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BRICS
Basic Research in Computer Science

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BRICS Report Series RS-94-9
ISSN 0909-0878 May 1994

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Bistructures, Bidomains and Linear Logic

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Abstract

Bistructures are a generalisation of event structures to represent spaces of functions at higher types; the partial order of causal dependency is replaced by two orders, one associated with input and the other output in the behaviour of functions. Bistructures form a categorical model of Girard's classical linear logic in which the involution of linear logic is modelled, roughly speaking, by a reversal of the roles of input and output. The comonad of the model has associated co-Kleisli category which is equivalent to a cartesian-closed full subcategory of Berry's bidomains.

1 Introduction

Girard has shown how to implement and analyse intuitionistic logic in his more primitive linear logic. When we look to models this is reflected in the fact that cartesian-closed categories (categorical models for intuitionistic logic) arise as co-Kleisli categories associated with the categorical models of linear logic. In particular, linear logic is leading to refined analyses of the categories of domains used in denotational semantics. For instance, recent work on sequentiality, obtaining *intensional* fully-abstract models of the programming language PCF [1, 6], has been informed by the insight that Berry and Curien's category of concrete data structures and sequential algorithms [3] is got as a co-Kleisli category from a games model of linear logic [3, 7].

Bistructures were introduced in [10] as a generalisation of event structures to represent a full subcategory of Berry's bidomains [2]. Bidomains possess an intensional, stable ordering, based on the method of computation, and an extensional ordering, inherited from Scott's domain theory; their morphisms are functions which respect both, a property shared by functions definable in PCF. Here we show that, with a small modification

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to their original definition, bistructures can be equipped with morphisms to form a model of classical linear logic. Through this observation we provide an explanation of a cartesian-closed full subcategory of bidomains as the co-Kleisli category got from a linear category of bistructures with a suitable comonad for the exponential !. This result fits into a more general programme. We are interested in weaker axioms for bistructures, variations in the exponential, and extensions of bistructures to models other than event structures, one motivation being to understand *extensional* fully abstract models of languages like PCF.

The work presented here can be viewed as generalising Girard's model of classical linear logic of webs $(E, \smile)$ [5]; as morphisms one considers relations $\alpha \subseteq E \times E'$ where it is required that for $a\ \alpha\ a'$ and $b\ \alpha\ b'$:[2]

$$a \stackrel{\frown}{\smile} b \Rightarrow a' \stackrel{\frown}{\smile} b' \quad \text{and} \quad a' \asymp b' \Rightarrow a \asymp b.$$

This category is equivalent to the category of coherence spaces and linear functions. The stable functions are obtained via the co-Kleisli construction associated with the model. Adding a partial order, we obtain event structures $(E, \leq, \smile)$ which represent coherent prime algebraic domains [8], and, with a finiteness axiom added, coherent dI-domains [2, 11, 12]. We can consider the corresponding categories of relations, again representing the linear stable functions; we must now add the condition:

$$a\ \alpha\ b \geq b' \Rightarrow \exists a' \leq a.\ \ a'\ \alpha\ b'$$

However, this only yields a model of *intuitionistic* linear logic [11, 13]. By passing to bistructures we add enough objects to obtain again a model of *classical* linear logic.

2 Motivation

Recall that an event structure is a structure $(E, \leq, \smile)$ where

- E is a set of *events*,
- $\leq$ is a partial order of *causal dependency*
- $\smile$ is a binary, irreflexive, symmetric relation of *conflict*

[2]Throughout this paper we use Girard's notation: $\asymp$ is the reflexive closure of the irreflexive relation $\smile$, and $\stackrel{\frown}{\smile}$, the complement of $\smile$, is the reflexive closure of the irreflexive relation $\frown$. It is clear that specifying one relation determines all the others.

satisfying

$$e \smile e' \leq e'' \Rightarrow e \smile e''.$$

The *configurations* (or states) of such an event structure are subsets $x \subseteq E$ which are

- *consistent:* $\forall e_1, e_2 \in x.\ e_1 \mathrel{\overset{\frown}{\smile}} e_2$,
- *secured:* $\forall e, e' \in E.\ e' \leq e \in x \Rightarrow e' \in x$.

Ordered by inclusion, the configurations $(\Gamma(E), \subseteq)$, form a coherent prime algebraic domain [8]; such domains are precisely the infinitely distributive, coherent Scott domains [12]. An instance of the causal dependency ordering $e' \leq e$ when e and e' are distinct, is understood as meaning that the event e causally depends on the event e', that the event e can only after e' has ocurred. Given this understanding it is reasonable to impose a finiteness axiom, expressing that an event has finite causes:

$\{e' \mid e' \leq e\}$ is finite, for all events e,

The event structures satisfying this axiom yield domains which are precisely the coherent dI-domains of Berry [2].

