and }\; e_1 = e_1',$$
$$(e_0, e_1) \leq^R (e_0', e_1') \Leftrightarrow e_0' = e_0 \;\text{ and }\; e_1 \leq_1 e_1'.$$

Then, it is clear that $\leq$ factors as

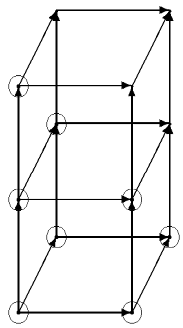
$$(e_0, e_1) \leq (e_0', e_1') \Leftrightarrow (e_0, e_1) \leq^L (e_0', e_1) \;\text{ and }\; (e_0', e_1) \leq^R (e_0', e_1'),$$

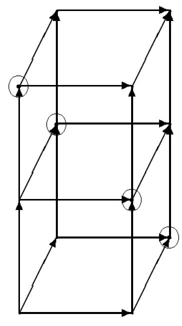
and that this factorisation is unique. Provided the orderings of E_0 and E_1 are finitary, then so are $\leq^R$ and $\geq^L$. This factorisation is the first step towards the definition of bistructures. To indicate its potential, and to further motivate bistructures, we study a simple example.

Let E_0 and E_1 be the event structures shown below. Both have empty conflict relations. Taking advantage of the factorisation we have drawn them alongside the additive function space $E_0 \multimap E_1$.

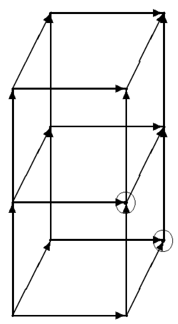


The conflict relation of $E_0 \multimap E_1$ is empty. So here an additive function from $\Gamma(E_0)$ to $\Gamma(E_1)$ is represented by a $\leq$-downwards-closed subset of events of $E_0 \multimap E_1$. For instance, the events in the diagram (below left) are associated with the function that outputs e on getting input event a, outputs f for input b or c, and outputs g for input d. The extensional ordering on functions corresponds to inclusion on $\leq$-downwards-closed subsets of events. It is clear that such a function is determined by specifying the minimal input events which yield some specific output (shown in the diagram below right). This amounts to the subset of $\leq^L$-maximal events of the function, and we can call this subset the *trace* of the function. Notice, though, that this particular function is not stable; output f can be obtained for two non-conflicting but distinct events b and c. A stable function should not have $\leq^L$-downwards compatible distinct events in its trace.

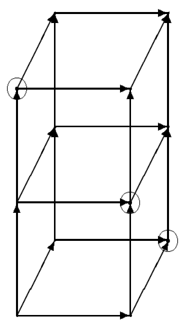




For stable functions, the stable ordering is obtained as inclusion of traces. For example:



is stable below



Notice that traces ϕ of additive functions from from $\Gamma(E_0)$ to $\Gamma(E_1)$ are secured, in the sense that:

$$(e \in \phi \text{ and } e' \leq^R e) \Rightarrow (\exists\, e'' \in \phi \;\; e' \leq^L e'')$$

or more concretely:

$$((e_0, e_1) \in \phi \text{ and } e_1' \leq e_1) \Rightarrow (\exists\, e_0' \;\; (e_0', e_1') \in \phi \text{ and } e_0' \leq e_0)$$

This is the same securedness condition that appeared in Section 2. Here we can understand the condition as saying that for any output, lesser output must arise through the same or lesser input.

Let us summarise this discussion.

- The graphs of additive functions are the $\leq$-downwards-closed, consistent subsets of events.
- The extensional order corresponds to inclusion of graphs.
- The traces of functions are the sets of $\leq^L$-maximal events of their graphs.
- The stable order corresponds to inclusion of traces.

These observations, based on the continuous model construction, will, as it turns out, also make sense in a biordered framework. They encourage us to consider bistructures $(E, \leq^L, \leq^R, \frown)$ and provide guidance as to which axioms we should impose on $\leq^L$,$\leq^R$, and $\frown$. One expects a function-space construction that maintains both stable and extensional orderings, corresponding to taking as morphisms those functions which are continuous with respect to the extensional ordering and stable with respect to the stable ordering.

We end the section with a remark. One might wonder why we have explicitly considered an ordering $\leq$ on events to describe a cartesian closed category of continuous functions, while webs suffice for the purpose of building a cartesian closed category of stable functions. The reason is that the treatment of stability is based on traces of functions, while the treatment of continuity is based on their graphs. Graphs of continuous functions [5] are upwards closed in their first component, even if the underlying event structure has a trivial partial order, and we need an order relation on events to capture that fact.

4 Bistructures

The following definition of bistructures allows us to fulfill the hopes expressed in the previous sections.

Definition 1 *A* (countable) bistructure *is a structure* $(E, \leq^L, \leq^R, \frown)$ *where* E *is a countable set of* events, $\leq^L, \leq^R$ *are partial orders on* E *and* $\frown$ *is a binary reflexive, symmetric relation on* E *such that:*

[5] The *graph* of a continuous function f from $\Gamma(E_0)$ to $\Gamma(E_1)$ consists of all pairs (x, e_1) with x compact such that $e_1 \in f(x)$.

1. *defining* $\leq = (\leq^L \cup \leq^R)^\star$, *we have the following factorisation property:*

$$e \leq e' \Rightarrow \exists\, e'' \;\; e \;\; \leq^L e'' \leq^R e'$$

2. *defining* $\preceq = (\geq^L \cup \leq^R)^\star$,

   (a) $\preceq$ *is finitary, i.e.,* $\{e' \mid \; e' \preceq \; e\}$ *is finite, for all* e,

   (b) $\preceq$ *is a partial order;*

3. (a) $\downarrow^L \subseteq \asymp$ $\qquad$ (b) $\uparrow^R \subseteq \overset{\frown}{\smile}$

   The two compatibility relations are defined by:

$$\begin{array}{lcl} e \downarrow^L e' & \Leftrightarrow & \exists\, e'' \;\; e'' \leq^L e \;\; \text{and } e'' \leq^L e', \\ e \uparrow^R e' & \Leftrightarrow & \exists\, e'' \;\; e \leq^R e'' \;\; \text{and } e' \leq^R e''. \end{array}$$

Notice the symmetry of the axioms. They are invariant under the "duality":

$$\begin{array}{lcl} \leq^L & \mapsto & \geq^R, \\ \leq^R & \mapsto & \geq^L, \\ \overset{\frown}{\smile} & \mapsto & \asymp \end{array}$$

which is why we obtain a model of the *classical* logic. Bistructures of the form $(E, id_E, \leq, \overset{\frown}{\smile})$, *i.e.,* such that the $\leq^L$ order is degenerate, are essentially the ordinary, countable event structures, $(E, \leq, \smile)$, satisfying the axiom of finite causes. We say "essentially" because Axiom 3(b) is not part of the above definition of event structure, but does not restrict the class of domains represented.

Remark 1 *In the presence of Axiom 2(a), Axiom 2(b) is equivalent to requiring that* $e \prec e'$ *is well-founded, where* $e \prec e'$ *means* $e \preceq e'$ *and* $e \neq e'$.

The axioms of bistructures are strong enough to imply the uniqueness of the decomposition of $\leq = (\leq^L \cup \leq^R)^\star$, and also that $\leq$ is a partial order.

Lemma 1 *Let* E *be a bistructure. The following properties hold, for all* e, e' *in* E:

$$\begin{array}{lcl} (e \downarrow^L e' \text{ and } e \uparrow^R e') & \Rightarrow & e = e', \\ e \leq e' & \Rightarrow & \exists !\, e'' \;\; e \leq^L e'' \leq^R e' \end{array}$$

PROOF. (1) If $e \downarrow^L e'$ and $e \uparrow^R e'$, then $e \asymp e'$ and $e \smile e'$, which implies $e = e'$ by definition of $\asymp$.

(2) Suppose that $e \leq^L e'' \leq^R e'$ and $e \leq^L e''' \leq^R e'$. Then $e'' \downarrow^L e'''$ and $e'' \uparrow^R e'''$, therefore $e'' = e'''$ by (1). $\square$

The unique factorisation property enables a diagrammatic style of proof.

Lemma 2 *The relation $\leq$ defined in Axiom 1 of bistructures is a partial order.*

PROOF. The relation $\leq$ is certainly reflexive and transitive. With the aim of proving antisymmetry, suppose $e \leq e'$ and $e' \leq e$. Then pictorially by factorising $\leq$, for some events ε and ε', we have:

$$\begin{tikzcd}
\varepsilon \arrow[d, "R"'] & e' \arrow[l, "L"'] \\
e \arrow[r, "L"] & \varepsilon' \arrow[u, "R"']
\end{tikzcd}$$

From

$$\begin{tikzcd}
\varepsilon & e' \arrow[l, "L"'] \\
e \arrow[r, "L"] & \varepsilon' \arrow[u, "R"']
\end{tikzcd}$$

we know $e \leq \varepsilon$. Thus by factorising $e \leq \varepsilon$ we get $e \leq^L \varepsilon'' \leq^R e$, for some ε'':

$$\begin{tikzcd}
 & \varepsilon \arrow[d, "R"] \\
\varepsilon'' \arrow[ur, "R"] & e \arrow[l, "L"]
\end{tikzcd}$$

But $e \leq^L e \leq^R e$ so the uniqueness of factorisation gives $e = \varepsilon''$. Then as $\leq^R$ is a partial order $e = \varepsilon$. Therefore the first picture collapses to:

$$\begin{tikzcd}
 & e' \arrow[dl, "L"'] \\
e \arrow[r, "L"] & \varepsilon' \arrow[u, "R"']
\end{tikzcd}$$

The uniqueness of the factorisation of $e' \leq e'$ gives $\varepsilon' = e'$, so as $\leq^L$ is a partial order, $e = e'$, as required. $\Box$

As with Girard's webs, bistructures provide a concrete level of description of abstract points, or configurations, which we define next.

