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Introduction to Linear Logic

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Preface

The main concern of this report is to give an introduction to Linear Logic. For pedagogical purposes we shall also have a look at Classical Logic as well as Intuitionistic Logic. Linear Logic was introduced by J.-Y. Girard in 1987 and it has attracted much attention from computer scientists, as it is a logical way of coping with resources and resource control. The focus of this technical report will be on proof-theory and computational interpretation of proofs, that is, we will focus on the question of how to interpret proofs as programs and reduction (cut-elimination) as evaluation. We first introduce Classical Logic. This is the fundamental idea of the proofs-as-programs paradigm. Cut-elimination for Classical Logic is highly non-deterministic; it is shown how this can be remedied either by moving to Intuitionistic Logic or to Linear Logic. In the case on Linear Logic we consider Intuitionistic Linear Logic as well as Classical Linear Logic. Furthermore, we take a look at the Girard Translation translating Intuitionistic Logic into Intuitionistic Linear Logic. Also, we give a brief introduction to some concrete models of Intuitionistic Linear Logic. No proofs will be given except that a proof of cut-elimination for the multiplicative fragment of Classical Linear Logic is included in an appendix.

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Chapter 1

Classical and Intuitionistic Logic

This chapter introduces Classical Logic and Intuitionistic Logic. Also, the Curry-Howard interpretation of Intuitionistic Logic, the λ-calculus, is dealt with.

1.1 Classical Logic

The presentation of Classical Logic given in this section is based on the book [GLT89]. Formulas of Classical Logic are given by the grammar

$$s \ ::= \ 1 \mid s \wedge s \mid 0 \mid s \vee s \mid s \Rightarrow s.$$

The meta-variables A, B, C range over formulae. Proof-rules for a Gentzen style presentation of the logic are given in Appendix A.1; they are used to derive sequents

$$A_1, ..., A_n \vdash B_1, ..., B_m.$$

Such a sequent amounts to the formula expressing that the conjunction of $A_1, ..., A_n$ implies the disjunction of $B_1, ..., B_m$. The meta-variables Γ, Δ range over lists of formulae and π, τ range over derivations as well as proofs. The Γ and Δ parts of a sequent $\Gamma \vdash \Delta$ are called *contexts*. The presence of contraction and weakening proof-rules allows us to consider the contexts of a sequent as *sets* of formulae rather than *multisets* of formulae, which is a feature distinguishing Classical Logic and Intuitionistic Logic from Linear Logic. The Gentzen style proof-rules were originally introduced in [Gen34]. This presentation is characterised by the presence of two different forms of rules for each connective, depending on which side of the turnstile the involved connective is introduced. Note that in Appendix A.1 the rules that introduce a connective on the left hand side have been positioned in the left hand side column, and similarly, the rules that introduce a connective on the right hand side have been positioned in the right hand side column. Note also that the rules for conjunction are symmetric to those for disjunction.

In the system above, negation is defined as $\neg A = A \Rightarrow 0$. An alternative formulation of Classical Logic can be obtained by leaving out implication and having negation as a builtin connective together with the proof-rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\,\neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\,\neg_R$$

Implication is then defined as $A \Rightarrow B = \neg A \vee B$. We would then have a perfectly symmetric system. However, we have chosen the system of Appendix A.1 with the aim of making clear the connection to Intuitionistic Logic.

One of the most important properties of the proof-rules for Classical Logic is that the cut-rule is redundant; this was originally proved by Gentzen in [Gen34]. The idea is that an application of the cut rule can either be pushed upwards in the surrounding proof or it can be replaced by cuts involving simpler formulae. The latter situation amounts to the following key-cases in which the cut formula is introduced in the last used rules of both immediate subproofs:

- The $(\wedge_R, \wedge_{L1})$ key-case

$$\frac{\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\vdots\,\pi_2}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \qquad \dfrac{\dfrac{\vdots\,\pi'_1}{\Gamma'', A \vdash \Delta''}}{\Gamma'', A \wedge B \vdash \Delta''}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\vdots\,\pi'_1}{\Gamma'', A \vdash \Delta''}}{\Gamma'', \Gamma \vdash \Delta'', \Delta}}{\overline{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta', \Delta}}$$

- The $(\wedge_R, \wedge_{L2})$ key-case

$$
\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\vdots\ \pi_2}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \quad \dfrac{\dfrac{\vdots\ \pi_1'}{\Gamma'', B \vdash \Delta''}}{\Gamma'', A \wedge B \vdash \Delta''}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\vdots\ \pi_2}{\Gamma' \vdash B, \Delta'} \quad \dfrac{\vdots\ \pi_1'}{\Gamma'', B \vdash \Delta''}}{\Gamma'', \Gamma' \vdash \Delta'', \Delta'}}{\overline{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta', \Delta}}
$$

- The $(1_R, 1_L)$ key-case

$$
\dfrac{\dfrac{}{\vdash 1} \quad \dfrac{\dfrac{\vdots\ \pi_1'}{\Gamma' \vdash \Delta'}}{\Gamma', 1 \vdash \Delta'}}{\Gamma' \vdash \Delta'}
\quad \rightsquigarrow \quad
\dfrac{\vdots\ \pi_1'}{\Gamma' \vdash \Delta'}
$$

- The $(\Rightarrow_R, \Rightarrow_L)$ key-case

$$
\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\Gamma, A \vdash B, \Delta}}{\Gamma \vdash A \Rightarrow B, \Delta} \quad \dfrac{\dfrac{\vdots\ \pi_1'}{\Gamma' \vdash A, \Delta'} \quad \dfrac{\vdots\ \pi_2'}{\Gamma'', B \vdash \Delta''}}{\Gamma', \Gamma'', A \Rightarrow B \vdash \Delta'', \Delta'}}{\Gamma', \Gamma'', \Gamma \vdash \Delta'', \Delta', \Delta}
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\vdots\ \pi_1'}{\Gamma' \vdash A, \Delta'} \quad \dfrac{\dfrac{\vdots\ \pi_1}{\Gamma, A \vdash B, \Delta} \quad \dfrac{\vdots\ \pi_2'}{\Gamma'', B \vdash \Delta''}}{\Gamma'', \Gamma, A \vdash \Delta'', \Delta}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}
$$

A double bar denotes a number of applications of rules. We also have $(\vee_{R1}, \vee_L)$, $(\vee_{R2}, \vee_L)$, and $(0_R, 0_L)$ key-cases, but they are left out as they are symmetric to the mentioned $(\wedge_R, \wedge_{L1})$, $(\wedge_R, \wedge_{L2})$, and $(1_R, 1_L)$ cases, respectively. We omit the full cut-elimination proof here; the reader is referred to [GLT89] for the details. A notable feature of the system is that all formulae occuring in a cut-free proof are subformulae of the formulae occuring in the end-sequent. This is called the *subformula* property.

The fundamental idea in the proofs-as-programs paradigm is to consider a proof as a program and reduction of the proof (cut-elimination) as evaluation of the program. This makes it desirable that the same reduced proof is obtained independent of the choices of reductions. However, this is not possible with Classical Logic where cut-elimination is highly non-deterministic, as pointed out in [GLT89]. The problem is witnessed by the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash A, \Delta} \quad \dfrac{\begin{matrix}\vdots\,\pi'\\ \Gamma' \vdash \Delta'\end{matrix}}{\Gamma', A \vdash \Delta'}}{\Gamma', \Gamma \vdash \Delta', \Delta}$$

which, because of its symmetry, can be reduced to both of the proofs

$$\begin{array}{c}\vdots\,\pi\\ \Gamma \vdash \Delta\\ \hline\hline \Gamma', \Gamma \vdash \Delta', \Delta\end{array} \qquad \begin{array}{c}\vdots\,\pi'\\ \Gamma' \vdash \Delta'\\ \hline\hline \Gamma', \Gamma \vdash \Delta', \Delta\end{array}$$

The example implies that Classical Logic as given above has no non-trivial sound denotational semantics; all proofs of a given sequent will simply have the same denotation. This deficiency can be remedied by breaking the symmetry; two ways of doing so can be pointed out:

- Each right hand side context is subject to the restriction that it has to contain exactly one formula. This amounts to Intuitionistic Logic which will be dealt with in Section 1.2.