If we ignore for the moment the intended interpretation of event structures, temporarily forgetting the axiom of finite causes, we can move quickly to a model of intuitionistic linear logic. The categorical model we have in mind is equivalent to the category of coherent prime algebraic domains, with additive functions (i.e. functions preserving arbitrary lubs). The category has as objects event structures (without the axiom of finite causes) and morphisms configurations of a "function space" of event structures, constructed as follows:
Let $E_i = (E_i, \leq_i, \smile_i)$, $i = 0, 1$, be event structures. Define

$$\begin{array}{rl} & E_0 \multimap E_1 = (E_0 \times E_1, \leq, \smile) \\ \text{where} & (e_0, e_1) \leq (e_0', e_1') \Leftrightarrow e_0' \leq_0 e_0 \ \&\ e_1 \leq_1 e_1' \\ \text{and} & (e_0, e_1) \smile (e_0', e_1') \Leftrightarrow e_0 \mathrel{\overset{\frown}{\smile}}_0 e_0' \ \&\ e_1 \smile_1 e_1'. \end{array}$$

The configurations of $E_0 \multimap E_1$ correspond to additive functions from $\Gamma(E_0)$ to $\Gamma(E_1)$—additive functions are determined by their action on complete primes[3] which correspond to events. The inclusion ordering

[3] A complete prime of a Scott domain $(D, \sqsubseteq)$ is an element p for which whenever $p \sqsubseteq \bigsqcup X$ then $p \sqsubseteq x$ for some $x \in X$.

on configuration reflects the Scott pointwise ordering on functions; in particular, the function events (e_0, e_1) correspond to special step functions and the order $\leq$ to the Scott order between them.

A morphism $E_0 \to E_1$ is defined to be a configuration of $E_0 \multimap E_1$. As such it is a relation between the events of E_0 and E_1. Composition in the category is that of relations. The category is a model of intuitionistic linear logic, as defined in [9]; for instance, its tensor is given in a coordinatewise fashion: For event structures $E_i = (E_i, \leq_i, \smile_i)$, for $i = 0, 1$, define

$$
\begin{aligned}
& E_0 \otimes E_1 = (E_0 \times E_1, \leq, \smile) \\
\text{where} \quad & (e_0, e_1) \leq (e_0', e_1') \Leftrightarrow e_0 \leq_0 e_0' \ \& \ e_1 \leq_1 e_1' \\
\text{and} \quad & (e_0, e_1) \frown (e_0', e_1') \Leftrightarrow e_0 \frown_0 e_0' \ \& \ e_1 \frown_1 e_1'.
\end{aligned}
$$

Monoidal-closure follows from the isomorphism

$$(E_0 \otimes E_1 \multimap E_2) \cong (E_0 \multimap (E_1 \multimap E_2))$$

natural in event structures E_0, E_1, E_2. Product and coproduct are obtained by disjoint juxtaposition of event structures, extending conflict across the two event sets in the case of coproduct. The comonad operation is

$$!E = (\Gamma(E)^0, \subseteq, \Uparrow)$$

for an event structure E, with *finite* configurations $\Gamma(E)^0$, on which incompatibility with respect to inclusion is denoted by $\Uparrow$. The continuous functions $\Gamma(E_0) \to \Gamma(E_1)$, between configurations of event structures E_0, E_1, are in 1-1 correspondence with configurations of $!E_0 \multimap E_1$

But a price has been paid. In this model of linear logic all hope of considering the order $\leq$ as causal dependency is lost. The difficulty stems from the definition of the order $\leq$ for $(E_0 \multimap E_1)$ of event structures $E_i = (E_i, \leq_i, \smile_i), i = 0, 1$. Its events are ordered by:

$$(e_0, e_1) \leq (e_0', e_1') \ \Leftrightarrow \ e_0' \leq_0 e_0 \ \& \ e_1 \leq_1 e_1'$$

The reversal in the $\leq_0$ order can lead to $\leq$ violating the axiom of finite causes, even though $\leq_0$ and $\leq_1$ do not: an infinite, ascending chain of events in E_0 can give rise to an infinite, *descending* chain in $E_0 \multimap E_1$. Of course, the extensional/Scott ordering on functions never made any pretence of being a relation of causal dependency, so it is not to be expected that its restriction to step functions $\leq$ should be finitary.