Definition 2 *A* configuration *of a bistructure* $(E, \leq^L, \leq^R, \frown)$ *is a subset* $x \subseteq E$ *which is:*

- consistent: $\forall\, e, e' \in x \;\; e \frown e'$*, and*
- secured: $\forall\, e \in x \;\; \forall\, e' \leq^R e \;\; \exists\, e'' \in x \;\; e' \leq^L e''$*.*

[Notice that e'' is unique in any consistent set because of Axiom 3(a) on bistructures.] Write $\Gamma(E)$ for the set of configurations of a bistructure E, and $\Gamma(E)^0$ for the set of finite configurations (see Proposition 2).

When $\leq^L = id$, the securedness condition amounts to $\leq^R$-downwards-closure, hence in that case configurations are just the configurations of the underlying event structure $(E, \leq^R, \smile)$.

Definition 3 *Let E be a bistructure. We define the* stable *ordering $\sqsubseteq^R$ and the* extensional *ordering $\sqsubseteq$ on configurations by:*

$$\begin{array}{l} \sqsubseteq^R \textit{ is set-theoretic inclusion,} \\ x \sqsubseteq y \Leftrightarrow \forall\, e \in x \;\; \exists\, e' \in y \;\; e \leq^L e' \end{array}$$

It follows from these definitions and from the reflexivity of $\leq^L$ that $\sqsubseteq^R$ is included in $\sqsubseteq$. We define a third relation $\sqsubseteq^L$ as follows:

$$x \sqsubseteq^L y \Leftrightarrow x \sqsubseteq y \;\; \textit{and} \;\; (\forall\, z \in \Gamma(E) \;\; (x \sqsubseteq z \;\; \textit{and} \; z \sqsubseteq^R y) \Rightarrow y = z)$$

Thus, $x \sqsubseteq^L y$ means that y is a $\sqsubseteq^R$-minimal configuration such that $x \sqsubseteq y$. Write $x \uparrow^R y$ for $(\exists\, z \in \Gamma(E) \;\; x, y \sqsubseteq^R z)$.

Thus bistructures E yield biorders $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$; some of their properties are examined in the rest of this section. In particular, Proposition 1 concerns factorisation—preparing the ground for the definition of exponentials in Section 6, while Propositions 2 and 3 concern finiteness and completeness properties.

Lemma 3 *Let E be a bistructure, and suppose that x, y are in $\Gamma(E)$. If $x \uparrow^R y$, $e \in x$, $e' \in y$, and $e \downarrow^L e'$, then $e = e'$.*

PROOF. Let z be such that $x \sqsubseteq^R z$ and $y \sqsubseteq^R z$. Then we have that $e \in z$ and $e' \in z$, hence $e \frown e'$. On the other hand, by Axiom 3(a) we find that $e \asymp e'$, hence $e = e'$. $\square$

Lemma 3 has two interesting consequences.

- If x is a configuration and $e \in x$, then e is $\leq^L$-maximal in x.
- If $x \uparrow^R y$, then the set intersection $x \cap y$ is the glb of x and y with respect to both $\sqsubseteq^R$ and $\sqsubseteq$.

Lemma 4 *Let E be a bistructure, and suppose that $x \in \Gamma(E)$. If e is in the $\leq$-downwards-closure of x, then it is in the $\leq^L$-downwards-closure of x.*

PROOF. Let e' in x be such that $e \leq e'$:

$$\begin{array}{ll} \exists\, e'' \;\; e \leq^L e'' \leq^R e' & \text{by factorisation,} \\ \exists\, e''' \in x \;\; e'' \leq^L e''' & \text{by securedness} \end{array}$$

Then $e \leq^L e'''$, which completes the proof. $\square$

It follows from Lemma 4 that the relation $\sqsubseteq$ is equivalently defined by stating that the $\leq$-downwards-closure of x is included in the $\leq$-downwards-closure of y. This characterisation is in accordance with the discussion in Section 3: compare $\sqsubseteq^R$ and $\sqsubseteq$ with graph and trace inclusion, respectively.

Definition 4 *For x in $\Gamma(E)$, we define the relativised relation $\preceq_x$ as the reflexive, transitive closure of $\preceq_x^1$ where:*

$$e \preceq_x^1 e' \Leftrightarrow_{def} e \in x \textit{ and } e' \in x \textit{ and } \; \exists\, e'' \;\; e \geq^L e'' \leq^R e'.$$

Lemma 5 *Let E be a bistructure. The following property holds, for all x, y in $\Gamma(E)$:*

$$(x \uparrow^R y \;\; \textit{and} \;\; e \in x \cap y) \;\Rightarrow\; (\forall\, e' \in E \;\; e' \preceq_x e \Leftrightarrow e' \preceq_y e)$$

PROOF. It is clearly enough to show this for the one step relations $\preceq_x^1$ and $\preceq_y^1$. Suppose $e' \preceq_x^1 e$, and let e'' be such that $e' \geq^L e'' \leq^R e$. Since y is secured, and since $e \in y$, there exists e''' in y such that $e'' \leq^L e'''$. By Lemma 3 applied to x, y, e', and e''', we get $e' = e'''$. Hence $e' = e''' \preceq_y e$. □

Proposition 1 *Let $\sqsubseteq^R, \sqsubseteq$, and $\sqsubseteq^L$ be the relations on configurations defined above. The following properties hold:*

1. *$\sqsubseteq$ is $(\sqsubseteq^L \cup \sqsubseteq^R)^\star$, and satisfies Axiom 1,*

2. *for all configurations x, y:*

$$x \sqsubseteq^L y \Leftrightarrow (x \sqsubseteq y \text{ and } \forall\, e \in y \ \exists\, e_0 \in x, e_1 \in y \ \ e \preceq_y e_1 \geq^L e_0)$$

3. *$\sqsubseteq^L$ is a partial order, and*

4. *for all configurations x, y with $x \sqsubseteq y$ there is a unique z such that $x \sqsubseteq^L z \sqsubseteq^R y$.*

PROOF. (1) Suppose $x \sqsubseteq y$. The subset $\{e_1 \in y \ \mid \exists\, e_0 \in x \ \ e_0 \leq^L e_1\}$ represents the part of y actually used to check $x \sqsubseteq y$. But we have to close this subset to make it secured. Thus define:

$$y_1 = \{e \in y \ \mid \exists\, e_0 \in x, e_1 \in y \ \ e \preceq_y e_1 \geq^L e_0\}$$

By construction, this set satisfies the following property: If $e \in y_1$ and $e' \preceq_y e$, then $e' \in y_1$. We show that y_1 is a configuration. It is clearly consistent, as it is a subset of y. If $e \in y_1$ and $e_1 \leq^R e$, since y is secured, there exists e_2 in y such that $e_1 \leq^L e_2$, and $e_2 \in y_1$ by construction. Thus y_1 is a configuration. We show $x \sqsubseteq^L y_1$. Suppose that $x \sqsubseteq y_1' \sqsubseteq^R y_1$, and let e be an element of y_1. By the construction of y_1, there are e_0 in x and e_1 in y such that $e \preceq_y e_1 \geq^L e_0$. Since $x \sqsubseteq y_1'$, $e_0 \leq^L e_1'$ for some e_1' in y_1'. Applying Lemma 3 to y, y_1', e_1, and e_1', we get $e_1 = e_1'$, hence $e_1 \in y_1'$, which implies $e \in y_1'$ by Lemma 5. Therefore $y_1 \sqsubseteq^R y_1'$, which completes the proof that $x \sqsubseteq^L y_1$. The decomposition $x \sqsubseteq^L y_1 \sqsubseteq^R y$ shows that $\sqsubseteq$ is contained in $(\sqsubseteq^L \cup \sqsubseteq^R)^\star$. The converse inclusion is obvious.

(2) follows immediately from the proof of (1).

(3) Reflexivity and antisymmetry follow from the inclusion of $\sqsubseteq^L$ in $\sqsubseteq$. Suppose that $x \sqsubseteq^L y' \sqsubseteq^L y$. Clearly $x \sqsubseteq y$, so with an eye to using (2) to show $x \sqsubseteq^L y$ suppose $e \in y$. By (2), there exist e_0 in y', e_1 in y, e'_0 in x and e'_1 in y' such that

$$e_0 \leq^L e_1 \text{ and } e \preceq_y e_1,$$
$$e'_0 \leq^L e'_1 \text{ and } e_0 \preceq_{y'} e'_1.$$

Or in full:

$$\begin{array}{ll} e_0 \leq^L e_1 \geq^R e_2 \cdots \leq^L e_{2i+1} = e & \text{with } e_{2j+1} \in y \text{ for all } 0 \leq j \leq i, \\ e'_0 \leq^L e'_1 \geq^R e'_2 \cdots \leq^L e'_{2i'+1} = e_0 & \text{with } e'_{2j+1} \in y' \text{ for all } 0 \leq j \leq i' \end{array}$$

Since $y' \sqsubseteq y$ and $e'_1 \in y'$, there exists e''_1 such that $e'_1 \leq^L e''_1$ and $e''_1 \in y$. Since $e'_2 \leq^R e'_1 \leq^L e''_1$, there exists e''_2 such that $e'_2 \leq^L e''_2 \leq^R e''_1$, by factorisation. Since y is secured, there exists e''_3 in y such that $e''_2 \leq^L e''_3$. In order to continue this lifting of the e'_i relative to y' to a sequence of the e''_i relative to y, we have to make sure that $e'_3 \leq^L e''_3$: But

$$e'_3 \leq^L e'''_3 \in y \text{ for some } e'''_3 \in y \text{ since } y' \sqsubseteq y, \text{ and}$$
$$e'''_3 = e''_3 \text{ since } e'_2 \leq^L e''_3, e'_2 \leq^L e'''_3, \text{ and } e''_3, e'''_3 \in y$$

Continuing in this way, we get:

$$e'_0 \leq^L e''_1 \geq^R e''_2 \cdots \leq^L e''_{2i'+1} = e_1 \geq^R e_2 \cdots \leq^L e_{2i+1} = e$$

where $e''_{2i'+1} = e_1$ follows from Lemma 3 applied to y, y, $e''_{2i'+1}$, and e_1. Since $e_{2j+1} \in y$ for all $0 \leq j \leq i$ and $e''_{2j+1} \in y$ for all $0 \leq j \leq i'$, by (2), we conclude that $x \sqsubseteq^L y$.