- Contraction and weakening is marked explicitly using additional modalities ! and ? on formulae. The !-modality corresponds to contraction and weakening on the left hand side, and similarly, the ?-modality corresponds to contraction and weakening on the right hand side. This amounts to Classical Linear Logic which will be dealt with in Section 2.1.

This dichotomy goes back to [GLT89]. Since the publication of this book a considerable amount of work has been devoted to giving Classical Logic a constructive formulation in the sense that proofs can be considered as programs. This has essentially been achieved by "decorating" formulas with information controlling the process of cut-elimination. The work of Parigot, [Par91, Par92], Ong, [Ong96] and Girard, [Gir91] seems especially promising. A notable feature of the latter paper is the presentation of a categorical model of Classical Logic where A is not isomorphic to $\neg\neg A$. Thus, the dichotomy above should not be considered as excluding other solutions. The lesson to learn is that constructiveness, in the sense that proofs can be considered as programs, is not a property of certain logics, but rather a property of certain *formulations* of logics.

1.2 Intuitionistic Logic

The presentation of Intuitionistic Logic given in this section is based on the book [GLT89]. Formulae of Intuitionistic Logic are the same as the formulae of Classical Logic. The proof-rules of Intuitionistic Logic in Gentzen style occur as those of Classical Logic given in Appendix A.1 where $\vee_L$ is written

$$\frac{\Gamma, A \vdash C \qquad \Gamma', B \vdash C}{\Gamma, \Gamma', A \vee B \vdash C}\ \vee_L$$

and the remaining rules are subject to the restriction that each right hand side context contains exactly one formula.

We shall here consider also an equivalent Natural Deduction presentation of Intuitionistic Logic which has cleaner dynamic properties than the presentation in Gentzen style. Proof-rules for this formulation of the logic are given in Appendix A.2; they are used to derive sequents

$$A_1, ..., A_n \vdash B.$$

The Natural Deduction style proof-rules were originally introduced by Gentzen in [Gen34] and later considered by Pravitz in [Pra65]. This style of presentation is characterised by the presence of two different forms of rules for each connective, namely *introduction* and *elimination* rules. Note that in Appendix A.2 the introduction rules have been positioned in the left hand side column, and the elimination rules have been positioned in the right hand side column. Note also that the contraction and weakening proof-rules are explicitly part of the Gentzen style formulation whereas they are admissible in the Natural Deduction formulation.

A notable feature of Intuitionistic Logic is the so-called Brouwer-Heyting-Kolmogorov functional interpretation where formulae are interpreted by means of their proofs:

- A proof of a conjunction $A \wedge B$ consists of a proof of A together with a proof of B,

- a proof of an implication $A \Rightarrow B$ is a function from proofs of A to proofs of B,

- a proof of a disjunction $A \vee B$ is either a proof of A or a proof of B together with a specification of which of the disjuncts is actually proved.

The proof-rules for Intuitionistic Logic can then be considered as methods for defining functions such that a proof of a sequent $\Gamma \vdash B$ gives rise to a function which assigns a proof of the formula B to a list of proofs proving the respective formulae in the context Γ. Note that *tertium non datur*, $A \vee \neg A$, which distinguishes Classical Logic from Intuitionistic Logic, cannot be interpreted in this way. It turns out that the λ-calculus is an appropriate language for expressing the Brouwer-Heyting-Kolmogorov interpretation. We shall come back to the λ-calculus in the next section, and in Section 1.4 we will introduce the Curry-Howard isomorphism that makes explicit the relation between the λ-calculus and Intuitionistic Logic.

Now, a Natural Deduction proof may be rewritten into a simpler form using a reduction rule. Reduction of a Natural Deduction proof corresponds to cut-eliminating in a Gentzen style formulation. The reduction rules are as follows:

- The $(\wedge_I, \wedge_{E1})$ case

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\vdash B\end{matrix}}{\Gamma\vdash A\wedge B}}{\Gamma\vdash A}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}$$

- The $(\wedge_I, \wedge_{E2})$ case

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}\qquad\begin{matrix}\vdots\\ \Gamma\vdash B\end{matrix}}{\Gamma\vdash A\wedge B}}{\Gamma\vdash B}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \Gamma\vdash B\end{matrix}$$

- The $(\Rightarrow_I, \Rightarrow_E)$ case

$$\dfrac{\dfrac{\begin{matrix}\overline{\Gamma, A, \Lambda\vdash A}\\ \vdots\\ \Gamma, A\vdash B\end{matrix}}{\Gamma\vdash A\Rightarrow B}\qquad\begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash B}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \Gamma, \Lambda\vdash A\\ \vdots\\ \Gamma\vdash B\end{matrix}$$

- The $(\vee_{I1}, \vee_E)$ case

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash A\vee B}\qquad\begin{matrix}\overline{\Gamma, A, \Lambda\vdash A}\\ \vdots\\ \Gamma, A\vdash C\end{matrix}\qquad\begin{matrix}\overline{\Gamma, B, \Delta\vdash B}\\ \vdots\\ \Gamma, B\vdash C\end{matrix}}{\Gamma\vdash C}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \Gamma, \Lambda\vdash A\\ \vdots\\ \Gamma\vdash C\end{matrix}$$

- The $(\vee_{I2}, \vee_E)$ case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \vee B} \quad \begin{array}{c}\overline{\Gamma, A, \Lambda \vdash A}\\ \vdots\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\overline{\Gamma, B, \Delta \vdash B}\\ \vdots\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ \Gamma, \Delta \vdash B\\ \vdots\\ \Gamma \vdash C\end{array}$$

Note how a reduction rule removes a "detour" in the proof created by the introduction of a connective immediately followed by its elimination.

The Natural Deduction presentation of Intuitionistic Logic satisfies the *Church-Rosser* property which means that whenever a proof π reduces to π' as well as to π'', there exists a proof π''' to which both of the proofs π' and π'' reduce, and moreover, it satisfies the *strong normalisation* property which means that all reduction sequences originating from a given proof are of finite length. Church-Rosser and strong normalisation implies that any proof π reduces to a unique proof with the property that no reductions can be applied; this is called the *normal form* of π. Via the Curry-Howard isomorphism this corresponds to analogous results for reduction of terms of the λ-calculus which we will come back to in the next two sections.

1.3 The λ-Calculus

The presentation of the λ-calculus given in this section is based on the book [GLT89]. In the next section we shall see how the λ-calculus occurs as a Curry-Howard interpretation of Intuitionistic Logic. Note that we consider products and sums as part of the λ-calculus; this convention is not followed by all authors. Types of the λ-calculus are given by the grammar

$$s \ ::= \ 1 \mid s \times s \mid s \Rightarrow s \mid 0 \mid s + s$$

Figure 1.1: Type Assignment Rules for the λ-Calculus

$$\frac{}{x_1 : A_1, ..., x_n : A_n \vdash x_q : A_q}$$

$$\frac{}{\Gamma \vdash \mathtt{true} : 1} \qquad \frac{\Gamma \vdash u : A \quad \Gamma \vdash v : B}{\Gamma \vdash (u, v) : A \times B} \qquad \frac{\Gamma \vdash u : A \times B}{\Gamma \vdash \mathtt{fst}(u) : A} \qquad \frac{\Gamma \vdash u : A \times B}{\Gamma \vdash \mathrm{snd}(u) : B}$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A.u : A \Rightarrow B} \qquad \frac{\Gamma \vdash f : A \Rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash fu : B}$$

$$\frac{\Gamma \vdash w : 0}{\Gamma \vdash \mathtt{false}^C(w) : C} \qquad \frac{\Gamma \vdash u : A}{\Gamma \vdash \mathtt{inl}^{A+B}(u) : A + B} \qquad \frac{\Gamma \vdash u : B}{\Gamma \vdash \mathtt{inr}^{A+B}(u) : A + B}$$

$$\frac{\Gamma \vdash w : A + B \quad \Gamma, x : A \vdash u : C \quad \Gamma, y : B \vdash v : C}{\Gamma \vdash \mathtt{case}\ w\ \mathtt{of}\ \mathtt{inl}(x).u \mid \mathtt{inr}(y).v : C}$$

and terms are given by the grammar

$$\begin{array}{lll} t & ::= & x \mid \\ & & \mathtt{true} \mid (t, t) \mid \mathtt{fst}(t) \mid \mathtt{snd}(t) \mid \\ & & \lambda x^A.t \mid tt \mid \\ & & \mathtt{false}^C(t) \mid \mathtt{inl}^{A+B}(t) \mid \mathtt{inr}^{A+B}(t) \mid \mathtt{case}\ t\ \mathtt{of}\ \mathtt{inl}(x).t \mid \mathtt{inr}(y).t \end{array}$$

where x is a variable ranging over terms. The set of free variables, denoted $FV(u)$, of a term u is defined by structural induction on u as follows:

$$\begin{array}{rcl}
FV(x) & = & \{x\} \\
FV(\texttt{true}) & = & \emptyset \\
FV((u,v)) & = & FV(u) \cup FV(v) \\
FV(\texttt{fst}(u)) & = & FV(u) \\
FV(\lambda x.u) & = & FV(u) - \{x\} \\
FV(fu) & = & FV(f) \cup FV(u) \\
FV(\texttt{false}(u)) & = & FV(u) \\
FV(\texttt{inl}(u)) & = & FV(u) \\
FV(\texttt{case}\ w\ \texttt{of}\ \texttt{inl}(x).u|\texttt{inr}(y).v) & = & FV(w) \cup FV(u) - \{x\} \cup FV(v) - \{y\}
\end{array}$$