However, if we factor $\leq$ into two orderings, one associated with input (on the left) and output (on the right) we can expose two finitary orderings. Define

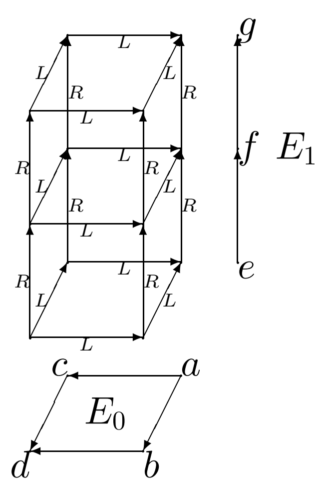
$$(e_0, e_1) \leq^L (e_0', e_1') \Leftrightarrow e_0' \leq_0 e_0 \ \& \ e_1 = e_1'$$
$$(e_0, e_1) \leq^R (e_0', e_1') \Leftrightarrow e_0' = e_0 \ \& \ e_1 \leq_1 e_1'.$$

Then, it is clear that $\leq$ factors as

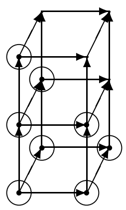
$$(e_0, e_1) \leq (e_0', e_1') \Leftrightarrow (e_0, e_1) \leq^L (e_0', e_1) \ \& \ (e_0', e_1) \leq^R (e_0', e_1'),$$

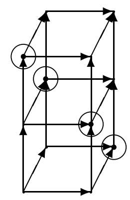
and that this factorisation is unique. Provided the orderings of E_0 and E_1 are finitary, then so are $\leq^R$ and the converse ordering $\geq^L$. This factorisation is the first step towards the definition of bistructures. To indicate its potential, and to further motivate bistructures, we study a simple example.

Let E_0 and E_1 be the event structures shown below. Both have empty conflict relations. Taking advantage of the factorisation we have drawn them alongside the function space $E_0 \multimap E_1$.



The conflict relation of $E_0 \multimap E_1$ is empty. So here an additive function from $\Gamma(E_0)$ to $\Gamma(E_1)$ is represented by a $\leq$-downwards-closed subset of events of $E_0 \multimap E_1$. For instance, the events in the diagram (below left) are associated with the function that outputs e on getting input event a, outputs f for input b or c, and outputs g for input d. The extensional/Scott ordering on functions corresponds to inclusion on $\leq$-downwards-closed subsets of events. It is clear that such a function is determined by specifying the minimal input events which yield some specific output (shown in the diagram below right). This amounts to the subset of $\leq^L$-maximal events of the function. Following Girard, we can call this subset the *trace* of the function. Notice, though, that this particular function is not stable; output f can be obtained for two non-conflicting but distinct events b and c. A stable function should not have $\leq^L$-downwards compatible distinct events in its trace. For stable functions, the stable/Berry ordering is obtained as inclusion of traces.





To summarise:

- additive functions correspond to $\leq$-downwards-closed, consistent subsets of events of $E_0 \multimap E_1$,
- the extensional/Scott order corresponds to inclusion of $\leq$-downwards-closed subsets of events,
- the trace of a function corresponds to its $\leq^L$-maximal events,
- if a function is stable then no two distinct events in its trace are $\leq^L$-downwards compatible, i.e. for e_1, e_2 in its trace, $e_1 \downarrow^L e_2$ implies $e_1 = e_2$, where $e_1 \downarrow^L e_2 \Leftrightarrow_{def} \exists e.\ e \leq^L e_1\ \&\ e \leq_L e_2$.
- the stable/Berry order corresponds to inclusion of traces.

But, of course, several of these observations are based on a special case, that where we construct the function space of ordinary event structures. We have not yet specified how to repeat the function-space construction at higher orders, when the event structures come already equipped with $\leq^L$ and $\leq^R$ orders. And what axioms should such structures satisfy?

3 Bistructures

The following definition of bistructures provides an answer to the question just raised as well as yielding a model of classical linear logic.

Definition: A *bistructure* consists of

$$(E, \leq^L, \leq^R, \smile)$$

where E is a countable set, $\leq^L, \leq^R$ are partial orders on E and $\smile$ is a binary irreflexive symmetric relation on E for which:

1. Defining $\leq = (\leq^L \cup \leq^R)^*$, we have the unique factorisation property:
$$e \leq e' \Rightarrow \exists!\ e''.\ e \ \leq^L e'' \leq^R e'$$
[It follows that $\leq$ is a partial order.]