(4) Only the uniqueness of z is in question, so suppose that $x \sqsubseteq^L z \sqsubseteq^R y$ and $x \sqsubseteq^L z' \sqsubseteq^R y$. By symmetry, it is enough to show that $z \subseteq z'$. So suppose that $e \in z$. Then by (2) there are e_0 in x and e_1 in z such that $e \preceq_z e_1 \sqsupseteq^L e_0$. We begin by showing that $e_1 \in z'$. Since $e_0 \in x \sqsubseteq z'$ there is an e'_1 in z' such that $e_0 \sqsubseteq^L e'_1$. So $e_1 \downarrow^L e'_1$ and therefore, by Lemma 3 applied to z, z', e_1, e'_1, we have $e_1 = e'_1 \in z'$.

We now consider the chain:

$$e = e_n \preceq^1_z \ldots \preceq^1_z e_2 \preceq^1_z e_1$$

and show by induction on j $(1 \leq j \leq n)$ that $e_j \in z'$. This holds for $j = 1$. Suppose it holds for $j < n$. Then $e_{j+1} \geq^L e'' \leq^R e_j$ for some e''. Then as z'

is a configuration, and $e_j \in z$, we have that $e'' \leq^L e'_{j+1}$ for some e'_{j+1} in z'. But then $e_{j+1} = e'_{j+1} \in z'$ follows as before, concluding the proof.

□

Definition 5 *For x in $\Gamma(E)$ and e in x, define*

$$[e]_x = \{e_0 \in x \ \mid e_0 \preceq_x e\}$$

Lemma 6 *Let E be a bistructure, and suppose that $e \in x \in \Gamma(E)$. Then $[e]_x$ is a finite configuration such that:*

$$e \in [e]_x \sqsubseteq^R x \quad \textit{and} \quad (\forall\, y \in \Gamma(E)\ \ (e \in y \uparrow^R x) \Rightarrow ([e]_x \sqsubseteq^R y))$$

Proof. The set $[e]_x$ is clearly a configuration (cf. the proof of Proposition 1). The finiteness of $[e]_x$ follows from Axiom 2(a). The rest of the statement is an immediate consequence of Lemma 5. □

Remark 2 *For any e the following "canonical" set*

$$[e] = \{e' \ \mid e' \leq^R e\}$$

is a configuration containing e, and if x is any other such, then $[e] \sqsubseteq x$. In contrast, there need be no $\subseteq$-least configuration containing a given e (cf. Lemma 6).

Proposition 2 *Let E be a bistructure. The following properties hold of the associated biorder:*

1. *all $\sqsubseteq$-directed lubs and $\sqsubseteq^R$-bounded lubs exist,*
2. *all $\sqsubseteq^R$-lubs of $\sqsubseteq^R$-directed sets exist, coinciding with their $\sqsubseteq$-lubs, and*
3. *a configuration is $\sqsubseteq$-compact iff it is $\sqsubseteq^R$-compact iff it is finite.*

It follows that $(\Gamma(E), \sqsubseteq^R)$ and $(\Gamma(E), \sqsubseteq)$ are ω-algebraic cpos (with common least element $\emptyset$), and that, moreover, $(\Gamma(E), \sqsubseteq^R)$ is bounded complete, i.e., is a Scott domain.

PROOF. (1) Let Δ be $\sqsubseteq$-directed. We show:

$z = \{e \in \bigcup \Delta \ \mid e$ is $\leq^L$-maximal in $\bigcup \Delta\}$ is the $\sqsubseteq$-lub of Δ

We first check that z is a configuration. If $e_1, e_2 \in z$, then $e_1 \in \delta_1$ and $e_2 \in \delta_2$ for some δ_1, δ_2 in Δ. Let δ in Δ be such that $\delta_1, \delta_2 \sqsubseteq \delta$. Then by definition of z and $\sqsubseteq$, it follows that $e_1, e_2 \in \delta$. Therefore $e_1 \frown e_2$. If $e \in z$ and $e_1 \leq^R e$, let δ in Δ be such that $e \in \delta$. Since δ is secured, there exists e_2 in δ such that $e_1 \leq^L e_2$. By definition of z and by Axiom 2 (cf. Remark 1), we can find e_3 in z such that $e_2 \leq^L e_3$. Hence z is indeed a configuration. It is obvious from the definition of z that $\delta \sqsubseteq z$ holds for any δ in Δ, and that if z_1 is an $\sqsubseteq$-upper bound of Δ then $z \sqsubseteq z_1$.

The $\sqsubseteq^R$-bounded lubs exist: if $X \subseteq \Gamma(E)$ and if x is an $\sqsubseteq^R$-upper bound of X, then $\bigcup X$ is consistent as a subset of x and secured as a union of secured sets of events.

(2) Let Δ be $\sqsubseteq^R$-directed (and hence *a fortiori* $\sqsubseteq$-directed). Clearly $\bigsqcup^R \Delta$ exists and is $\bigcup \Delta$; we prove that $\bigsqcup \Delta = \bigcup \Delta$ (where $\bigsqcup^R$ and $\bigsqcup$ are relative to $\sqsubseteq^R$ and $\sqsubseteq$, respectively). We have to show that any e in $\bigcup \Delta$ is $\leq^L$-maximal. Suppose there exists e_1 in $\bigcup \Delta$ such that $e \leq^L e_1$. Then, applying Lemma 3 to $\bigcup \Delta$, $\bigcup \Delta$, e, and e_1, we get $e = e_1$.

(3) We prove three implications.

- x finite $\Rightarrow$ x $\sqsubseteq$-compact: Let $\{e_1, \ldots, e_n\} \sqsubseteq \bigsqcup \Delta$. There exist $e'_1, \ldots, e'_n$ in $\bigsqcup \Delta$ such that $e_i \leq^L e'_i$ for all i. Let $\delta_1, \ldots \delta_n$ in Δ be such that $e'_i \in \delta_i$ for all i, and let δ in Δ be such that $\delta_i \sqsubseteq \delta$ for all i. Then by the $\leq^L$-maximality of $e'_1, \ldots, e'_n$ we get that $e'_i \in \delta$ for all i. Hence $\{e_1, \ldots, e_n\} \sqsubseteq \delta$.

- x $\sqsubseteq$-compact $\Rightarrow$ x $\sqsubseteq^R$-compact: If $x \sqsubseteq^R \bigsqcup^R \Delta$, then, *a fortiori*, $x \sqsubseteq \bigsqcup \Delta$, therefore $x \sqsubseteq \delta$ for some δ in Δ. We show that actually $x \sqsubseteq^R \delta$ holds. Suppose $e \in x$ and let e_1 in δ be such that $e \leq^L e_1$. Then we get $e = e_1$ by Lemma 3 applied to x, δ, e, and e_1.

- x $\sqsubseteq^R$-compact $\Rightarrow$ x finite : We claim that, for any z:

$$\{y \ \mid y \text{ finite and } y \sqsubseteq^R z\}$$

  is $\sqsubseteq^R$-directed and has z as lub. The directedness is obvious. We have to check that $z \sqsubseteq^R \bigsqcup^R \{y \ \mid y$ finite and $y \sqsubseteq^R z\}$, *i.e.,* for all e in z, there exists a finite y such that $y \sqsubseteq^R z$ and $e \in y$. The configuration $[e]_z$ (cf. Lemma 6) does the job. $\square$

Proposition 3 *Let E be a bistructure. Then the following properties hold:*

1. *$(\Gamma(E), \sqsubseteq^R)$ is a dI-domain, and*
2. *the complete primes of $(\Gamma(E), \sqsubseteq^R)$ are the configurations of the form $[e]_x$.*

PROOF. (1) A dI-domain is a Scott domain which is distributive (see Definition 7) and satisfies Axiom I, which states that a compact element dominates finitely many elements. Axiom I follows from the finiteness of compacts, proved in Proposition 2. Distributivity is then equivalent to prime-algebraicity, *i.e.,* the property that any element is the lub of the complete primes that it dominates. (We refer to [31, 27] for a proof.) Prime-algebraicity is an immediate consequence of (2).

(2) Consider a configuration $[e]_x$. We show it is a complete prime. If Y is bounded above and $[e]_x \sqsubseteq^R \bigsqcup^R Y = \bigcup Y$, then $e \in y$ for some y in Y. Since $[e]_x \uparrow^R y$, we infer that $[e]_x \subseteq y$, by Lemma 5. Conversely, every complete prime is of this form, since for any configuration x we have $x = \bigcup\{[e]_x \mid e \in x\}$. $\square$

The properties proved in Proposition 2 and Proposition 3 correspond to the most interesting structure of Berry's bidomains. However, to show its configurations form a bidomain we will require a bistructure to fulfill extra axioms; these assure the existence of enough meets. We pursue these matters in Section 7.

