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Bootstrapping the Primitive Recursive Functions by 47 Colours

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Abstract

I construct a concrete colouring of the 3 element subsets of $\mathbb{N}$. It has the property that each homogeneous set $\{s_0, s_1, s_2.., s_r\}, r \geq s_0 - 1$ has its elements spread so much apart that $F_\omega(s_i) < s_{i+1}$ for $i = 1, 2, ..., r-1$. It uses only 47 colours. This is more economical than the approximately 160000 colours used in [1].

1 Introduction and preliminaries

In the famous paper [2] L.Harrington and J.Paris showed that a certain finitary version **PH** of Ramseys Theorem is true, but unprovable in the celebrated system of Peanos Arithmetic. This is an example of Gödels incompleteness theorem. However, unlike Gödels consistency statement **PH** has generally been accepted to be a natural statement from Arithmetic. In

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[1] Ketonen and Solovay gave a careful analysis of the underlying growth-rate of **PH**. As a first step in this analysis it was shown that for each increasing primitive recursive function f there exists n and a colouring of the 3 element subsets of $\{n, n+1, n+2, ..., f(n)\}$ such that there are no homogeneous sets $\{s_0, s_1, s_2, ..., s_r\}$ with $r \geq s_0 - 1$. The real point is that the number of colours can always be chosen to be less than a number fixed in advance. Ketonen and Solovay defined various algebras and took a series of products, in order to obtain the required colouring. An examination of their proof shows that they used approximately 160000 colours. However they clearly did not try to be economical. Actually in the work of Ketonen and Solovay the important point is that the number is finite. In this paper I construct a concrete colouring which uses only 47 colours.

Recall that the first functions in the Wainer hierarchy [3] are defined by $F_0(n) := n+1$, $F_k^1(n) := F_k(n)$, $F_k^{m+1}(x) := F_k^m(F_k(n))$, $F_{k+1}(n) := F_k^n(n)$, $F_\omega(n) := F_n(n)$. The function F_ω is the first function in this hierarchy which growth faster than each primitive recursive function.

Let $S^{[k]}$ denote the collection of k element subsets of S. We use the convention that the elements in displayed in sets $S = \{s_0, s_1, ..., s_r\} \subseteq \mathbb{N}$ are listed after size (i.e. $s_0 < s_1 < ....s_r$). Let $g : \mathbb{N}^{[k]} \rightarrow C$. We say that $S \subseteq \mathbb{N}$ is homogeneous (for g) if $u \geq k+1$ and g takes a constant value on $S^{[k]}$. The elements in C are called *colours.* If $g_1 : \mathbb{N}^{[k]} \rightarrow C_1, g_2 : \mathbb{N}^{[k]} \rightarrow C_2, ..., g_u : \mathbb{N}^{[k]} \rightarrow C_u$ we define the *product colouring* $g := g_1 \times g_2 \times ... \times g_r$ as the product map $g : \mathbb{N}^{[k]} \rightarrow C_1 \times C_2 \times ... \times C_u$. Notice that S is homogeneous for g if and only if S is homogeneous for all the maps $g_1, ..., g_u$.

2 Definition of the colouring

Let $j(x, y)$ be the smallest j such that $y \leq F_j(x)$. Consider the following 7 open propositions:

$\psi_1(\{x_0, x_1\}) := x_1 \leq F_\omega(x_0)$

$\psi_2(\{x_0, x_1\}) := j(x_0, x_1) > x_0$
$\psi_3(\{x_0, x_1\}) := j(x_0, x_1) \geq \lfloor \frac{x_0}{2} \rfloor$
$\psi_4(\{x_0, x_1, x_2\}) := j(x_0, x_1) \neq j(x_0, x_2)$
$\psi_5(\{x_0, x_1\}) := x_1 < F_{j-1}^{x_0-1}(x_0)$ where $j := j(x_0, x_1)$.
$\psi_6(\{x_0, x_1, x_2\}) := j(x_0, x_1) > j(x_1, x_2)$
$\psi_7(\{x_0, x_1\}) := j(x_0, x_1) \geq 2$.