2. Defining $\preceq = (\geq^L \cup \leq^R)^*$,
   (a) $\{e' \mid \ e' \preceq \ e\}$ is finite, for all e,
   (b) $\preceq$ is a partial order.

3. (a) $\downarrow^L \subseteq \asymp$ (b) $\uparrow^R \subseteq \frown\!\!\!\!\smile$
The two compatibility relations mean
$$\begin{array}{lcl} e \downarrow^L e' & \Leftrightarrow & \exists e''.\ e'' \leq^L e \ \&\ e'' \leq^L e' \\ e \uparrow^R e' & \Leftrightarrow & \exists e''.\ e \leq^R e'' \ \&\ e' \leq^R e''. \end{array}$$

Ordinary, countable event structures, $(E, \leq, \smile)$, satisfying the axiom of finite causes, yield special bistructures $(E, 1_E, \leq, \smile)$, in which the $\leq^L$ order is degenerate.

Definition: A *configuration* of a bistructure $(E, \leq^L, \leq^R, \smile)$ is a subset $x \subseteq E$ which is

- *consistent:* $\forall e_1, e_2 \in x.\ \ e_1 \frown\!\!\!\!\smile e_2$,
- *secured:* $\forall e \in x \forall e' \leq^R e \exists e''.\ \ e' \leq^L e'' \in x$.

[Notice that e'' is unique in any consistent set because of Axiom 3(a) on bistructures.] Write $\Gamma(E)$ for the set of configurations of a bistructure E.

A helpful case to consider is that where the bistructure represents a space of functions, as in the introduction. Configurations then correspond to traces. Because $\downarrow^L$ is included in $\asymp$, consistency has the force of ensuring the functions represented are stable. The securedness condition says that for any output, lesser output must have arisen previously through the same or lesser input.

$\Gamma(E)$ possesses two orderings making it into a Berry bidomain; the inclusion order corresponds to the stable order while the Scott extensional order is obtained by:

$$x \sqsubseteq y \text{ iff } \forall e \in x \exists e' \in y.\ \ e \leq^L e'$$

This is equivalent to every event of x being $\leq$-below some event of y.

Proposition 1 $(\Gamma(E), \subseteq, \sqsubseteq)$ *is a Berry bidomain.*

Proof: The proof is essentially as in [10] (Theorem 9.9.8—relying on 9.4.8). $\square$

4 A category of bistructures

Morphisms between bistructures will correspond to configurations of a function-space construction. They will determine (special) extensional, linear (= stable and additive) functions on bidomains. Assuming $E_i = (E_i, \leq_i^L, \leq_i^R, \smile_i)$, for $i = 0, 1$, are bistructures, define their *linear function space* by

$$\begin{array}{rl} & E_0 \multimap E_1 = (E_0 \times E_1, \leq^L, \leq^R, \smile) \\ \text{where} & (e_0, e_1) \leq^L (e_0', e_1') \Leftrightarrow e_0' \leq^R e_0 \ \&\ e_1 \leq^L e_1' \\ & (e_0, e_1) \leq^R (e_0', e_1') \Leftrightarrow e_0' \leq^L e_0 \ \&\ e_1 \leq_1^R e_1' \\ \text{and} & (e_0, e_1) \asymp (e_0', e_1') \Leftrightarrow e_0 \overline{\smile}_0\, e_0' \ \&\ e_1 \asymp_1 e_1'. \end{array}$$

We define the category of bistructures **BS** by taking morphisms E_0 to E_1 to be configurations of $E_0 \multimap E_1$, composed as relations. Of course, we should show that this composition is well-defined and has identities. First we observe that with this definition, morphisms of bistructures generalise those of Girard on webs $(E, \smile)$—see the introduction:

Proposition 2 *A configuration of $E_0 \multimap E_1$, for bistructures E_0, E_1, consists of a morphism α from the web of E_0 to the web of E_1 which satisfies:*

$$e_0' \geq^L e_0 \alpha e_1 \geq^R e_1' \Rightarrow \exists e_0'', e_1''.\ e_0' \geq^R e_0'' \alpha e_1'' \geq^L e_1'$$

Proof: Consistency of a configuration of a function space amounts to the relation being a morphism between the underlying webs. Securedness is equivalent to the condition of the proposition. □

Lemma 3 *Let α be a configuration of $E_0 \multimap E_1$ and β a configuration of $E_1 \multimap E_2$. Then their relational composition $\beta \circ \alpha$ is a configuration of $E_0 \multimap E_2$. A bistructure $E \multimap E$ has the identity relation as a configuration.*