5 A category of bistructures

Morphisms between bistructures correspond to configurations of the function-space construction given below. They determine (certain—see Remark 3) extensional, linear (= stable and additive) functions on domains of configurations. Given bistructures $E_i = (E_i, \leq_i^L, \leq_i^R, \smile_i)$, for $i = 0, 1$, their *linear function space* is defined by:

$$E_0 \multimap E_1 = (E_0 \times E_1, \leq^L, \leq^R, \smile)$$

$$
\begin{array}{ll}
\text{where} & (e_0, e_1) \leq^L (e'_0, e'_1) \Leftrightarrow e'_0 \leq^R e_0 \;\text{ and }\; e_1 \leq^L e'_1, \\
 & (e_0, e_1) \leq^R (e'_0, e'_1) \Leftrightarrow e'_0 \leq^L e_0 \;\text{ and }\; e_1 \leq^R_1 e'_1 \\
 & \\
\text{and} & (e_0, e_1) \asymp (e'_0, e'_1) \Leftrightarrow e_0 \smile_0 e'_0 \;\text{ and }\; e_1 \asymp_1 e'_1.
\end{array}
$$

It is straightforward to show that this is a bistructure. We define the category of bistructures **BS** by taking the morphisms from E_0 to E_1 to be configurations of $E_0 \multimap E_1$, with composition being that of relations. We must show that this composition is well-defined and has identities.

Proposition 4 *Let α be a configuration of $E \multimap E'$ and β be a configuration of $E' \multimap E''$. Then their relational composition $\beta \circ \alpha$ is a configuration of $E \multimap E''$. Also the identity relation on a bistructure E is a configuration of $E \multimap E$.*

PROOF. That identity relations are configurations relies, for securedness, on the factorisation property (1) of bistructures. For the relational composition $\beta \circ \alpha$ to be a configuration we require it to be consistent and secured.

Consistency: From the definition of $\asymp$ on function space we require that for $(e_1, e''_1), (e_2, e''_2)$ in $\beta \circ \alpha$ that

$$(i)\ e_1 \smile e_2 \Rightarrow e''_1 \smile e''_2 \text{ and } (ii)\ e''_1 \asymp e''_2 \Rightarrow e_1 \asymp e_2,$$

facts which hold of the composition $\beta \circ \alpha$ because they hold of α and β.

Securedness: Suppose $(e, e'') \in \beta \circ \alpha$ and that

$$(e_0, e''_0) \leq^R (e, e''),$$

i.e., $e \leq^L e_0$ and $e''_0 \leq^R e''$. It is required that there is

$$(e^\star, e''^\star) \in \beta \circ \alpha$$

such that

$$(e_0, e''_0) \leq^L (e^\star, e''^\star),$$

i.e., $e^\star \leq^R e_0$ and $e''_0 \leq^L e''^\star$. [In the following argument, it is helpful to refer to the diagram below.]

As $(e, e'') \in \beta \circ \alpha$ there is $e_0'^{\star}$ such that $(e, e_0'^{\star}) \in \alpha$ and $(e_0'^{\star}, e'') \in \beta$. Because $e_0'' \leq^R e''$, we obtain that

$$(e_0'^{\star}, e_0'') \leq^R (e_0'^{\star}, e'')$$

As β is secured there is (e_1', e_1'') in β for which $(e_0'^{\star}, e_0'') \leq^L (e_1', e_1'')$, *i.e.*,

$$(e_1', e_1'') \in \beta \quad \text{and} \quad e_1' \leq^R e_0'^{\star} \quad \text{and} \quad e_0'' \leq^L e_1''. \tag{1β}$$

As $e \leq^L e_0$ and $e_1' \leq^R e_0'^{\star}$, we have $(e_0, e_1') \leq^R (e, e_0'^{\star})$. But α is secured, so there is $(e_1, e_1'^{\star})$ in α for which $(e_0, e_1') \leq^L (e_1, e_1'^{\star})$, *i.e.*,

$$(e_1, e_1'^{\star}) \in \alpha \quad \text{and} \quad e_1 \leq^R e_0 \quad \text{and} \quad e_1' \leq^L e_1'^{\star}. \tag{1α}$$

From (1α) and (1β) we obtain:

$$e_0 \geq^R e_1 \quad (e_1, e_1'^{\star}) \in \alpha \quad e_1'^{\star} \geq_L e_1' \quad (e_1', e_1'') \in \beta \quad e_1'' \geq^L e_0'' \tag{1}$$

We now show that this pattern in $e_1, e_1'^{\star}, e_1', e_1''$, relative to e_0 and e_0'', must repeat.

It follows from $e_1' \leq^L e_1'^{\star}$ that $(e_1'^{\star}, e_1'') \leq^R (e_1', e_1'') \in \beta$. As β is secured, there is an (e_2', e_2'') in β for which $(e_1'^{\star}, e_1'') \leq^L (e_2', e_2'')$, *i.e.*,

$$(e_2', e_2'') \in \beta \quad \text{and} \quad e_2' \leq^R e_1'^{\star} \quad \text{and} \quad e_1'' \leq^L e_2'' \tag{2β}$$

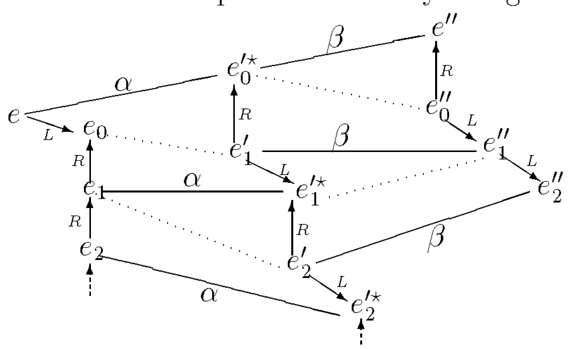
As $e_2' \leq^R e_1'^{\star}$, we have $(e_1, e_2') \leq^R (e_1, e_1'^{\star}) \in \alpha$. But α is secured, so there is an $(e_2, e_2'^{\star})$ in α for which $(e_1, e_2') \leq^L (e_2, e_2'^{\star})$, *i.e.*,

$$(e_2, e_2'^{\star}) \in \alpha \quad \text{and} \quad e_2 \leq^R e_1 \quad \text{and} \quad e_2' \leq^L e_2'^{\star} \tag{2α}$$

and the pattern in (1) repeats in (2) below—obtained directly from (2α) and (2β):

$$e_0 \geq^R e_2 \quad (e_2, e_2'^{\star}) \in \alpha \quad e_2'^{\star} \geq_L e_2' \quad (e_2', e_2'') \in \beta \quad e_2'' \geq^L e_0'' \tag{2}$$

where $e_2' \leq^R e_1'^{\star}$. This can be repeated infinitely. Diagrammatically:



The chain

$$e_0'^{\star} \geq^R e_1' \leq^L e_1'^{\star} \geq^R e_2' \leq^L e_2'^{\star} \geq^R \cdots$$

must eventually be constant by Axiom 2(a) on bistructures. Hence we obtain

$$(e_n, e_n'^{\star}) \in \alpha \quad \text{and} \quad e_n' = e_n'^{\star} \quad \text{and} \quad (e_n', e_n'') \in \beta$$

yielding (e_n, e_n'') in $\beta \circ \alpha$ with $e_n \leq^R e_0$ and $e_0'' \leq^L e_n''$, *i.e.,* $(e_0, e_0'') \leq^L (e_n, e_n'')$; so (e_n, e_n'') fulfills the requirements we seek for $(e^{\star}, e''^{\star})$. □

Proposition 5 *Let F be a configuration of $E \multimap E'$ and let x be a configuration of E. Defining*

$$F \cdot x = \{e' \mid \exists\, e \in x \;\; (e, e') \in F\}$$

yields a configuration of E'. The function $x \mapsto F \cdot x : \Gamma(E) \to \Gamma(E')$ is linear with respect to $\sqsubseteq^R$ and continuous with respect to $\sqsubseteq$.

Proof. The first part of the statement is a consequence of Proposition 4, since $F \cdot x$ can be read as $F \circ x$ if x is viewed as a configuration from I (the tensor unit, defined at the beginning of the next section) to E. The verification that $x \mapsto F \cdot x$ is linear is easy. We check only that it preserves compatible binary lubs. Suppose that $e' \in F \cdot (x \sqcup^R y)$, *i.e.,* $(e, e') \in F$ for some e in $x \sqcup^R y$. Then $e \in x$ or $e \in y$, hence $e' \in (F \cdot x) \sqcup^R (F \cdot y)$.