Now we define 7 auxiliary colourings $h_1, h_2, ..., h_7$ as follows. The colouring $h_i : \mathbb{N}^{[2]} \to \{0, 1\};\ i = 1, 2, 3, 5, 7$ takes the value 1 exactly when ψ_i holds. The colouring $h_j : \mathbb{N}^{[3]} \to \{0, 1\};\ j = 4, 6$ takes the value 1 exactly when ψ_j holds.

Lemma: *Suppose that* $S = \{s_0, s_1, ..., s_r\} \subseteq \mathbb{N}$ *contains at least* s_0 *elements,* $s_0 \geq 5$ *and* S *is homogeneous for the colourings* $h_1, h_2, ..., h_7$*. Then* $F_\omega(s_i) < s_{i+1}$ *for* $i = 1, 2, ..., r - 1$.

Proof:

(1) If $h_1 \equiv 0$ on $S^{[2]}$ then $F_\omega(s_i) < s_{i+1}$ for $i = 0, 1, 2, ..., r - 1$. This is what we want to show.

(2) So assume that $h_1 \equiv 1$ on $S^{[2]}$. According to the definition $F_\omega(x) := F_x(x)$. So $s_{i+1} \leq F_\omega(s_i) = F_{s_i}(s_i),\ i = 0, 1, 2, ..., r - 1$.

(3) For $i = 0$ this gives $s_1 \leq F_{s_0}(s_0)$.

(4) According to the definition $j(s_0, s_1) \leq s_0$.

(5) This shows that $h_2 \equiv 0$ on $S^{[2]}$.

In particular $j(s_0, s_1), j(s_0, s_2), ..., j(s_0, s_r) \leq s_0$.

(6) Now whether $h_3 \equiv 0$ or $h_3 \equiv 1$ on $S^{[2]}$ by (5) we know that $j(s_0, s_1), j(s_0, s_2), ..., j(s_0, s_r)$ takes at most $\lfloor \frac{s_0}{2} \rfloor + 1$ different values.

(7) Now $h_4 \equiv 0$ on $S^{[3]}$, because otherwise $j(s_0, s_1), j(s_0, s_2), ..., j(s_0, s_r)$ would all take different values. This is impossible because $r \geq s_0 - 1 > \lfloor \frac{s_0}{2} \rfloor + 1$ and $s_0 \geq 5$.

(8) But if $h_4 \equiv 0$ on $S^{[3]}$, then $j(s_0, s_1) = j(s_0, s_2) = ... = j(s_0, s_r)$. Let j_0 denote this value.

(9) The value j_0 cannot be 0, because then according to the definition of $j(s_0, s_r)$ we would have $s_0 + 4 \leq s_r \leq F_0(s_0) = s_0 + 1$.

(10) According to (9) $j_0 > 0$. By the definition of j_0 we have $F_{j_0-1}(s_0) < s_i \leq F_{j_0}(s_0)$ when $i = 0, 1, .., r$.

(11) Now h_6 cannot take the value 1 on $S^{[3]}$. To see this suppose that $h_6 \equiv 1$ on $S^{[3]}$. Then $s_0 \geq j(s_0, s_1) > j(s_1, s_2) > .... > j(s_{r-1}, s_r)$ and especially $j(s_0, s_1) > 2$. Then by the definition of h_7 this would have the consequence that $j(s_{r-1}, s_r) > 2$. But this is a contradiction because: $j(s_0, s_1) \geq j(s_{r-1}, s_r) + r - 1$, so $j(s_0, s_1) \geq r + 1 > s_0 \geq j(s_0, s_1)$.

(12) So $h_6 \equiv 0$ on $S^{[3]}$. In particular $j_0 = j(s_0, s_1) \leq j(s_1, s_2) \leq ... \leq j(s_{r-1}, s_r)$.