We say that a term u is *closed* iff $FV(u) = \emptyset$. We also say that the variable x is *bound* in the term $\lambda x.u$. A similar remark applies to the case construction. We need a convention dealing with substitution: If a term v together with n terms $u_1, ..., u_n$ and n pairwise distinct variables $x_1, ..., x_n$ are given, then $v[u_1, ..., u_n/x_1, ..., x_n]$ denotes the term v where simultaneously the terms $u_1, ..., u_n$ have been substituted for free occurrences of the variables $x_1, ..., x_n$ such that bound variables in v have been renamed to avoid capture of free variables of the terms $u_1, ..., u_n$. Occasionally a list $u_1, ..., u_n$ of n terms will be denoted $\overline{u}$ and a list $x_1, ..., x_n$ of n pairwise distinct variables will be denoted $\overline{x}$. Given the definition of free variables above, it should be clear how to formalise substitution.

Rules for assignment of types to terms are given in Figure 1.1. Type assignments have the form of sequents

$$x_1 : A_1, ..., x_n : A_n \vdash u : B$$

where $x_1, ..., x_n$ are pairwise distinct variables. It can be shown by induction on the derivation of the type assignment that

$$FV(u) \subseteq \{x_1, ..., x_n\}.$$

The λ-calculus satisfies the following properties:

Lemma 1.3.1 If the sequent $\Gamma \vdash u : A$ is derivable, then for any derivable sequent $\Gamma \vdash u : B$ we have $A = B$.

Proof: Induction on the derivation of $\Gamma \vdash u : A$. □

The following proposition is the essence of the Curry-Howard isomorphism:

Proposition 1.3.2 If the sequent $\Gamma \vdash u : A$ is derivable, then the rule instance above the sequent is uniquely determined.

Proof: Use Lemma 1.3.1 to check each case. □

We need a small lemma dealing with expansion of contexts.

Lemma 1.3.3 If the sequent $\Delta, \Lambda \vdash u : A$ is derivable and the variables in the contexts Δ, Λ and Γ are pairwise distinct, then the sequent $\Delta, \Gamma, \Lambda \vdash u : A$ is also derivable.

Proof: Induction on the derivation of $\Delta, \Lambda \vdash u : A$. □

Now comes a lemma dealing with substitution.

Lemma 1.3.4 (Substitution Property) If both of the sequents $\Gamma \vdash u : A$ and $\Gamma, x : A, \Lambda \vdash v : B$ are derivable, then the sequent $\Gamma, \Lambda \vdash v[u/x] : B$ is also derivable.

Proof: Induction on the derivation of $\Gamma, x : A, \Lambda \vdash v : B$. We need Lemma 1.3.3 for the case where the derivation is an axiom

$$\frac{}{x_1 : A_1, ..., x_n : A_n \vdash x_q : A_q}$$

such that the variable x is equal to x_q. □

The λ-calculus has the following β-reduction rules each of which is the image under the Curry-Howard isomorphism of a reduction on the proof corresponding to the involved term:

$$\begin{array}{rcl}
\mathtt{fst}((u,v)) & \leadsto & u \\
\mathrm{snd}((u,v)) & \leadsto & v \\
(\lambda x.u)w & \leadsto & u[w/x] \\
\mathtt{case}\ \mathtt{inl}(w)\ \mathtt{of}\ \mathtt{inl}(x).u|\mathtt{inr}(y).v & \leadsto & u[w/x] \\
\mathtt{case}\ \mathtt{inr}(w)\ \mathtt{of}\ \mathtt{inl}(x).u\ |\mathtt{inr}(y).v & \leadsto & v[w/y]
\end{array}$$

We shall not be concerned with η-reductions or commuting conversions. The properties of Church-Rosser and strong normalisation for proofs of Intuitionistic Logic correspond to analogous notions for terms of the λ-calculus via the Curry-Howard isomorphism, and in [LS86] it is shown that these properties are indeed satisfied. First strong normalisation is proved. By König's Lemma, this implies that any term t is *bounded*, that is, there exists a number n such that no sequence of one-step reductions originating from t has more than n steps. Given the result that all terms are bounded, Church-Rosser is proved by induction on the bound.

1.4 The Curry-Howard Isomorphism

The original Curry-Howard isomorphism, [How80], relates the Natural Deduction formulation of Intuitionistic Logic to the λ-calculus; formulae correspond to types, proofs to terms, and reduction of proofs to reduction of terms. This is dealt with in [GLT89] and in [Abr90]; the first emphasises the logic side of the isomorphism, the second the computational side. In what follows, we will consider the Natural Deduction presentation of Intuitionistic Logic given in Appendix A.2. The relation between formulae of Intuitionistic Logic and types of the λ-calculus is obvious. The idea of the Curry-Howard isomorphism on the level of proofs is that proof-rules can be "decorated" with terms such that the term induced by a proof encodes the proof. In the case of Intuitionistic Logic an appropriate term language for this purpose is the λ-calculus. If we decorate the proof-rules of Intuitionistic Logic with terms in the appropriate way we get the rules for assigning types to terms of the λ-calculus, and moreover, if we take the typing rules of the λ-calculus and remove the variables and terms we can recover the proof-rules. We get the Curry-Howard isomorphism on the level of proofs as follows: Given a proof of the sequent $A_1, ..., A_n \vdash B$, that is, a proof of the formula B on assumptions $A_1, ..., A_n$, one can inductively construct a derivation of a sequent $x_1 : A_1, ..., x_n : A_n \vdash u : B$, that is, a term u of type B with free variables $x_1, ..., x_1$ of respective types $A_1, ..., A_n$. Conversely, if one has a derivable sequent $x_1 : A_1, ..., x_n : A_n \vdash u : B$, there is an easy way of getting a proof of $A_1, ..., A_n \vdash B$; erase all terms and variables in the derivation of the type assignment. The two processes are each other's inverses modulo renaming of variables. The isomorphism on the level of proofs is essentially given by

Proposition 1.3.2.

On the level of reduction the Curry-Howard isomorphism says that a reduction on a proof followed by application of the Curry-Howard isomorphism on the level of proofs, yields the same term as application of the Curry-Howard isomorphism on the level of proofs followed by the term-reduction corresponding to the proof-reduction. This can be verified by applying the Curry-Howard isomorphism to the proofs involved in the reduction rules of Intuitionistic Logic. For example, in the case of a $(\Rightarrow_I, \Rightarrow_E)$ reduction we get

$$
\dfrac{\dfrac{\begin{array}{c}\overline{\Gamma, x : A, \Lambda \vdash x : A} \\ \vdots \\ \Gamma, x : A \vdash u : B\end{array}}{\Gamma \vdash \lambda x.u : A \Rightarrow B} \qquad \begin{array}{c}\vdots \\ \Gamma \vdash v : A\end{array}}{\Gamma \vdash (\lambda x.u)v : B}
\quad \rightsquigarrow \quad
\begin{array}{c}\vdots \\ \Gamma, \Lambda \vdash v : A \\ \vdots \\ \Gamma \vdash u[v/x] : B\end{array}
$$

We see that a β-reduction has taken place on the term encoding the proof on which the reduction is performed. In fact all β-reductions appear as Curry-Howard interpretations of reductions on the corresponding proofs.