For continuity, by Lemma 7 below, it is enough to check $\sqsubseteq$-monotonicity, which is proved using a "staircase argument" similar to that of the previous proof: if $x \sqsubseteq y$ and $e' \in F \cdot x$, let e in x be such that $(e, e') \in F \cdot x$, and let e_1 in y be such that $e \leq^L e_1$. Since $(e_1, e') \leq^R (e, e')$, there exists (e_2, e_1') in F such that $e' \leq^L e_1'$ and $e_2 \leq^R e_1$. From there we find an e_3 such that $e_3 \in y$ and $e_2 \leq^L e_3$, which leads to an (e_4, e_2') in F such that $(e_3, e_1') \leq^L (e_4, e_2')$. In this way, we construct $e' \leq^L e_1' \leq^L e_2' \cdots$, where the e_i' are in y. The sequence eventually ends, yielding an e_i' that fits. □

Lemma 7 *Let E and E' be bistructures. Suppose that $f : \Gamma(E) \to \Gamma(E')$ is $\sqsubseteq^R$-continuous and $\sqsubseteq$-monotonic. Then it is also $\sqsubseteq$-continuous.*

Proof. Suppose $\{e_1, \ldots, e_n\} \sqsubseteq f(x)$. Then there exist $e_1', \ldots, e_n'$ in $f(x)$ such that $e_i \leq^L e_i'$ for all i. By $\sqsubseteq^R$-continuity, there exists a finite $x_1 \sqsubseteq^R x$ such that $e_1', \ldots, e_n' \in f(x_1)$, hence $\{e_1, \ldots, e_n\} \sqsubseteq f(x_1)$. □

Remark 3 *Not all $\sqsubseteq^R$-linear and $\sqsubseteq$-continuous functions are represented by a morphism. The represented functions also satisfy a uniformity requirement, where a $\sqsubseteq^R$-stable and $\sqsubseteq$-continuous function f from $\Gamma(E)$ to $\Gamma(E')$ is* uniform *iff for all e in E and configurations x containing e and for all e' in E' we have that $([e]_x, e') \in \mathrm{tr}(f)$ implies $([e], e') \in \mathrm{tr}(f)$ (see Definition 6 below, and cf. Remark 2); unfortunately, even this condition is not sufficient for representability. On a more positive note, one can show that the assignment of functions to configurations is 1-1.*

6 A model of classical linear logic

Here we give the constructions showing that **BS** is a (non-compactly closed) model of classical linear logic. The constructions extend those of Girard (recalled in Section 2). Define *linear negation*, the involution of linear logic, by

$$E^\perp = (E, \geq^R, \geq^L, \asymp)$$

where $E = (E, \leq^L, \leq^R, \smile)$. Clearly $(E^\perp)^\perp = E$. The remaining multiplicatives, $\wp$ (*par*) and $\otimes$ (*tensor*), are determined by the usual isomorphisms of classical linear logic:

$$E_0 \wp E_1 \cong (E_0^\perp \multimap E_1), \qquad E_0 \otimes E_1 \cong (E_0 \multimap E_1^\perp)^\perp$$

In more detail, the tensor product is defined as follows:

$$E_0 \otimes E_1 = (E_0 \times E_1, \leq^L, \leq^R, \smile)$$

$$\begin{array}{ll} \text{where} & (e_0, e_1) \leq^L (e_0', e_1') \Leftrightarrow e_0 \leq^L e_0' \ \text{ and } e_1 \leq^L e_1', \\ & (e_0, e_1) \leq^R (e_0', e_1') \Leftrightarrow e_0 \leq^R e_0' \ \text{ and } e_1 \leq^R_1 e_1' \\ & \\ \text{and} & (e_0, e_1) \smile (e_0', e_1') \Leftrightarrow e_0 \smile_0 e_0' \ \text{ and } e_1 \smile_1 e_1'. \end{array}$$

The construction $E^\perp$ is isomorphic to $(E \multimap I)$ where $I = (\{\bullet\}, id, id, id)$ is the unit of $\otimes$. *Product* and *coproduct* in the category **BS** are (again) obtained by disjoint juxtaposition (now of bistructures), extending conflict across the two event sets in the case of coproduct. The terminal object is $1 = (\emptyset, \emptyset, \emptyset, \emptyset)$.

We define the *exponential* $!E$ of a bistructure E by:

$$!E = (\Gamma(E)^0, \sqsubseteq^L, \sqsubseteq^R, \uparrow^R)$$

(recall $\sqsubseteq^L$ and $\sqsubseteq^R$, and $\uparrow^R$ from Definition 3).

Lemma 8 *$!E$ is a bistructure.*

PROOF. Obviously, $\sqsubseteq^R$ is a partial order. By Proposition 1, $\sqsubseteq^L$ is a partial order and so Axiom 1 is verified. Given the definition of $\smile_!$, Axiom 3(b) holds *a fortiori*, and we can rephrase Axiom 3(a) as:

$$(x_1 \downarrow^L x_2 \text{ and } x_1 \uparrow^R x_2) \Rightarrow x_1 = x_2$$

which is an immediate consequence of Proposition 1.4.

The main difficulty in the proof is in showing Axiom 2(b), *i.e.,* that the relation $\preceq_! = (\sqsupseteq^L \cup \sqsubseteq^R)^\star$ of $!E$ is a partial order. We need only show antisymmetry. Thus suppose for x_i, x'_i in $\Gamma(E)^0$ we have:

$$x_0 \sqsubseteq^R x'_0 \sqsupseteq^L x_1 \sqsubseteq^R x'_1 \sqsupseteq^L \cdots \sqsupseteq^L x_n \sqsubseteq^R x'_n \tag{1}$$

with $x_n = x_0$ and $x'_n = x'_0$. We shall show $x_i = x'_i = x_j = x'_j$ for all i, j. Then by the definition of $\preceq_!$ on $!E$ it follows that $\preceq_!$ is antisymmetric.

Define $\mathit{fix} = \bigcap_i x_i$. We first show $\mathit{fix} \in \Gamma(E)$. Consistency is obvious. Suppose $e \in \mathit{fix}$ and $e_1 \leq^R e$. Since $e \in x_i$, there exist e_1^i in x_i such that $e_1 \leq^L e_1^i$, for all $i \geq 1$. Since $x_i \sqsubseteq x'_{i-1}$, there exist $e_1'^i$ in x'_{i-1} such that $e_1^i \leq^L e_1'^i$, for all $i \geq 1$. We now apply Lemma 3 twice.

- From $e_1 \leq^L e_1'^i$, $e_1 \leq^L e_1^{i-1}$ and $x_{i-1} \sqsubseteq^R x'_{i-1}$ we conclude $e_1'^i = e_1^{i-1}$, therefore $e_1^{i-1} \geq^L e_1^i$.
- From $e_1 \leq^L e_1^0$, $e_1 \leq^L e_1^n$ and $x_0 = x_n$, we obtain $e_1^0 = e_1^n$.

Since $\leq^L$ is a partial order, we get: $e_1^0 = \cdots = e_1^i = \cdots = e_1^n$, hence $e_1^0 \in \mathit{fix}$, which shows that fix is secured. Consequently $\mathit{fix} \in \Gamma(E)$ and clearly $\mathit{fix} \sqsubseteq^R x_i, x'_i$ for all i. It remains to show $\mathit{fix} = x_i = x'_i$ for all i.

Notice that in chain (1) we can bring any index i to the head position, by a circular permutation. Thus, it suffices to show $x_0 = x'_0 = \mathit{fix}$.

Take e in x'_0. Then by repeated use of Proposition 1, we deduce from (1) that

$$e = e_0 \preceq_{x'_0} e'_0 \geq^L e_1 \preceq_{x'_1} e'_1 \geq^L e_2 \cdots \geq^L e_m \preceq_{x'_{[m]_n}} e'_m \geq^L e_{m+1} \cdots \qquad (2)$$

for some e_i in $x_{[i]_n}$ and e'_i in $x'_{[i]_n}$ where $i \in \omega$ (here $[m]_n$ is m modulo n).

The sequence has been continued infinitely by going around and around the loop (1). As x'_0 is finite and the sequence (2) visits x'_0 infinitely often there must be e_m, e_q in x'_0 such that $m < q$ and $[m]_n = [q]_n = 0$ and $e_m = e_q$. Then as $\preceq$ is a partial order, $e_m = e'_m = e_{m+1} = \cdots = e_q$. Thus $e_m \in \mathit{fix}$ so the sequence (2) eventually contains an element of fix. We know $\mathit{fix} \sqsubseteq^R x_i, x'_i$, for all i. Now, working backwards along the chain (2), starting at e_m, we show that all elements of the chain are in fix.

- From e_{i+1} to e'_i: this follows from Lemma 3 applied to fix, x'_i, e_{i+1}, and e'_i.
- From e'_i to e_i: by Lemma 5.

In particular, $e_0\ (= e) \in \mathit{fix}$. We have proved $x'_0 \sqsubseteq^R \mathit{fix}$, which together with $\mathit{fix} \sqsubseteq^R x_0 \sqsubseteq^R x'_0$ implies $x_0 = x'_0 = \mathit{fix}$ as required. Thus the relation $\preceq_!$ on $!E$ is a partial order.