(13) According to (12) $F_{j_0-1}(s_i) \leq F_{j(s_i,s_{i+1})}(s_i)$. The definition of the function j shows that $F_{j(s_i,s_{i+1})-1}(s_i) < s_{i+1}$.

Combining this shows that $F_{j_0-1}(s_i) < s_{i+1}$.

(14) According to (13) $s_r > F_{j_0-1}(s_{r-1}) > F_{j_0-1}(F_{j_0-1}(s_{r-2})) > ... > F^{(r)}_{j_0-1}(s_0)$.

(15) Now $r \geq s_0 - 1$ so by (14) $s_r > F^{(s_0-1)}_{j_0-1}(s_0)$ so $h_5(\{s_0, s_r\}) = 0$.

(16) So $h_5 \equiv 0$ on $S^{[2]}$, and then $s_{i+1} > F^{(s_i-1)}_{j(s_i,s_{i+1})}(s_i),\ i = 0, 1, 2, ..., r - 1$.

(17) Now $s_{i-1} \geq s_0 + 1$ so according to (12) $j(s_i, s_{i+1}) \geq j_0$, and thus $F^{(s_i-1)}_{j(s_i,s_{i+1})-1}(s_i) \geq F^{(s_0-1)}_{j_0-1}(s_i)$.

(18) This shows that $s_r > F^{s_0-1}_{j_0-1}(s_{r-1}) > ... > F^{(r\cdot(s_0-1))}_{j_0-1}(s_0)$.

(19) Now $r \cdot (s_0 - 1) > s_0 + 1$ $(s_0 \geq 5)$ so $s_r > F^{(s_0+1)}_{j_0-1}(s_0) = F_{j_0}(s_0)$. This shows that $j(s_0, s_r) > j_0$ which violates (8) $j(s_0, s_r) = j_0$.

(20) The contradiction in (19) shows that the assumption in (2) is impossible. Thus $h_1 \equiv 0$ and we are back to (1). $\square$

Lemma: *There is a colouring* $U : \mathbb{N}^{[3]} \to \{1, 2, ..., 44\}$ *using* 44 *different colours such that if* S *is homogeneous for* h *then* S *is simultaneously homogeneous for the maps* $h_1, h_2, ..., h_7$

Proof: Now $1 + 5 \cdot 2 = 11$ so by [1] there exists a colouring $U_1 : \mathbb{N}^{[3]} \to \{1, 2, ..., 11\}$ such that if S is homogeneous for U_1 then S is simultaneously homogeneous for h_1, h_2, h_3, h_5 and h_7. Now let $U : \mathbb{N}^{[3]} \to \{1, 2, ..., 11\} \times \{0, 1\} \times \{0, 1\}$ be the product of U_1, h_4 and h_6. It uses 44 colours. $\square$

Theorem: *There is a colouring* $W : \mathbb{N}^{[3]} \to \{1, 2, ..., 47\}$ *such that if* $S := \{s_0, ..., s_u\}$ *is homogeneous for* W *then* $F_\omega(s_i) < s_{i+1}$.

Proof: Define W as U except that $W(\{s_0, s_1, s_2\})$ gets colour 45 if $s_0 < 5$ and $s_1 \geq 5$ or $s_0, s_1, s_2 < 5$ and $s_2 = 4$, and colour 46 if $s_0, s_1 < 5$ and $s_2 \geq 5$, and colour 47 if $s_0, s_1, s_2 < 5$ and $s_2 \neq 4$. It is straightforward to show that any set $S := \{s_0, s_1, s_2, s_3\}$ which is homogeneous for W must have $s_0 \geq 5$.

3 Final remarks and open questions

There is no reason to believe that 47 is a natural constant. Actually by a slight change in the problem I can show that 12 colours suffice. This suggests that the following question might be critical:

Problem 1: *Is it possible to use only* 12 *colours?*

One can also ask for the asymptotic answer. Here I think the critical question could be whether:

Problem 2: *Is it possible to use only* 3 *colours?*

To my knowledge the 47 colours used in this paper provides the best known lower bound to both of these questions.

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