Chapter 2

Linear Logic

This chapter introduces Classical Linear Logic and Intuitionistic Linear Logic. We make a detour to Russell's Paradox with the aim of illustrating the difference between Intuitionistic Logic and Intuitionistic Linear Logic. Also, the Curry-Howard interpretation of Intuitionistic Linear Logic, the linear λ-calculus, is dealt with. Furthermore, we take a look at the Girard Translation translating Intuitionistic Logic into Intuitionistic Linear Logic. Finally, we give a brief introduction to some concrete models of Intuitionistic Linear Logic.

2.1 Classical Linear Logic

Linear Logic was discovered by J.-Y. Girard in 1987 and published in the now famous paper [Gir87]. In the abstract of this paper, it is stated that "a completely new approach to the whole area between constructive logics and computer science is initiated". In [Gir89] the conceptual background of Linear Logic is worked out. The fundamental idea of Linear Logic is to control the use of resources which is witnessed by the fact that the contraction and weakening proof-rules are not admissible in general. Rather, Linear Logic occurs essentially as Classical Logic with the restriction that contraction and weakening is marked explicitly using additional modalities ! and ? on formulae. The !-modality corresponds to contraction and weakening on the left hand side, and similarly, the ?-modality corresponds to contraction and weakening on the right hand side. A proof of $!A$ amounts to having a proof of A that can be used an arbitrary number of times.

Here we shall only consider the multiplicative fragment of Classical Linear Logic. Formulae are given by the grammar

$$s \ ::= \ I \mid s \otimes s \mid \ \bot \ \mid s \parr s \mid s \multimap s \mid !s \mid ?s \mid \& \mid 1 \mid \ \oplus \ \mid 0.$$

Proof-rules for a Gentzen style presentation of the logic are given in Appendix A.3; they are used to derive sequents

$$A_1, ..., A_n \Vdash B_1, ..., B_m.$$

A Girardian turnstile $\Vdash$ is used to distinguish sequents of Classical Linear Logic from sequents of Classical Logic, where the usual turnstile $\vdash$ is used. The &,1,$\oplus$ and 0 connectives are called additive. The system obtained by excluding the additives is called the multiplicative fragment. Note that, unlike most presentations of Classical Linear Logic, we use two-sided sequents, which will make the connection to Intuitionistic Linear Logic more explicit. Negation is then defined as $A^\bot = A \multimap \bot$. The absence of contraction and weakening prevents us from considering the contexts of a sequent as *sets* of formulae, but we have to consider them to be *multisets* instead. This should be compared to Classical and Intuitionistic Logic where we do have contraction and weakening which implies that contexts can be considered as sets of formulae. The fact that contexts are considered as multisets means that every formula occuring in the context of a sequent has to be used exactly once. Therefore the two conjunctions & and $\otimes$ of Linear Logic are very different constructs: A proof of $A\&B$ consists of a proof of A together with a proof of B where exactly one of the proofs has to be used. A proof of $A \otimes B$ also consists of a proof of A together with a proof of B but here both of the proofs have to be used.

Now, as with Classical Logic, the cut-rule of Classical Linear Logic is redundant. Again the idea is that an application of the cut rule can either be pushed upwards in the surrounding proof or it can be replaced by cuts involving simpler formulae. In Classical Linear Logic we have the following key-cases (excluding the additive key-cases which are similar to the corresponding key-cases for Classical Logic):

- The $(\otimes_R, \otimes_L)$ key-case

$$
\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\vdots\ \pi_2}{\Gamma' \vdash B, \Delta'}}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \qquad \dfrac{\dfrac{\vdots\ \pi'_1}{\Gamma'', A, B \vdash \Delta''}}{\Gamma'', A \otimes B \vdash \Delta''}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\vdots\ \pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\dfrac{\vdots\ \pi_2}{\Gamma' \vdash B, \Delta'} \quad \dfrac{\vdots\ \pi'_1}{\Gamma'', A, B \vdash \Delta''}}{\Gamma'', A, \Gamma' \vdash \Delta'', \Delta'}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta', \Delta}
$$

- The (I_R, I_L) key-case

$$
\dfrac{\dfrac{}{\vdash I} \qquad \dfrac{\dfrac{\vdots\ \pi'_1}{\Gamma' \vdash \Delta'}}{\Gamma', I \vdash \Delta'}}{\Gamma' \vdash \Delta'} \qquad \rightsquigarrow \qquad \dfrac{\vdots\ \pi'_1}{\Gamma' \vdash \Delta'}
$$

- The $(\multimap_R, \multimap_L)$ key-case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma, A \vdash B, \Delta\end{array}}{\Gamma \vdash A \multimap B, \Delta} \quad \dfrac{\begin{array}{c}\vdots\ \pi'_1\\ \Gamma' \vdash A, \Delta'\end{array} \quad \begin{array}{c}\vdots\ \pi'_2\\ \Gamma'', B \vdash \Delta''\end{array}}{\Gamma', \Gamma'', A \multimap B \vdash \Delta'', \Delta'}}{\Gamma', \Gamma'', \Gamma \vdash \Delta'', \Delta', \Delta}$$

$$\rightsquigarrow$$

$$\dfrac{\begin{array}{c}\vdots\ \pi'_1\\ \Gamma' \vdash A, \Delta'\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma, A \vdash B, \Delta\end{array} \quad \begin{array}{c}\vdots\ \pi'_2\\ \Gamma'', B \vdash \Delta''\end{array}}{\Gamma'', \Gamma, A \vdash \Delta'', \Delta}}{\Gamma'', \Gamma, \Gamma' \vdash \Delta'', \Delta, \Delta'}$$

- The $(!_R, !_L)$ key-case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ !\Gamma \vdash A, ?\Delta\end{array}}{!\Gamma \vdash !A, ?\Delta} \quad \dfrac{\begin{array}{c}\vdots\ \pi'_1\\ \Gamma', A \vdash \Delta'\end{array}}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\ \pi_1\\ !\Gamma \vdash A, ?\Delta\end{array} \quad \begin{array}{c}\vdots\ \pi'_1\\ \Gamma', A \vdash \Delta'\end{array}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

- The $(!_R, W_L)$ key-case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ !\Gamma \vdash A, ?\Delta\end{array}}{!\Gamma \vdash !A, ?\Delta} \quad \dfrac{\begin{array}{c}\vdots\ \pi'_1\\ \Gamma' \vdash \Delta'\end{array}}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\ \pi'_1\\ \Gamma' \vdash \Delta' \\ \hline\hline \Gamma', !\Gamma \vdash \Delta', ?\Delta\end{array}$$

We also have $(\parr_R, \parr_L)$, $(\bot_R, \bot_L)$, $(W_R, ?_L)$, and $(?_R, ?_L)$ key-cases, but they are left out as they are symmetric to the mentioned $(\otimes_R, \otimes_L)$, (I_R, I_L),

$(!_R, W_L)$, and $(!_R, !_L)$ cases, respectively. The key-cases have the property that a cut is replaced by cuts involving simpler formulas. Furthermore we have the following so-called pseudo key-case:

- The $(!_R, C_L)$ pseudo key-case

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{!\Gamma \vdash A, ?\Delta}}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\overset{\vdots\,\pi_1'}{\Gamma', !A, !A \vdash \Delta'}}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{!\Gamma \vdash A, ?\Delta}}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\dfrac{\overset{\vdots\,\pi_1}{!\Gamma \vdash A, ?\Delta}}{!\Gamma \vdash !A, ?\Delta} \qquad \overset{\vdots\,\pi_1'}{\Gamma', !A, !A \vdash \Delta'}}{\Gamma', !\Gamma, !A \vdash \Delta', ?\Delta}}{\Gamma', !\Gamma, !\Gamma \vdash \Delta', ?\Delta, ?\Delta}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

We also have a $(C_R, ?_L)$ pseudo key-case, but this is left out as it is symmetric to the mentioned $(!_R, C_L)$ case. Note that in a pseudo key-case the cut formula is replaced by cuts involving the same formula. Hence, the pseudo key-cases does not simplify the involved cut formulas, which distinguishes them from the key-cases (and this is indeed the reason why we call them *pseudo* key-cases). In Appendix B we shall give a proof of cut-elimination for the multiplicative fragment. As with Classical Logic, it is the case that Classical Linear Logic satisfies the subformula property, that is, all formulae occuring in a cut-free proof are subformulae of the formulae occuring in the end-sequent.