Finally, we prove Axiom 2(a), *i.e.,* $\{x' \in \Gamma(E)^0 \mid x' \preceq_! x\}$ is finite, for $x \in \Gamma(E)^0$. By Proposition 1 it is clear that

$$x' \preceq_! x \Rightarrow \forall\, e' \in x' \;\; \exists\, e \in x \;\; e' \preceq e$$

Thus:

$$x' \preceq_! x \Rightarrow x' \subseteq \bigcup \{e' \mid \exists\, e \in x \;\; e' \preceq e\}$$

As x is finite and $\preceq$ is finitary, we have $\{x' \in \Gamma(E)^0 \mid x' \preceq_! x\}$ is finite, as required. □

The configurations of the bistructure $!E_0 \multimap E_1$ are in 1-1 correspondence with the type of functions from $(\Gamma(E_0), \sqsubseteq^R, \sqsubseteq)$ to $(\Gamma(E_1), \sqsubseteq^R, \sqsubseteq)$ considered by Berry in his cartesian closed category of bidomains. We need notions of trace and extensional and stable orderings for such functions (cf. Section 3):

Definition 6 *Let f be a $\sqsubseteq$-continuous and $\sqsubseteq^R$-stable function from $\Gamma(E_0)$ to $\Gamma(E_1)$. Then:*

$$\mathrm{tr}(f) = \{(x, e) \in \Gamma(E_0)^0 \times E_1 \mid x \textit{ is } \sqsubseteq^R\textit{-minimal such that } e \in f(x)\}$$

The stable *ordering on such functions is defined by:*

$$f \leq g \Leftrightarrow \forall\, x \sqsubseteq^R y \;\; f(x) = f(y) \cap g(x)$$

The extensional *ordering on such functions is defined by:*

$$f \sqsubseteq g \Leftrightarrow \forall\, x \;\; f(x) \sqsubseteq g(x)$$

Proposition 6 *Let E_0, E_1 be bistructures. For R in $\Gamma(!E_0 \multimap E_1)$ and x in $\Gamma(E_0)$ define*

$$\bar{R}(x) = \{e \mid \exists\, x_0 \sqsubseteq^R x \;\; (x_0, e) \in R\}.$$

Then $\bar{R}$ is a function $\Gamma(E_0) \to \Gamma(E_1)$ which is continuous with respect to $\sqsubseteq$ and stable with respect to $\sqsubseteq^R$ on configurations. In fact, $R \mapsto \bar{R}$ is a 1-1 correspondence between configurations of $!E_0 \multimap E_1$ and such functions, with inverse $f \mapsto \mathrm{tr}(f)$. Further, for the stable ordering of functions we have:

$$f \leq g \Leftrightarrow \mathrm{tr}(f) \sqsubseteq^R \mathrm{tr}(g)$$

and for the extensional ordering we have:

$$f \sqsubseteq g \Leftrightarrow \mathrm{tr}(f) \sqsubseteq \mathrm{tr}(g)$$

PROOF. We first check that $\bar{R}(x)$ is a configuration. Consistency follows from the definition of $\frown$ in $!E_0 \multimap E_1$. Suppose $e \in \bar{R}(x)$ and $e' \leq^R e$. Choose $x_0 \sqsubseteq^R x$ such that $(x_0, e) \in R$. Then $(x_0, e') \leq^R (x_0, e)$, hence by securedness there exists (x'', e'') in R such that $(x_0, e') \leq^L (x'', e'')$. Then e'' fits since $x'' \sqsubseteq^R x_0 \sqsubseteq^R x$. Thus the function $\bar{R}$ is well-defined. It is $\sqsubseteq^R$-continuous by construction and $\sqsubseteq^R$-stable by the definition of $\frown$ in $!E_0 \multimap E_1$. We next show that $\bar{R}$ is $\sqsubseteq$-monotonic, hence $\sqsubseteq$-continuous, by Lemma 7. Suppose $x \sqsubseteq y$ and $e \in \bar{R}(x)$. Choose $x_0 \sqsubseteq^R x$ such that $(x_0, e) \in R$. Let us factorise $x_0 \sqsubseteq y$ as $x_0 \sqsubseteq^L x_1 \sqsubseteq^R y$. Since $(x_1, e) \leq^R (x_0, e)$, there exists (x_2, e_1) in R such that $(x_1, e) \leq^L (x_2, e_1)$, by securedness. Then e_1 fits, since $e \leq^L e_1$ and $x_2 \sqsubseteq^R x_1 \sqsubseteq^R y$.

We now show that tr is the inverse of $R \mapsto \bar{R}$. Suppose $f : \Gamma(E_0) \to \Gamma(E_1)$ is $\sqsubseteq$-continuous and $\sqsubseteq^R$-stable. We need first that $\mathrm{tr}(f) \in \Gamma(!E_0 \multimap E_1)$, *i.e.*, that $\mathrm{tr}(f)$ is consistent and secured:

Consistency: Suppose $(x,e),(x',e') \in \mathrm{tr}(f)$ and that $(x,e) \asymp (x',e')$, *i.e.*, $x \uparrow^R x'$ and $e \asymp e'$. We show $(x,e) = (x',e')$. As $e, e' \in f(x \cup x')$, we must have $e \frown e'$, which combined with $e \asymp e'$, entails $e = e'$. Now, (x,e), (x',e) are both in the trace of f. Because $x \uparrow^R x'$ and f is $\sqsubseteq^R$-stable we conclude that $x = x'$.

Securedness: Suppose $(x',e') \leq^R (x,e) \in \mathrm{tr}(f)$. Then $x \sqsubseteq^L x'$ and $e' \leq^R e$. As f is $\sqsubseteq$-monotonic, $f(x) \sqsubseteq f(x')$. Because $e' \leq^R e$ and $e \in f(x)$, we see that e' is in the $\leq$-downwards-closure of $f(x')$. Thus by Lemma 4, there exists e'' in $f(x')$ such that:

$$e' \leq^L e'' \tag{1}$$

By the definition of $\mathrm{tr}(f)$, there is

$$x_0 \sqsubseteq^R x' \tag{2}$$

such that

$$(x_0, e'') \in \mathrm{tr}(f) \tag{3}$$

Combining (1), (2), (3) we obtain, as required

$$(x',e') \leq^L (x_0, e'') \in \mathrm{tr}(f).$$

For f a $\sqsubseteq$-continuous, $\sqsubseteq^R$-stable function $\Gamma(E_0) \to \Gamma(E_1)$ its continuity with respect to $\sqsubseteq^R$ entails $\overline{\mathrm{tr}(f)} = f$. For R in $\Gamma(!E_0 \multimap E_1)$ a direct translation of the definitions yields $\mathrm{tr}(\bar{R}) = R$. Thus the map $R \mapsto \bar{R} = R$ is a 1-1 correspondence.

Turning to the two orderings of functions, the claim for the stable ordering is established exactly as in the case of webs [11]. For the extensional ordering, suppose first that $f \sqsubseteq g$ and choose (x,e) in $\mathrm{tr}(f)$. Then $e \in f(x) \sqsubseteq g(x)$, and so for some $e' \geq^L e$, $e' \in g(x)$. But then there is an $x_0 \sqsubseteq^R x$ such that $(x_0, e') \in \mathrm{tr}(g)$ and we have that $(x,e) \leq^L (x_0,e') \in \mathrm{tr}(g)$. Conversely, suppose that $\mathrm{tr}(f) \sqsubseteq \mathrm{tr}(g)$ and choose e, x with $e \in f(x)$. Then for some subset x_0 of x, $(x_0, e) \in \mathrm{tr}(f) \sqsubseteq \mathrm{tr}(g)$, and so there is a $y \sqsubseteq^R x_0$ and an $e' \geq^L e$ such that $(y, e') \in \mathrm{tr}(g)$. Then we have $e \leq^L e' \in g(y) \sqsubseteq^R g(x)$, concluding the proof. $\square$

This section has provided the key constructions for showing that **BS** is a model of classical linear logic, and that the associated co-Kleisli category is equivalent to one of biorders:

Theorem 1 *The category* **BS** *forms a linear category in the sense of [24]. The exponential* ! *forms a comonad on the category* **BS**. *Together they form a model of classical linear logic (a Girard category in the sense of [24]—see too [4]).*

The associated co-Kleisli category is (necessarily) cartesian closed and isomorphic to the category whose objects are the structures $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$, *where* E *is a bistructure, and whose morphisms are the* $\sqsubseteq$*-continuous and* $\sqsubseteq^R$*-stable functions.*

Proof. The proof that **BS** forms a linear category is a straightforward extension of the web case, while the facts about the co-Kleisli category largely follow from Proposition 6. For the monoidal structure, at the level of events, the canonical isomorphisms are given by:

$$\begin{array}{rcl} ((e_0, e_1), e_2) & \leftrightarrow & (e_0, (e_1, e_2)), \\ (e, \bullet) & \leftrightarrow & e, \\ (\bullet, e) & \leftrightarrow & e \end{array}$$

The first of these correspondences also serves to establish the closed structure. For example, we have:

$$\begin{aligned} &((e_0, e_1), e_2) \leq^L_{E_0 \otimes E_1 \multimap E_2} ((e_0', e_1'), e_2') \\ &\quad \Leftrightarrow e_0' \leq^R_{E_0} e_0 \text{ and } e_1' \leq^R_{E_1} e_1 \text{ and } e_2 \leq^L_{E_2} e_2' \\ &\quad \Leftrightarrow (e_0, (e_1, e_2)) \leq^L_{E_0 \multimap (E_1 \multimap E_2)} (e_0', (e_1', e_2')) \end{aligned}$$

The trace of the canonical morphism from E to $(E \multimap I) \multimap I$ is:

$$\{(e, ((e, \bullet), \bullet)) \mid e \in E\}$$

It is clear that it has as inverse the function whose trace is:

$$\{(((e, \bullet), \bullet), e) \mid e \in E\}$$

Turning to the comonadic structure, the counit $\varepsilon : !E \multimap E$ is given by:

$$\varepsilon = \{([e]_x, e) \mid e \in x \in \Gamma(E)\}$$

For any $\alpha : !E_0 \multimap E_1$, its *lift* $\alpha^\sharp : !E_0 \multimap !E_1$ is given by:

$$\alpha^\sharp = \{(\bigcup_{i=1,n} x_i, \{e_1, \ldots, e_n\}) \in !E_0 \times !E_1 \mid (x_i, e_i) \in \alpha \text{ (for } i = 1, n)\}$$

(Then the comultiplication $\delta : !E \multimap !!E$ is $(id_{!E})^\sharp$ and the functorial action of ! is $f \mapsto (f \circ \varepsilon)^\sharp$.) The canonical isomorphism $!(E_0 \times E_1) \cong (!E_0) \otimes (!E_1)$ is given by:

$$x \quad \leftrightarrow \quad (x \cap E_0, x \cap E_1)$$

The isomorphism $!1 \cong I$ follows immediately from the observation that $\Gamma(1)$ is a singleton.