Proof: That identity relations are configurations relies, for securedness, on the factorisation property (1) of bistructures. For the relational composition $\beta \circ \alpha$ to be a configuration we require it consistent and secured. *Consistent*: From the definition of $\asymp$ on function space we require that for $(a, c), (a', c') \in \beta \circ \alpha$ that

$$(i)\ a \frown a' \Rightarrow c \frown c' \text{ and } (ii)\ c = c' \Rightarrow a \asymp a',$$

facts which hold of the composition $\beta \circ \alpha$ because they hold of α and β. *Secured*: Suppose $(a, c) \in \beta \circ \alpha$ and that

$$(a_0, c_0) \leq^R (a, c),$$

i.e., $a \leq^L a_0$ & $c_0 \leq^R c$. It is required that there is

$$(a', c') \in \beta \circ \alpha$$

such that

$$(a_0, c_0) \leq^L (a', c'),$$

i.e., $a' \leq^R a_0$ & $c_0 \leq^L c'$. [In the following argument, it is helpful to refer to the diagram below.]

As $(a, c) \in \beta \circ \alpha$ there is b'_0 such that $(a, b'_0) \in \alpha$ and $(b'_0, c) \in \beta$. Because $c_0 \leq^R c$, we obtain that

$$(b'_0, c_0) \leq^R (b'_0, c).$$

As β is secured there is $(b_1, c_1) \in \beta$ for which $(b'_0, c_0) \leq^L (b_1, c_1)$, i.e.,

$$(b_1, c_1) \in \beta \ \&\ b_1 \leq^R b'_0 \ \&\ c_0 \leq^L c_1. \tag{1β}$$

As $a \leq^L a_0$ and $b_1 \leq^R b'_0$, we have $(a_0, b_1) \leq^R (a, b'_0)$. But α is secured, so there is $(a_1, b'_1) \in \alpha$ for which $(a'_0, b_1) \leq^L (a_1, b'_1)$, i.e.

$$(a_1, b'_1) \in \alpha \ \&\ a_1 \leq^R a_0 \ \&\ b_1 \leq^L b'_1. \tag{1α}$$

From (1α) and (1β) we obtain:

$$a_0 \geq^R a_1 \ \alpha \ b'_1 \geq_L b_1 \ \beta \ c_1 \geq^L c_0 \tag{1}$$

This pattern in a_1, b'_1, b_1, c_1 must repeat by the next argument.

It follows from (1) that $(b'_1, c_1) \leq^R (b_1, c_1) \in \beta$. As β is secured, there is $(b_2, c_2) \in \beta$ for which $(b'_1, c_1) \leq^L (b_2, c_2)$, i.e.,

$$(b_2, c_2) \in \beta \ \& \ b_2 \leq^R b'_1 \ \& \ c_1 \leq^L c_2 \tag{2β}$$

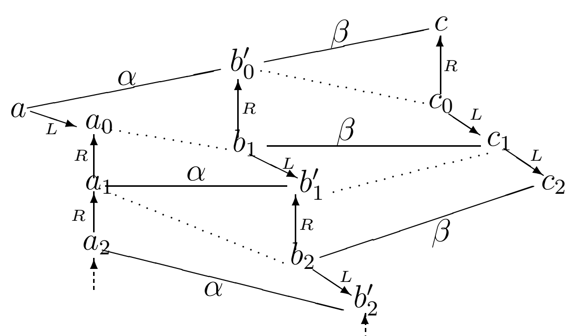
As $b_2 \leq^R b'_1$, we have $(a_1, b_2) \leq^R (a_1, b'_1) \in \alpha$. But α is secured, so there is $(a_2, b'_2) \in \alpha$ for which $(a_1, b_2) \leq^L (a_2, b'_2)$, i.e.

$$(a_2, b'_2) \in \alpha \ \& \ a_2 \leq^R a_1 \ \& \ b_2 \leq^L b'_2 \tag{2α}$$

and the pattern in (1) repeats in (2) below—got directly from (2α) and (2β):

$$a_0 \geq^R a_2 \ \alpha \ b'_2 \geq_L b_2 \ \beta \ c_2 \geq^L c_0 \tag{2}$$

where $b_2 \leq^R b'_1$. This can be repeated infinitely. In a diagram:



The chain

$$b'_0 \geq^R b_1 \leq^L b'_1 \geq^R b_2 \leq^L b'_2 \geq^R \cdots$$

must eventually be constant by Axiom 2(a) on bistructures. Hence we obtain

$$(a_n, b'_n) \in \alpha \ \& \ b_n = b'_n \ \& \ (b_n, c_n) \in \beta$$

making $(a_n, c_n) \in \beta \circ \alpha$ with $a_n \leq^R a_0$ and $c_0 \leq^L c_n$, i.e. $(a_0, b_0) \leq^L (a_n, c_n)$—and (a_n, c_n) fulfils the requirements we seek for (a', c'). $\square$

Using the same kind of well-foundedness argument as in the proof of Lemma 3 above, we can show:

Proposition 4 *Let F be a configuration of $E_0 \multimap E_1$ and x a configuration of E_0. Defining*

$$F.x = \{b \mid \exists a \in x.\ \ (a,b) \in F\}$$

yields a configuration of E_1. The function $x \mapsto F.x \ \ : \Gamma(E_0) \to \Gamma(E_1)$ is linear with respect to $\subseteq$ and continuous with respect to $\sqsubseteq$. [Not all such functions are represented, unless events of E_0 correspond to complete primes of $\Gamma(E_0)$.]

5 A model of classical linear logic

Here we give the constructions showing that **BS** is a model of classical linear logic. Define *linear negation*, the involution of linear logic, by

$$E^{\perp} = (E, \geq^R, \geq^L, \frown)$$

where $E = (E, \leq^L, \leq^R, \smile)$. Clearly $(E^{\perp})^{\perp} = E$. The remaining multiplicatives, $\wp$ (*par*)and $\otimes$ (*tensor*), are determined by the usual isomorphisms of classical linear logic

$$E_0 \wp E_1 \cong (E_0^{\perp} \multimap E_1) \qquad E_0 \otimes E_1 \cong (E_0 \multimap E_1^{\perp})^{\perp}$$

and have a form generalising those of Girard's constructions on coherence spaces. The construction $E^{\perp}$ is isomorphic to $(E \multimap \mathbf{1})$ where $\mathbf{1} = (\{\bullet\}, 1, 1, \emptyset)$ is the unit of $\otimes$. The *product* in the category **BS** is given by

$$E_0 \times E_1 = (E_0 \cup E_1, \leq^L, \leq^R, \smile)$$

obtained as the juxtaposition of the two bistructures, assumed disjoint, so for example $\smile = \smile_0 \cup \smile_1$. The coproduct $E_0 + E_1$, isomorphic to $(E_0^{\perp} \times E_1^{\perp})^{\perp}$, is again obtained as the disjoint juxtaposition of the two bistructures, but this time extending conflict across the two components.

To get the *exponential* $!E$, of a bistructure E, we first define orderings on the finite configurations $\Gamma(E)^0$. For $x, y \in \Gamma(E)^0$ define

$$\begin{aligned}
&x \sqsubseteq^R y \Leftrightarrow x \subseteq y \\
&x \sqsubseteq^L y \Leftrightarrow x \sqsubseteq y \ \& \ (\forall y' \in \Gamma(E).\ \ x \sqsubseteq y' \ \& \ y' \subseteq y \Rightarrow y = y').
\end{aligned}$$

Thus, $x \sqsubseteq^L y$ means y is a $\subseteq$-minimum configuration s.t. $x \sqsubseteq y$. Now define

$$!E = (\Gamma(E)^0, \sqsubseteq^L, \sqsubseteq^R, \not\uparrow^R)$$

where $x \uparrow^R y \Leftrightarrow_{def} \ \exists z \in \Gamma(E).\ x, y \subseteq z$.

Notation: Let $E = (E, \leq^L, \leq^R, \smile)$ be a bistructure.

Let $x \in \Gamma(E)$. Let $e \in E$ be in the $\leq$-downwards closure of x, i.e., $e \leq e'$, for some $e' \in x$. Factorising $\leq$, we obtain $e \leq^L e'' \leq^R e'$, and as x is a configuration there is a unique $e_{max} \in x$ such that $e'' \leq^L e_{max}$. This shows that any event e in the $\leq$-downwards closure of a configuration x is $\leq^L$-dominated by a unique event e_{max} in x. We write $m(e, x)$ for this event e_{max}.