Classical Linear Logic does not satisfy Church-Rosser, but on the other hand, it is possible to give a non-trivial sound denotational semantics using coherence spaces, see [GLT89]. Thus, the non-determinism of cut-elimination for Classical Linear Logic is limited. Note also that the example of Section 1.1 showing the non-determinism of cut-elimination for Classical Logic does not go through for Classical Linear Logic. It is, however, the case that the multiplicative fragment of Classical Linear Logic satisfies Church-Rosser, cf. [Laf96]. A proof can be found in [Dan90].

2.2 Intuitionistic Linear Logic

This section deals with Intuitionistic Linear Logic. The formulae are the same as with Classical Linear Logic except that those involving the connectives $\perp, \parr$ and ? are omitted. The proof-rules of Intuitionistic Linear Logic in Gentzen style occur as those of Classical Linear Logic given in Appendix A.3 where the proof-rules are subject to the restriction that each right hand side context contains exactly one formula. It is possible to deal with the $\perp, \parr$ and ? connectives intuitionistically by allowing sequents to have more than one conclusion together with an appropriate restriction on the $\multimap_R$ rule - see [BdP96, HdP93]. It turns out that the ! modality enables Intuitionistic Logic to be interpreted faithfully in Intuitionistic Linear Logic via the Girard Translation - see Section 2.6.

Here we shall also consider a Natural Deduction presentation of Intuitionistic Linear Logic which is equivalent to the Gentzen style formulation. Proof-rules for this formulation of the logic are given in Appendix A.4; they are used to derive sequents

$$A_1, ..., A_n \vdash B.$$

The Natural Deduction presentation is the same as the one given in the paper [BBdPH92], and later fleshed out in detail in [Bie94]. A historical account of the Natural Deduction presentation of Intuitionistic Linear Logic can be found in Section 4.4 of [Bra96].

As in Intuitionistic Logic, a proof might be rewritten into a simpler form using a reduction rule. Again, reduction of a Natural Deduction proof corresponds to cut-eliminating in a Gentzen style formulation. The reduction rules of the Natural Deduction presentation of Intuitionistic Linear Logic given here are as follows (excluding the additive reductions which are similar to the corresponding reductions for Intuitionistic Logic):

- The (I_I, I_E) case

$$\dfrac{\dfrac{}{\vdash I} \qquad \begin{array}{c}\vdots\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A} \quad \leadsto \quad \begin{array}{c}\vdots\\ \Gamma \vdash A\end{array}$$

- The $(\otimes_I, \otimes_E)$ case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\vdots\\ \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \otimes B} \qquad \begin{array}{c}\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}\\ \vdots\\ \Lambda, A, B \vdash C\end{array}}{\Lambda, \Gamma, \Delta \vdash C} \quad \leadsto \quad \begin{array}{c}\begin{array}{c}\vdots\\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\vdots\\ \Delta \vdash B\end{array}\\ \vdots\\ \Lambda, \Gamma, \Delta \vdash C\end{array}$$

- The $(\multimap_I, \multimap_E)$ case

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A \vdash A}\\ \vdots\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B} \qquad \begin{array}{c}\vdots\\ \Lambda \vdash A\end{array}}{\Gamma, \Lambda \vdash B} \quad \leadsto \quad \begin{array}{c}\vdots\\ \Lambda \vdash A\\ \vdots\\ \Gamma, \Lambda \vdash B\end{array}$$

- The $(Promotion, Dereliction)$ case

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash !A_1\end{array} \;,\;\ldots\;,\; \begin{array}{c}\vdots\\ \Gamma_n \vdash !A_n\end{array} \qquad \begin{array}{c}\overline{!A_1 \vdash !A_1}\;,...,\;\overline{!A_n \vdash !A_n}\\ \vdots\\ !A_1,...,!A_n \vdash B\end{array}}{\Gamma_1,...,\Gamma_n \vdash !B}}{\Gamma_1,...,\Gamma_n \vdash B}$$

$$\rightsquigarrow$$

$$\begin{array}{c} \begin{array}{c}\vdots\\ \Gamma_1 \vdash !A_1\end{array}\;,...,\;\begin{array}{c}\vdots\\ \Gamma_n \vdash !A_n\end{array}\\ \vdots\\ \Gamma_1,...,\Gamma_n \vdash B\end{array}$$

- The $(Promotion, Contraction)$ case

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\\ \Gamma_1 \vdash\, !A_1\end{array}\ ,...,\ \begin{array}{c}\vdots\\ \Gamma_n \vdash\, !A_n\end{array}
\qquad
\begin{array}{c}\overline{!A_1 \vdash\, !A_1}\ ,...,\ \overline{!A_n \vdash\, !A_n}\\ \vdots\\ !A_1,...,!A_n \vdash B\end{array}
}{\Gamma_1,...,\Gamma_n \vdash\, !B}
\qquad
\begin{array}{c}\overline{!B \vdash\, !B}\quad \overline{!B \vdash\, !B}\\ \vdots\\ \Lambda, !B, !B \vdash C\end{array}
}{\Lambda, \Gamma_1,...,\Gamma_n \vdash C}
$$

$$\rightsquigarrow$$

$$
\dfrac{
\begin{array}{c}\vdots\\ \Gamma_1 \vdash\, !A_1\end{array}\ ,...,\ \begin{array}{c}\vdots\\ \Gamma_n \vdash\, !A_n\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\overline{!A_1 \vdash\, !A_1}\ ,...,\ \overline{!A_n \vdash\, !A_n}\\ \vdots\\ !A_1,...,!A_n \vdash B\end{array}}{!A_1,...,!A_n \vdash\, !B}
\quad
\dfrac{\begin{array}{c}\overline{!A_1 \vdash\, !A_1}\ ,...,\ \overline{!A_n \vdash\, !A_n}\\ \vdots\\ !A_1,...,!A_n \vdash B\end{array}}{!A_1,...,!A_n \vdash\, !B}
\\ \vdots\\
\Lambda, !A_1,...,!A_n, !A_1,...,!A_n \vdash C
\end{array}
}{\Lambda, \Gamma_1,...,\Gamma_n \vdash C}
$$

where the last used rule is derivable by using the *Contraction* and *Exchange* rules. Note that a special case of the *Contraction* rule is used.

- The $(Promotion, Weakening)$ case

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash\, !A_1\end{array} \;,\; \dots \;,\; \begin{array}{c}\vdots\\ \Gamma_n \vdash\, !A_n\end{array} \qquad \begin{array}{c}\overline{!A_1 \vdash\, !A_1} \;, \dots, \; \overline{!A_n \vdash\, !A_n}\\ \vdots\\ !A_1, \dots, !A_n \vdash B\end{array}}{\Gamma_1, \dots, \Gamma_n \vdash\, !B} \qquad \Lambda \vdash C}{\Lambda, \Gamma_1, \dots, \Gamma_n \vdash C}
$$

$$\rightsquigarrow$$

$$
\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash\, !A_1\end{array} \;,\; \dots \;,\; \begin{array}{c}\vdots\\ \Gamma_n \vdash\, !A_n\end{array} \qquad \Lambda \vdash C}{\Lambda, \Gamma_1, \dots, \Gamma_n \vdash C}
$$

where the last used rule is derivable by using the $Weakening$ rule.

If we think of the $Promotion$ rule as putting a "box" around the right hand side proof, then $(Promotion, Dereliction)$ reduction removes the box, whereas the $(Promotion, Contraction)$ and $(Promotion, Weakening)$ reductions respectively copy and discard the box.

Notions of Church-Rosser and strong normalisation for the Natural Deduction presentation of Intuitionistic Linear Logic are defined in analogy with the notions of Church-Rosser and strong normalisation for Intuitionistic Logic. Intuitionistic Linear Logic does indeed satisfy these properties; via a Curry-Howard isomorphism this corresponds to analogous results for reduction of terms of the linear λ-calculus which we will return to in Section 2.4 and Section 2.5.

2.3 A Digression - Russell's Paradox and Linear Logic

In this section we will make a digression with the aim of illustrating the fine grained character of Intuitionistic Linear Logic compared to Intuitionistic

Logic. We will take set-theoretic comprehension into account: In both of the logics unrestricted comprehension enables a contradiction to be proved via Russell's Paradox, but it turns out that Intuitionistic Linear Logic allows the presence of a restricted form of comprehension, which is not possible in the case of Intuitionistic Logic.