For the last part, the isomorphism is to send a bistructure E to the biorder $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$, and a morphism $R : !E_0 \multimap E_1$ to the function $\overline{R} : \Gamma(E_0) \to \Gamma(E_1)$. Given the results of Proposition 6, it only remains to show functoriality, and that is automatic from the comonadicity of the exponential, once we notice that $\overline{R}(x) = R \circ x^\sharp$, identifying elements of $\Gamma(E)$ with morphisms from !1 to E. $\square$

A fortiori, bistructures provide a model of simply-typed λ-calculi, and of PCF in particular. One interprets base types as event structures, *i.e.*, bistructures of the form $(E, id, \leq, \smile)$. At first order, $\Gamma(!E_0 \multimap E_1)$ is up to bijection the set of stable functions from $\Gamma(E_0)$ to $\Gamma(E_1)$, represented, not as an event structure, but as a bistructure with a non-trivial $\leq^L$-order which captures the extensional order between stable functions. At higher orders, as discussed in Section 2, the model diverges from the stable model by enforcing both $\sqsubseteq^R$-stability and $\sqsubseteq$-continuity of the morphisms. In the next section we shall see that this model coincides with that obtained from Berry's category of bidomains.

7 Bidomains

Bidomains are not very widely known so we present their definition here. They were introduced in Gérard Berry's Thèse de Doctorat d'Etat [2]. More details and background motivation can be obtained either from Berry's thesis or the references [3, 25].

The most general biordered domains Berry considered were the bicpos.

Definition 7 *A* bicpo *is a biorder* $(D, \leq, \sqsubseteq)$ *such that:*

1. *the structure* $(D, \sqsubseteq)$ *is a cpo with a continuous greatest-lower-bound operation* $\sqcap$ *(the order* $\sqsubseteq$ *is called the* extensional *order),*

2. *the structure* $(D, \leq)$ *is a cpo, with the same least element as* $(D, \sqsubseteq)$ *and the identity on* D *is a continuous function from* $(D, \leq)$ *to* $(D, \sqsubseteq)$ *(the order* $\leq$ *is called the* stable *order),*

3. *the operation* $\sqcap$ *is* $\leq$*-continuous—in fact,* $\leq$*-monotonicity suffices, by 1. and 2. (it follows that the* $\leq$*-greatest lower bound,* $x \wedge y$*, of points* x, y*, bounded above in* $(D, \leq)$*, exists and coincides with* $x \sqcap y$*), and*

4. *for all* $\sqsubseteq$*-directed subsets* $S, S' \subseteq D$*, if for all* s *in* S*,* s' *in* S' *there are* t *in* S, t' *in* S' *such that*

$$s \sqsubseteq t \text{ and } s' \sqsubseteq t' \text{ and } t \leq t'$$

then $\bigsqcup S \leq \bigsqcup S'$.

A bicpo is distributive *iff*

- *whenever* x, y *are bounded above in* $(D, \leq)$ *then their least upper bounds,* $x \vee y$*, in* $(D, \leq)$*, and* $x \sqcup y$*, in* $(D, \sqsubseteq)$*, exist and coincide, and*

- *the cpo* $(D, \leq)$ *is distributive, i.e., for all* x, y, z *in* D*, if* $\{x, y, z\}$ *is bounded above, then* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Bicpos form a category where the morphisms are taken to be those functions which are continuous with respect to the extensional order and conditionally multiplicative with respect to the stable order, *i.e.,* the functions preserve binary compatible glbs. This category is cartesian closed; products are given by cartesian products of the underlying sets, the two orders being obtained in a coordinatewise fashion; the function space $[D_0 \rightarrow D_1]$ consists of the set of morphisms from D_0 to D_1 with the extensional order $\sqsubseteq$ being the pointwise order, based on the extensional order of D_1, and with the order $\leq$ being the stable order of functions, based on the stable orders of D_0 and D_1.

Here are brief hints on how the axioms of Definition 7 are used. Lubs and glbs in the function spaces are defined pointwise. The continuity properties of the meet operation are put to use when proving that the pointwise lubs are conditionally multiplicative. Axiom 4 serves to establish that the pointwise extensional lubs are monotonic with respect to the stable ordering.

The smallest category of biordered domains Berry considered was that of bidomains. They are a form of distributive bicpo, satisfying a restriction which ensures that they are dI-domains with respect to the stable order.

Definition 8 *A* finite projection *on a bicpo D is a morphism $\psi : D \to D$ such that $h \circ h = h$ and $h(x) \leq x$, for all x in D, and which is compact with respect to the stable order on $[D \to D]$.*

A bicpo D is said to be a bidomain *iff it is distributive and there is a sequence of finite projections ψ_i ($i \in \omega$) on it, increasing with respect to the stable order, whose least upper bound is the identity function $1_D : D \to D$.*

The category of bidomains and of $\sqsubseteq$-continuous and $\leq$-stable functions is a full sub-cartesian closed category of that of bicpos. [In the subcategory, we have stable = conditionally multiplicative.]

We now make good our earlier claim that configurations of (suitably restricted) bistructures form a bidomain. Let E be a bistructure. The problem is that the axioms of bistructures adopted here are too weak to ensure the existence of enough $\sqsubseteq$-glbs to yield a bicpo. We can prove the existence of the required glbs provided we add two further axioms on bistructures:

Definition 9 *A B-bistructure ("B" stands for "Bidomain") is a bistructure $(E, \leq^L, \leq^R, \stackrel{\frown}{\smile})$ that satisfies the following two additional axioms:*

4. *if $e \geq^L e'$ and $e \frown e''$, then $e' \frown e''$, and*
5. *bounded binary $\leq^L$-lubs exist, i.e., if $e_1, e_2 \leq^L e$, for some event e, then the lub, $e_1 \vee^L e_2$, with respect to $\leq^L$ exists.*

B-bistructures are quite similar to the original version of bistructures in [25] —for example, Axioms 1, 2 and 5 are common, factorisation is unique in both, and all the other axioms relate to the interaction between coherence and the two partial orders. Axiom 3(a) is missing in [25]; instead, configurations are defined in a way equivalent to requiring for any elements e, e' that not only $e \stackrel{\frown}{\smile} e'$ but also $\neg(e \downarrow^L e')$. Pleasantly, as is straightforward to show, the class of biorders obtained from the original version is included in that obtained from B-bistructures.

Proposition 7 *Let E be a B-bistructure. Then $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$ is a bidomain.*

PROOF. We must first show that $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$ is a distributive bicpo. We already know from Proposition 2 that $(\Gamma(E), \sqsubseteq)$ and $(\Gamma(E), \sqsubseteq^R)$ are cpos, and that $\sqsubseteq^R$-directed lubs are $\sqsubseteq$-lubs, which is the same as requiring that the identity function is continuous from $(\Gamma(E), \sqsubseteq^R)$ to $(\Gamma(E), \sqsubseteq)$. We also know that $(\Gamma(E), \sqsubseteq)$ is algebraic, which is easily seen to imply the continuity of $\sqcap$, provided binary glbs exist, which we prove now, using Axioms 4 and 5. Let x and y be two configurations. We define $x \sqcap y$ as the set of all events e such that e is in the $\leq^L$-downwards-closure of both x and y, and is $\leq^L$-maximal with that property. This is clearly the glb of x and y if only it is a configuration. The definition ensures securedness by construction: if $e \in x \sqcap y$ and $e' \leq^R e$, then e' is in the $\leq^L$-downwards-closure of both x and y by Lemma 4, and by definition of $x \sqcap y$ there exists e'' in $x \sqcap y$ such that $e' \leq^L e''$. Suppose now that e_1, e_2 are in $x \sqcap y$. Let e'_1, e'_2 in x be such that $e_1 \leq^L e'_1$ and $e_2 \leq^L e'_2$, and let e''_1, e''_2 in y be such that $e_1 \leq^L e''_1$ and $e_2 \leq^L e''_2$. Since $e'_1, e'_2 \in x$, we have $e'_1 \smile e'_2$, and similarly $e''_1 \smile e''_2$. We distinguish several cases.

- $e'_1 \neq e'_2$. Here, by two applications of Axiom 4 we get first $e_1 \frown e'_2$ then $e_1 \frown e_2$.
- $e''_1 \neq e''_2$. We obtain $e_1 \frown e_2$ similarly to the previous case.
- $e'_1 = e'_2$ and $e''_1 = e''_2$. By Axiom 5, $e_1 \vee^L e_2$ exists. It is in the $\leq^L$-downwards-closure of both x and y, which, by the definition of $x \sqcap y$, forces $e_1 = e_2 = e_1 \vee^L e_2$.

In all cases, we have $e_1 \smile e_2$. This completes the proof of Axioms 1 and 2.

We next check Axiom 3. Let $y \sqsubseteq^R z$ and $e \in x \sqcap y$. Then e is in the $\leq^L$-downwards-closure of both x and z. We show that e is $\leq^L$-maximal with that property. Suppose that $e \leq^L e'$ and $e' \leq^L e'_1 \in x$ and $e' \leq^L e'_2 \in z$. Now, for some e'_3 in y, $e \leq^L e'_3$ and thus, by Lemma 3, we have that $e'_2 = e'_3 \in y$. Therefore, $e = e'$ by the maximality property of e with respect to x and y. Hence $e \in x \sqcap z$, and we have proved $x \sqcap y \sqsubseteq^R x \sqcap z$.