For $x \in \Gamma(E)^0$, we define the relativised relation $\preceq_x$ as the reflexive, transitive closure of $\preceq^1_x$ where, for $e, e' \in x$,

$$e \preceq^1_x e' \Leftrightarrow_{def} \exists e''.\ e'' \leq^L e \ \&\ e'' \leq^R e'.$$

Lemma 5 *For a bistructure E, let $x, y \in \Gamma(E)^0$,*

$$(i)\ x \uparrow^R y \ \&\ e \in x \cap y \Rightarrow (\forall e' \in E.\ e' \preceq_x e \Leftrightarrow e' \preceq_y e)$$
$$(ii)\ x \sqsubseteq^L y \Leftrightarrow x \sqsubseteq y \ \&\ \forall e' \in y \exists e \in x.\ e' \preceq_y m(e, y).$$

Lemma 6 *$!E$ is a bistructure.*

Proof: The main difficulty in the proof is in showing that the relation $\preceq_! = (\sqsupseteq^L \cup \sqsubseteq^R)^*$ of $!E$ is a partial order. We need only show antisymmetry. Thus suppose for x_i, x'_i in $\Gamma(E)^0$ we have:

$$x_0 \sqsubseteq^R x'_0 \sqsupseteq^L x_1 \sqsubseteq^R x'_1 \sqsupseteq^L \cdots \sqsupseteq^L x_n \sqsubseteq^R x'_n \ \&\ x_n = x_0 \ \&\ x'_n = x'_0 \quad (1)$$

We shall show $x_i = x'_i = x_j = x'_j$ for all i, j. Then by the definition of $\preceq_!$ on $!E$ it follows that $\preceq_!$ is antisymmetric.

Define $fix = \bigcap_i x_i$. We first show $fix \in \Gamma(E)$. Consistency is obvious. We show that it is secured. Suppose $e \in fix$ and $\varepsilon \leq^R e$. Consider the chain (1). As $x_0 \sqsubseteq^R x'_0$ we have $m(\varepsilon, x_0) = m(\varepsilon, x'_0)$ by the consistency of x'_0. At the next link in the chain $x'_0 \sqsupseteq^L x_1$ with ε in the $\leq$-downwards closure of x'_0 and x_1 so $m(\varepsilon, x'_0) \geq^L m(\varepsilon, x_1)$ by the consistency of x'_0. Continuing in this way along the chain (1) we get:

$$m(\varepsilon, x_0) = m(\varepsilon, x'_0)\ \geq^L m(\varepsilon, x_1) = m(\varepsilon, x'_1) \geq^L m(\varepsilon, x_2) = m(\varepsilon, x'_2) \cdots \geq^L m(\varepsilon, x_n)$$

But $x_0 = x_n$ so $m(\varepsilon, x_0) = m(\varepsilon, x_n)$. As $\leq^L$ is a partial order, $m(\varepsilon, x_i) = m(\varepsilon, x_0)$ for all i. Thus $m(\varepsilon, x_0) \in fix$, so fix is secured.

Consequently $fix \in \Gamma(E)$ and clearly $fix \sqsubseteq^R \quad x_i, x_i'$ for all i. It remains to show $fix = x_i = x_i'$ for all i. By symmetry, it suffices to show $x_0 = x_0' = fix$.

Take $e \in x_0'$. Then by repeated use of Lemma 5(ii), characterising $\sqsubseteq^L$, we deduce from (1) that

$$e = e_0 \preceq_{x_0'} e_0' \geq^L e_1 \preceq_{x_1'} e_1' \geq^L e_2 \cdots \geq^L e_m \preceq_{x'_{[m]n}} e_m' \geq^L \dot{e}_{m+1} \cdots \qquad (2)$$

for some e_i in $x_{[i]_n}$ and e_i' in $x_{[i]_n}$ where $i \in \omega$ (here $[m]_n$ is m modulo n).

The sequence has been continued infinitely by going around and around the loop (1). As x_0 is finite and the sequence (2) visits x_0 infinitely often there must be e_m, e_q in x_0 such that $m < q$ and $[m]_n = [q]_n = 0$ and $e_m = e_q$. Then as $\preceq$ is a p.o., $e_m = e_m' = e_{m+1} = \cdots = e_q$. Thus $e_m \in fix$ so the sequence (2) eventually contains an element of fix. We know $fix \sqsubseteq^R x_i, x_i'$, for all i. Now working backwards along the chain (2), starting at e_m, it follows that all elements of the chain are in fix—at $\geq^L$-links this follows from the consistency of each x_i' and at $\preceq_{x_i'}$-links by Lemma 5(i). In particular, $e_0 (= e)$ must be the earliest element of (2) in fix. But e was chosen to be an arbitrary event in x_0'. Thus $x_0' = fix$. Therefore $x_0 = x_0' = fix$ as required. □

The bistructure, $!E_0 \multimap E_1$, has configurations in 1-1 correspondence with elements in the function space $[(\Gamma(E_0), \subseteq, \sqsubseteq) \to (\Gamma(E_1), \subseteq, \sqsubseteq)]$ in the cartesian-closed category of Berry's bidomains:

Proposition 7 *Let E_0, E_1 be bistructures. For $R \in \Gamma(!E_0 \multimap E_1)$ and $x \in \Gamma(E_0)$ define*

$$\bar{R}(x) = \{e \mid \exists x_0 \subseteq x.\ (x_0, e) \in R\}.$$

Then $\bar{R}$ is a function $\Gamma(E_0) \to \Gamma(E_1)$ which is continuous with respect to $\sqsubseteq$ and stable with respect to $\subseteq$ on configurations. In fact, $R \mapsto \bar{R}$ is a 1-1 correspondence between configurations of $!E_0 \multimap E_1$ and such functions.

Proof: First observe that the configurations of $!E_0$ are precisely those sets

$$x^+ = \{y \in \Gamma(E_0)^0 \mid y \subseteq x\}$$

for some $x \in \Gamma(E_0)$—again $\Gamma(E_0)^0$ denotes the finite configurations of E_0. Noting

$$\bar{R}(x) = R.(x^+), \text{ for } x \in \Gamma(E_0)$$

it follows by Proposition 4 that $\bar{R}(x) \in \Gamma(E_1)$, so that $\bar{R}$ is well-defined as a function $\Gamma(E_0) \to \Gamma(E_1)$. Moreover, the $\sqsubseteq$-continuity and $\subseteq$-stability of $\bar{R}$ follow directly from the corresponding properties of the linear function $x \mapsto R.x : \Gamma(!E_0) \to \Gamma(E_1)$.

We now construct an inverse to $R \mapsto \bar{R}$. Suppose $f : \Gamma(E_0) \to \Gamma(E_1)$ is $\sqsubseteq$-continuous and $\subseteq$-stable. Define $\varphi(f)$ to be the trace of f, i.e.

$$\varphi(f) = \{(x, e) \in \Gamma(E_0)^0 \times E_1 \mid x \text{ is minimal s.t. } e \in f(x)\}$$

We need first that $\varphi(f) \in \Gamma(!E_0 \multimap E_1)$, i.e. that $\varphi(f)$ is consistent and secured:

Consistent: Suppose $(x, e), (x', e') \in \varphi(f)$ and that $(x, e) \asymp (x', e')$i.e.$x \uparrow^R x'$ & $e \asymp e'$. We show $(x, e) = (x', e')$. As $e, e' \in f(x \cup x')$, we must have $e \frown e'$, which combined with $e \asymp e'$, entails $e = e'$. Now, $(x, e), (x', e)$ are both in $\varphi(f)$ the trace of f. Because $x \uparrow^R x'$ and f is $\subseteq$-stable we conclude that $x = x'$.

Secured: Suppose $(x', e') \leq^R (x, e) \in \varphi(f)$. Then $x \sqsubseteq^L x'$ and $e' \leq^R e$. As f is $\sqsubseteq$-monotonic, $f(x) \sqsubseteq f(x')$. Because $e' \leq^R e$ and $e \in f(x)$, we see that e' is in the $\leq$-downwards closure of $f(x')$. Thus

$$e' \leq^L e'' \tag{1}$$

where $e'' = m(e', f(x')) \in f(x)$. As f is $\subseteq$-stable, there is

$$x_0 \subseteq x' \tag{2}$$

with (x_0, e'') in its trace, i.e.

$$(x_0, e'') \in \varphi(f) \tag{3}$$

Combining (1), (2), (3) we obtain, as required

$$(x', e') \leq^L (x_0, e'') \in \varphi(f).$$

For f a $\sqsubseteq$-continuous, $\subseteq$-stable function $\Gamma(E_0) \to \Gamma(E_1)$ its continuity with respect to $\subseteq$ entails $\overline{\varphi(f)} = f$. For $R \in \Gamma(!E_0 \multimap E_1)$ a direct translation of the definitions yields $\varphi(\bar{R}) = R$. Thus the map $R \mapsto \bar{R} = R$ is a 1-1 correspondence. $\square$

More completely, this section provides the key constructions in showing:

Theorem 8 *The category* **BS** *forms a linear category in the sense of [9]. The exponential* ! *forms a comonad on the category* **BS**. *Together they form a model of classical linear logic (a Girard category in the sense of [9]). The associated co-Kleisli category is equivalent to a cartesian-closed full subcategory of Berry's bidomains, where morphisms are continuous with respect to the Scott order and stable with respect to the Berry order.*

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