Unrestricted comprehension says that for any predicate $A(x)$ there is a set $\{x \mid A(x)\}$ with the property that

$$t \in \{x \mid A(x)\} \quad \Leftrightarrow \quad A(t).$$

This is a very strong axiom; it has all the axioms of Zermelo-Fraenkel set theory except the Axiom of Choice as special cases. An informal proof of Russell's Paradox now goes as follows: Using comprehension we define a set u as

$$u = \{x \mid \neg(x \in x)\}.$$

Thus, u is the famous set of all those sets that are not elements of themselves. We then define a formula R to be

$$u \in u.$$

We then have $R \Leftrightarrow \neg R$ which enables a contradiction to be proved as follows: *Assume* R, this entails $\neg R$, which *together with* R entails a contradiction. Thus we have proved $\neg R$. The proof of $\neg R$ also gives a proof of R. But a proof of $\neg R$ together with a proof of R entails a contradiction.

Note how the contradiction is proved in two stages: First a formula R such that $R \Leftrightarrow \neg R$ is defined, then a contradiction is derived in a proof where the assumption R is used twice. The two applications of the assumption R are emphasised in the informal proof above. The are two ways of remedying this inconsistency:

- Unrestricted comprehension is replaced by weaker axioms such that it is impossible to define the set u and hence the formula R with the property that $R \Leftrightarrow \neg R$ cannot be defined either. This was the option taken historically and which gave rise to Zermelo-Fraenkel set theory.

- Unrestricted comprehension is kept but the surrounding logic is weakened such that the existence of a formula R with the property that $R \Leftrightarrow \neg R$ does not imply a contradiction. The !-free fragment of Linear

Logic is one such option as assumptions here can be used only once (recall that in deriving a contradiction from $R \Leftrightarrow \neg R$ the assumption R is used twice).

We will now flesh out some details of the second option. First we give a formal proof of Russell's Paradox in Intuitionistic Logic extended with unrestricted comprehension as prescribed in [Pra65]. Recall that in Intuitionistic Logic negation $\neg A$ is defined as $A \Rightarrow 0$. The grammar for formulae of Intuitionistic Logic is extended with an additional clause

$$s \ ::= \ldots \mid t \in t$$

and a grammar for terms is added

$$t \ ::= \ x \mid \{x \,|\, s\}$$

where x is a variable ranging over terms. Terms are to be thought of as sets (they should not be confused with terms of the λ-calculus). Furthermore, proof-rules for introduction and elimination of the connective $\in$ are added

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash t \in \{x \,|\, A\}} \in_I \qquad \frac{\Gamma \vdash t \in \{x \,|\, A\}}{\Gamma \vdash A[t/x]} \in_E$$

The first stage of Russell's Paradox is formalised as follows: We define the term u and the formula R as above and get

$$\frac{\Gamma \vdash R}{\Gamma \vdash \neg R}$$

and vice versa, which is equivalent to provability of $\vdash R \Rightarrow \neg R$ and vice versa. The second stage of the paradox is formalised as follows: Two copies of the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{R \vdash R}}{R \vdash \neg R} \qquad \overline{R \vdash R}}{R, R \vdash 0}}{R \vdash 0}}{\vdash \neg R}$$

are applied in the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \vdash \neg R\end{matrix}}{\vdash R} \qquad \begin{matrix}\vdots\\ \vdash \neg R\end{matrix}}{\vdash 0}$$

of a contradiction. Note that we have used the admissible contraction proof-rule together with a multiplicative version of the $\Rightarrow_E$ rule which is also admissible in Intuitionistic Logic. The presence of contraction is crucial for the proof of inconsistency to go through. In [Pra65] the following observation is made: It can be shown that the sequent $\vdash 0$ is not provable by a normal proof; this means that no reduction sequences originating from the proof of $\vdash 0$ above end in a normal proof. Indeed, the proof reduces in two stages to itself by carrying out the only performable reductions.

Now, above we have shown that unrestricted comprehension in the context of Intuitionistic Logic is inconsistent. But it turns out that we do not get inconsistency if we extend the !-free fragment of Intuitionistic Linear Logic with unrestricted comprehension as above. Negation $\neg A$ is here defined as $A \multimap 0$[1]. We still have a formula R such that $R \multimap \neg R$ and vice versa, but now we cannot prove $\vdash 0$ as before because contraction is forbidden. The system was proved consistent in [Gri82] using the following two observations: A proof essentially shrinks under normalisation and there is no normal proof of $\vdash 0$. The !-free fragment of Intuitionistic Linear Logic with unrestricted comprehension is, however, very unexpressive. A partial solution to this lack of expressiveness is to extend Intuitionistic Linear Logic with comprehension as above but with the restriction that the ! modality is not allowed to occur in the involved formula $A(t)$. We still have the formula R such that $R \multimap \neg R$ and vice versa, but it turns out that this system is consistent, which was proved in [Shi94].

Hence, the fine-grainedness of Intuitionistic Linear Logic allows the presence of a restricted form of comprehension, which is not possible in the context of Intuitionistic Logic. It should be mentioned that considerations on Russell's Paradox in the context of Linear Logic have been crucial for Girard's discovery of Light Linear Logic - see [Gir94].

[1] This is not the same as the negation $A^\perp$ of Classical Linear Logic which is defined as $A^\perp = A \multimap \perp$. However, this difference is not of importance here.

2.4 The Linear λ-Calculus

The presentation of the linear λ-calculus which we give in this section is the same as the one given in [BBdPH92, BBdPH93], and later fleshed out in detail in [Bie94]. We have omitted the additive fragment as it is similar to the corresponding fragment of the λ-calculus. The next section shows how the linear λ-calculus occurs as a Curry-Howard interpretation of Intuitionistic Linear Logic in the same way as the λ-calculus occurs as a Curry-Howard interpretation of Intuitionistic Logic. Types of the linear λ-calculus are given by the grammar

$$s \;::=\; I \mid s \otimes s \mid s \multimap s \mid !s$$

and terms are given by the grammar

$$\begin{array}{lll}
t & ::= & x \mid \\
 & & * \mid \mathtt{let}\ t\ \mathtt{be}\ *\ \mathtt{in}\ t \mid t \otimes t \mid \mathtt{let}\ t\ \mathtt{be}\ x \otimes y\ \mathtt{in}\ t \mid \\
 & & \lambda x^A.t \mid tt \mid \\
 & & \mathtt{promote}\ t, ..., t\ \mathtt{for}\ x_1, ..., x_n\ \mathtt{in}\ t \mid \mathtt{derelict}(t) \mid \\
 & & \mathtt{discard}\ t\ \mathtt{in}\ t \mid \mathtt{copy}\ t\ \mathtt{as}\ x, y\ \mathtt{in}\ t
\end{array}$$

where x is a variable ranging over terms and $t, ..., t$ denotes a list of n occurrences of the symbol t. The set of free variables, denoted $FV(u)$, of a term u is defined by induction on u as follows:

$$\begin{array}{rcl}
FV(x) & = & \{x\} \\
FV(*) & = & \emptyset \\
FV(\mathtt{let}\ w\ \mathtt{be}\ *\ \mathtt{in}\ u) & = & FV(w) \cup FV(u) \\
FV(u \otimes v) & = & FV(u) \cup FV(v) \\
FV(\mathtt{let}\ w\ \mathtt{be}\ x \otimes y\ \mathtt{in}\ u) & = & FV(w) \cup FV(u) - \{x, y\} \\
FV(\lambda x.u) & = & FV(u) - \{x\} \\
FV(fu) & = & FV(f) \cup FV(u) \\
FV(\mathtt{promote}\ v_1, ..., v_n\ \mathtt{for}\ x_1, ..., x_n\ \mathtt{in}\ u) & = & FV(v_1) \cup ... \cup FV(v_n) \\
FV(\mathtt{derelict}(u)) & = & FV(u) \\
FV(\mathtt{discard}\ w\ \mathtt{in}\ u) & = & FV(w) \cup FV(u) \\
FV(\mathtt{copy}\ w\ \mathtt{as}\ x, y\ \mathtt{in}\ u) & = & FV(w) \cup FV(u) - \{x, y\}
\end{array}$$