To check Axiom 4, let subsets S, S' be $\sqsubseteq$-directed subsets of $\Gamma(E)$ such that whenever $s \in S$ and $s' \in S'$ there are t in S and t' in S' such that

$$s \sqsubseteq t \text{ and } s' \sqsubseteq t' \text{ and } t \sqsubseteq^R t'.$$

By the proof of Proposition 2,

$$\bigsqcup S = \{e \in \bigcup S \ \mid e \text{ is } \leq^L\text{-maximal in } \bigcup S\}$$

Equivalently, $\bigsqcup S$ consists of the events e such that $e \in s$ for some s in S with the property that

$$\forall t \in S \ (s \sqsubseteq t \Rightarrow e \in t). \tag{1}$$

Indeed, suppose that e and s satisfy property (1), and let $e_1 \geq^L e$ be such that $e_1 \in \bigcup S$, *i.e.,* $e_1 \in t$ for some t in S. Let s' in S be such that $s \sqsubseteq s'$ and $t \sqsubseteq s'$. Then $e \in s'$ and e_1 is in the $\leq^L$-downwards-closure of s'. This combined with $e \leq^L e_1$ yields $e = e_1$. Conversely, if $e \in s \in S$ and e is $\leq^L$-maximal in $\bigcup S$, then e, s obviously satisfy (1).

Given e and s satisfying (1), we wish to show $e \in \bigsqcup S'$. By assumption, taking s' an arbitrary element of S', we have for some t in S and t' in S' that

$$t \sqsubseteq^R t' \text{ and } s \sqsubseteq t \text{ and } s' \sqsubseteq t'.$$

By (1), we get that $e \in t'$. Suppose $t' \sqsubseteq t'' \in S'$. Replacing s' by t'', and reasoning as above, we get that $e \in t'''$ for some t''' such that $t'' \sqsubseteq t'''$. Finally, one easily derives that $e \in t''$ from $t' \sqsubseteq t'' \sqsubseteq t'''$, $e \in t'$ and $e \in t'''$. Thus e and t' satisfy property (1) relative to S', *i.e.,* $e \in \bigsqcup S'$. We conclude that $\bigsqcup S \sqsubseteq^R \bigsqcup S'$, as required. Thus $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$ is a bicpo.

For distributivity, by Proposition 3, we know that $(\Gamma(E), \sqsubseteq^R)$ is a distributive cpo. Now remark that if $x \uparrow^R y$, $x \sqsubseteq z$ and $y \sqsubseteq z$, then also $x \cup y \sqsubseteq z$ by the definition of $\sqsubseteq$.

Finally, to show that $(\Gamma(E), \sqsubseteq^R, \sqsubseteq)$ is a bidomain we need to produce an ω-chain of finite projections on it whose least upper bound is the identity on $\Gamma(E)$. Here we refer to Proposition 6, showing how continuous stable functions on $\Gamma(E)$ correspond to configurations of the bistructure $!E \multimap E$; a configuration F of $!E \multimap E$ corresponds to a continuous, stable function $\overline{F} : \Gamma(E) \to \Gamma(E)$ so that inclusion of configurations corresponds to the stable ordering of functions. Let X be the configuration of $!E \multimap E$ corresponding to the identity function on $\Gamma(E)$. Because the events of $!E \multimap E$ form a countable set we may enumerate them as $e_0, e_1, \cdots, e_n, \cdots$. Define X_n as the configuration

$$[e_0]_X \cup \cdots \cup [e_n]_X.$$

Each X_n is a finite set and so compact with respect to $\sqsubseteq^R$. By Proposition 6, we obtain a chain of functions $\overline{X}_n$, n in ω, compact and increasing with respect to the stable order, with least upper bound the identity function on $\Gamma(E)$. Being stably less than the identity, each function $\overline{X}_n$ must be a projection.

We conclude that $\Gamma(E)$ is a bidomain. $\square$

Proposition 8 *If E_0 and E_1 are B-bistructures, then so are $E_0 \times E_1$ and $!E_0 \multimap E_1$.*

Proof. The verification for the product is immediate. We check Axioms 4 and 5 for $!E_0 \multimap E_1$. Let $(x, e_1) \geq^L (y, e_1')$ and $(x, e_1) \frown (z, e_1'')$. Then $x \subseteq y$, $e_1 \geq^L e_1'$ and either $x \smile z$ or $e_1 \frown e_1''$. If $x \smile z$ then $y \smile z$ (as $x \subseteq y$); if $e_1 \frown e_1''$ then $e_1' \frown e_1''$, by Axiom 4 for E_1. In either case it follows that $(y, e_1') \frown (z, e_1'')$. The verification of Axiom 5 is straightforward: $(x, e_1) \vee^L (y, e_1') = (x \sqcap^R y, e_1 \vee^L e_1')$, where we use Axiom 5 for E_1 and the fact that $\Gamma(E_0)$ is $\sqsubseteq^R$-bounded complete (cf. Proposition 2). [In bounded-complete cpos, all non-empty glbs exist.] $\square$

Remark 4 *On the other hand, Axiom 4 is not preserved by any of $\otimes$, $(\cdot)^\perp$ or $!$. Therefore we do not see how to get a model of linear logic with B-bistructures.*

Theorem 2 *The B-bistructures yield a full sub-cartesian closed category of the co-Kleisli category of the exponential comonad on* **BS**. *It is equivalent to a full sub-cartesian closed category of bidomains.*

Proof. The first assertion is immediate from Proposition 8. By Proposition 7 and the last part of Theorem 1, the subcategory is equivalent to a full subcategory of the category of bidomains. It remains to show that there are isomorphisms of bidomains between $\Gamma(E_0) \times \Gamma(E_1)$ and $\Gamma(E_0 \times E_1)$, and also between $\Gamma(E_0) \to \Gamma(E_1)$ and $\Gamma(!E_0 \multimap E_1)$. The first is easy to see (it is $(x, y) \mapsto x \cup y$, assuming E_0 and E_1 disjoint). The second is given by $f \mapsto \mathrm{tr}(f)$, by virtue of the remarks above on function spaces of bidomains and Proposition 6. $\square$

Notice that if $\leq^L = id$, Axioms 4 and 5 hold vacuously, hence the category of B-bistructures is large enough to contain the models of typed λ-calculi where the base types are interpreted by event structures.

8 Discussion

In this section, we discuss possible variations and connections with other work.

Variations. Besides coherence spaces there are a few other "web-based" models of intuitionistic linear logic or full classical linear logic. Most notably, there are Ehrhard's *hypercoherences* with a Kleisli category of *strongly stable functions*, which can be seen as an extensional (although not order-extensional) account of sequentiality. At first order, the strongly stable model contains exactly the sequential functions. At higher orders, it is the extensional collapse of the model of sequential algorithms [8, 9]. Generalisations encompassing both Girard's webs and hypercoherences have been proposed independently by Lamarche [18] (based on quantale-valued sets) and by Winskel [30] (based on a notion of indexing inspired by logical relations). We believe that our biordering treatment can be applied to all these categories. We checked this for hypercoherences. There, the binary coherence relation $\smile$ is replaced by a coherence hypergraph Γ. Thus, a hypercoherence is given by a set E of events and a set Γ of finite non-empty subsets of E. We axiomatise "hypercoherence-bistructures" exactly like bistructures, replacing $\smile$ by Γ, and Axiom 3 by:

$$
\begin{array}{ll}
3(a) & (X \in \Gamma \text{ and } X \text{ is } \leq^L -\text{lower bounded}) \Rightarrow X \text{ is a singleton,} \\
3(b) & X \text{ is } \leq^R -\text{upper bounded} \Rightarrow X \in \Gamma
\end{array}
$$

The variation works smoothly because there is very little interaction between the axiomatisation of the coherence structure and that of the orders. The two propagate smoothly in the construction of the connectives, and are related only through Axiom 3 or its variants.

Although the bidomain model incorporates both the stable and pointwise order, its PCF theory (those inequalities on terms which hold in the bidomain model) does not include that of the Scott model. The argument follows Jim and Meyer's in [13], and is based on the bidomain model failing to eliminate an "or-tester", the first of Curien's examples in [5], Proposition 4.4.2. The same argument applies to the variation sketched above, where coherence is replaced by hypercoherence. In [29] it is shown how to adjoin an additional conflict relation to bistructures to obtain a model of PCF whose theory strictly includes that of the Scott model. Several strengthenings of this idea are possible; the conflict relation can be combined with hypercoherences on bistructures, or even replaced by coherence relations of a more complex kind (akin to logical relations, as in [30]).

As remarked in the introduction, by Loader's result [19] there is no hope of finding a direct presentation of the fully abstract model for PCF in this way, adjoining a suitable coherence structure to the two orders of bistructures, unless, as in [22], this is of an infinitary nature. However, the fact that there are hypercoherence-based bistructure models refining the Scott model shows (adapting the "Definability Lemma", *e.g.,* Proposition 4.1.10 of [5], to a slightly broader setting) that there are extensions of PCF, having some claims to being sequential and functional, but which nevertheless have an effectively-given fully abstract model. Such extensions to PCF escape Loader's proof. But the precise programming constructs associated with these extensions is a mystery, and would need a deeper understanding of the operational nature of bistructure models.

Connections. Lamarche [17] (followed by Kegelmann [15]) has proposed a large cartesian closed category of "interpolative domains" encompassing, as full sub-cartesian closed categories, categories of continuous functions on one hand, and categories of stable functions on the other. Lamarche's framework has some technical similarities with ours; two orders are axiomatised, with a factorisation property. But the approach and goals are quite different. In Lamarche's setting, once the standard interpretation of the basic flat domains of, say, PCF, is fixed appropriately, the larger category induces exactly the continuous or stable model, whereas we obtain a model simultaneously refining both the stable and continuous models.

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