Figure 2.1: Type Assignment Rules for the Linear λ-Calculus

$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma, x : A, y : B, \Delta \vdash u : C}{\Gamma, y : B, x : A, \Delta \vdash u : C}$$

$$\frac{}{\vdash * : I} \qquad \frac{\Lambda \vdash w : I \quad \Gamma \vdash u : A}{\Gamma, \Lambda \vdash \texttt{let } w \texttt{ be } * \texttt{ in } u : A}$$

$$\frac{\Gamma \vdash u : A \quad \Delta \vdash v : B}{\Gamma, \Delta \vdash u \otimes v : A \otimes B} \qquad \frac{\Lambda \vdash w : A \otimes B \quad \Gamma, x : A, y : B \vdash u : C}{\Gamma, \Lambda \vdash \texttt{let } w \texttt{ be } x \otimes y \texttt{ in } u : C}$$

$$\frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash \lambda x^A . u : A \multimap B} \qquad \frac{\Lambda \vdash f : A \multimap B \quad \Delta \vdash u : A}{\Lambda, \Delta \vdash fu : B}$$

$$\frac{\Gamma_1 \vdash v_1 :!A_1, \ \ldots \ , \Gamma_n \vdash v_n :!A_n \quad x_1 :!A_1, \ldots, x_n :!A_n \vdash u : B}{\Gamma_1, \ldots, \Gamma_n \vdash \texttt{promote } v_1, \ldots, v_n \texttt{ for } x_1, \ldots, x_n \texttt{ in } u :!B}$$

$$\frac{\Lambda \vdash u :!B}{\Lambda \vdash \texttt{derelict}(u) : B}$$

$$\frac{\Lambda \vdash w :!A \quad \Gamma, x :!A, y :!A \vdash u : B}{\Gamma, \Lambda \vdash \texttt{copy } w \texttt{ as } x, y \texttt{ in } u : B} \qquad \frac{\Lambda \vdash w :!A \quad \Gamma \vdash u : B}{\Gamma, \Lambda \vdash \texttt{discard } w \texttt{ in } u : B}$$

We use the same convention concerning substitution as for terms of the λ-calculus: If a term v together with n terms $u_1, ..., u_n$ and n pairwise distinct variables $x_1, ..., x_n$ are given, then $v[u_1, ..., u_n/x_1, ..., x_n]$ denotes the term v where simultaneously the terms $u_1, ..., u_n$ have been substituted for free occurrences of the variables $x_1, ..., x_n$ such that bound variables in v have been renamed to avoid capture of free variables of the terms $u_1, ..., u_n$.

Rules for assignment of types to terms are given in Appendix A.4. Type assignments have the form of sequents

$$x_1 : A_1, ..., x_n : A_n \vdash u : A$$

where $x_1, ..., x_n$ are pairwise distinct variables. Note that the definition of sequents implicitly restricts use of the rules. For example, it is not possible to use the rule for introduction of $\otimes$ if the contexts Γ and Δ have common variables. By induction on the derivation of the type assignment it can be shown that

$$FV(u) = \{x_1, ..., x_n\}.$$

Note that this is different from the λ-calculus where we did not have equality, but only an inclusion. By induction on the derivation of the type assignment it can be shown that every free variable of the term u occurs exactly once[2]. The linear λ-calculus satisfies the following properties:

Lemma 2.4.1 If the sequent $\Gamma \vdash u : A$ is derivable, then for any derivable sequent $\Gamma' \vdash u : B$, where the context Γ' is a permutation of the context Γ, we have $A = B$.

Proof: Induction on the derivation of $\Gamma \vdash u : A$. $\square$

The following proposition is the essence of the Curry-Howard isomorphism:

Proposition 2.4.2 If the sequent $\Gamma \vdash u : A$ is derivable, then the first rule instance above the sequent which is different from an instance of the *Exchange* rule is uniquely determined up to permutation of the context Γ.

Proof: Use Lemma 2.4.1 to check each case. $\square$

Now comes a lemma dealing with substitution.

[2]This does not hold if the additive fragment is included.

Lemma 2.4.3 (Substitution Property) If both of the sequents $\Gamma \vdash u : A$ and $\Delta, x : A, \Lambda \vdash v : B$ are derivable and the variables in the contexts Γ and Δ, Λ are pairwise distinct, then the sequent $\Delta, \Gamma, \Lambda \vdash v[u/x] : B$ is also derivable.

Proof: Induction on the derivation of $\Delta, x : A, \Lambda \vdash v : B$. $\square$

We now need a couple of conventions: The term

$$\texttt{copy}\ w_1\ \texttt{as}\ x_1, y_1\ \texttt{in}\ (\ldots\texttt{copy}\ w_n\ \texttt{as}\ x_n, y_n\ \texttt{in}\ u\ldots)$$

is denoted

$$\texttt{copy}\ \overline{w}\ \texttt{as}\ \overline{x}, \overline{y}\ \texttt{in}\ u$$

and the term

$$\texttt{discard}\ w_1\ \texttt{in}\ (\ldots\texttt{discard}\ w_n\ \texttt{in}\ u\ldots)$$

is denoted

$$\texttt{discard}\ \overline{w}\ \texttt{in}\ u.$$

The linear λ-calculus has the following β-reduction rules each of which is the image under the Curry-Howard isomorphism of a reduction on the proof corresponding to the involved term:

$$\begin{array}{rcl}
\texttt{let}\ *\ \texttt{be}\ *\ \texttt{in}\ u & \leadsto & u \\
u \otimes v\ \texttt{be}\ x \otimes y\ \texttt{in}\ w & \leadsto & w[u, v/x, y] \\
(\lambda x.u)w & \leadsto & u[w/x] \\
\texttt{derelict}(\texttt{promote}\ \overline{w}\ \texttt{for}\ \overline{x}\ \texttt{in}\ u) & \leadsto & v[\overline{w}/\overline{x}] \\
\texttt{discard}\ (\texttt{promote}\ \overline{w}\ \texttt{for}\ \overline{x}\ \texttt{in}\ u)\ \texttt{in}\ v & \leadsto & \texttt{discard}\ \overline{w}\ \texttt{in}\ v
\end{array}$$

$$\texttt{copy}\ (\texttt{promote}\ \overline{w}\ \texttt{for}\ \overline{x}\ \texttt{in}\ u)\ \texttt{as}\ y, z\ \texttt{in}\ v$$
$$\leadsto$$
$$\texttt{copy}\ \overline{w}\ \texttt{as}\ \overline{x'}, \overline{x''}\ \texttt{in}\ (v[\texttt{promote}\ \overline{x'}\ \texttt{for}\ \overline{x}\ \texttt{in}\ u, \texttt{promote}\ \overline{x''}\ \texttt{for}\ \overline{x}\ \texttt{in}\ u/y, z])$$

We shall only be concerned with β-reductions. The properties of Church-Rosser and strong normalisation for proofs of Intuitionistic Linear Logic correspond to analogous notions for terms of the linear λ-calculus via the Curry-Howard isomorphism. In [Ben93] it is shown that the linear λ-calculus satisfies strong normalisation by giving a reduction preserving translation into System F (which does satisfy strong normalisation). By König's Lemma, this implies that all terms are bounded. Church-Rosser can then be proved in the standard way by induction on the bound. The proof is straightforward, but tedious.

2.5 The Curry-Howard Isomorphism

In what follows, we will consider the Natural Deduction presentation of Intuitionistic Linear Logic given in Appendix A.4. Intuitionistic Linear Logic corresponds to the linear λ-calculus via a Curry-Howard isomorphism in the same way as Intuitionistic Logic corresponds to the λ-calculus. The formulae of Intuitionistic Linear Logic correspond to types of the linear λ-calculus in the obvious way. We get the rules for assigning types to terms of the linear λ-calculus if we decorate the proof-rules of Intuitionistic Linear Logic with terms in the appropriate way, and moreover, we can recover the proof-rules if we take the typing rules of the linear λ-calculus and remove the terms. The isomorphism on the level of proofs is essentially given by Proposition 2.4.2.

As with the λ-calculus, it is the case that all the β-reductions of the linear λ-calculus appear as Curry-Howard interpretations of reduction rules of Intuitionistic Linear Logic. For example, in the case of a $(\otimes_I, \otimes_E)$ reduction we get

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash u : A} \quad \dfrac{\vdots}{\Delta \vdash v : B}}{\Gamma, \Delta \vdash u \otimes v : A \otimes B} \qquad \dfrac{\overline{x : A \vdash x : A} \quad \overline{y : B \vdash y : B}}{\vdots \atop \Lambda, x : A, y : B \vdash wC}}{\Lambda, \Gamma, \Delta \vdash \texttt{let } u \otimes v \texttt{ be } x \otimes y \texttt{ in } w : C}$$

$$\leadsto$$

$$\begin{array}{c} \vdots \qquad\qquad \vdots \\ \Gamma \vdash u : A \quad \Delta \vdash v : B \\ \vdots \\ \Lambda, \Gamma, \Delta \vdash w[u, v/x, y] : C \end{array}$$

We see that a β-reduction has taken place on the term encoding the proof on which the reduction is performed. All β-reductions actually do appear as Curry-Howard interpretations of reductions on the corresponding proofs.

2.6 The Girard Translation

The [Gir87] paper introduced the Girard Translation which embeds Intuitionistic Logic into Intuitionistic Linear Logic. We will state the Girard Translation in terms of the Natural Deduction presentations of Intuitionistic Logic and Intuitionistic Linear Logic given in Appendix A.2 and Appendix A.4, respectively. The translation works at the level of formulae as well as at the level of proofs. At the level of formulae the Girard Translation is defined inductively as follows:

$$\begin{array}{rcl} 1^o & = & 1 \\ (A \wedge B)^o & = & A^o \& B^o \\ (A \Rightarrow B)^o & = & !A^o \multimap B^o \\ 0^o & = & 0 \\ (A \vee B)^o & = & !A^o \oplus !B^o \end{array}$$

At the level of proofs the Girard Translation inductively translates a proof of a sequent

$$A_1, ..., A_n \vdash B$$

into a proof of the sequent

$$!A_1^o, ..., !A_n^o \vdash B^o.$$

In what follows we consider each case except the symmetric ones. Special cases of rules are used when appropriate. Recall that a double bar means a number of applications of rules.

- A derivation

$$\frac{}{A_1, ..., A_n \vdash A_p}$$

  is translated into

$$\genfrac{}{}{1.2pt}{}{\dfrac{\dfrac{}{!A_q^o \vdash !A_p^o}}{!A_q^o \vdash A_p^o}}{!A_1^o, ..., !A_n^o \vdash A_p^o}$$

- A derivation

$$\frac{}{\Gamma \vdash 1}$$

  is translated into

$$\frac{}{!\Gamma^o \vdash 1}$$

- A derivation

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

  is translated into

$$\frac{!\Gamma^o \vdash A^o \quad !\Gamma^o \vdash B^o}{!\Gamma^o \vdash A^o \& B^o}$$

- A derivation

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

  is translated into

$$\frac{!\Gamma^o \vdash A^o \& B^o}{!\Gamma^o \vdash A^o}$$

- A derivation
$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$
is translated into
$$\frac{!\Gamma^o, !A^o \Vdash B^o}{!\Gamma^o \Vdash !A^o \multimap B^o}$$

- A derivation
$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$
is translated into
$$\dfrac{\dfrac{!\Gamma^o \Vdash !A^o \multimap B^o \qquad \dfrac{!\Gamma^o \Vdash A^o}{!\Gamma^o \Vdash !A^o}}{!\Gamma^o, !\Gamma^o \Vdash B^o}}{!\Gamma^o \Vdash B^o}$$

- A derivation
$$\frac{\Gamma \vdash 0}{\Gamma \vdash C}$$
is translated into
$$\frac{!\Gamma^o \Vdash 0}{!\Gamma^o \Vdash C^o}$$

- A derivation
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$
is translated into
$$\dfrac{\dfrac{!\Gamma^o \Vdash A^o}{!\Gamma^o \Vdash !A^o}}{!\Gamma^o \Vdash !A^o \oplus !B^o}$$

- A derivation
$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

is translated into

$$\cfrac{\cfrac{!\Gamma^o \vdash !A^o \oplus !B^o \quad !\Gamma^o, !A^o \vdash C^o \quad !\Gamma^o, !B^o \vdash C^o}{!\Gamma^o, !\Gamma^o \vdash C^o}}{!\Gamma^o \vdash C^o}$$

The translation is sound with respect to provability in the sense that the sequent $A_1, ..., A_n \vdash B$ is provable (in Intuitionistic Logic) iff $!A_1^o, ..., !A_n^o \vdash B^o$ is provable (in Intuitionistic Linear Logic). Moreover, it is shown in [Bie94] that the translation preserves β-reductions. In [Gir87] it is shown that the Girard Translation is sound with respect to the standard coherence space interpretation, that is, the interpretation of a proof in Intuitionistic Logic is essentially the same as the interpretation of its image under the Girard Translation. A categorical version of this result can be found in [Bra96].

2.7 Concrete Models

An example of a concrete model of Intuitionistic Linear Logic is the category of cpos and strict continuous functions. A cpo is a partial order with the property that each directed subset has a least upper bound. Note that this entails that a cpo has a bottom element. A monotone function between cpos is continuous if it preserves the least upper bound of any non-empty directed subset, and it is strict if it preserves the bottom element. The symmetric monoidal structure is given by the smash product, the internal-hom of two objects is given by the set of strict continuous functions with the pointwise order, and the comonad is given by the lift operation.

It should be mentioned that the category of cpos and strict continuous functions is actually a model for intuitionistic relevant logic in the sense of [Jac94].

Also the category of dI domains and linear stable functions is a model of Intuitionistic Linear Logic. We will first define a dI domain. Let $(D, \sqsubseteq)$ be a non-empty poset such that every finitely bounded subset X has a least upper bound $\sqcup X$ where a subset X is finitely bounded iff every finite subset of X has an upper bound. This entails that D has a bottom element, and moreover, every non-empty subset X has a greatest lower bound which we will denote by $\sqcap X$. A *prime* element of D is an element d such that

$$d \sqsubseteq \sqcup X \Rightarrow \exists x \in X.\, d \sqsubseteq x$$

for any finitely bounded subset X. We will denote the set of prime elements of D by D_p. The poset D is *prime algebraic* iff

$$\forall d \in D.\, d = \sqcup\{d' \in D_p \mid d' \sqsubseteq d)\}$$

A *finite* element of D is an element d such that

$$d \sqsubseteq \sqcup X \Rightarrow \exists x \in X.\, d \sqsubseteq x$$

for any non-empty directed subset X. We will denote the set of finite elements of D by D_o. The poset D is *finitary* iff

$$\forall d \in D_o.\, |\{d' \in D \mid d' \sqsubseteq d\}| < \infty$$

The poset D is a *dI domain* if it is prime algebraic and finitary. A monotone function f between dI domains is *stable* iff $f(\sqcap X) = \sqcap\{f(x) \mid x \in X\}$ for any non-empty finitely bounded subset X, and it is *linear* iff $f(\sqcup X) = \sqcup\{f(x) \mid x \in X\}$ for any finitely bounded subset X. The trace $tr(f)$ of a linear stable function $f : D \to E$ is a subset of $D_p \times E_p$ defined as

$$\{(d, e) \in D_p \times E_p \mid e \sqsubseteq f(d) \wedge \forall d' \sqsubseteq d.\, (e \sqsubseteq f(d') \Rightarrow d = d')\}$$

In what follows, $X \uparrow$ means that the subset X has an upper bound. The tensor product $D \otimes E$ of two dI domains D and E is defined as

$$\{t \subseteq D_p \times E_p \mid \pi_1(t) \uparrow \wedge \pi_2(t) \uparrow \wedge t \text{ is down-closed } \}$$

ordered by inclusion. The unit I is defined to be the dI domain with two elements. We define the internal-hom $D \multimap E$ as

$$\{tr(f) \mid f : D \to E\}$$

ordered by inclusion. This is called the *stable* order. The dI domain $!D$ is defined as

$$\{t \subseteq D_o \mid t \uparrow \wedge t \text{ is down-closed } \}$$

ordered by inclusion.

There are quite a few other concrete models of Intuitionistic Linear Logic around. The traditional coherence space model should be mentioned - see [Gir87] and [GLT89] for details. In another vein, there are the bistructure model of [PW94] and the category of hypercoherences and strongly stable linear functions of [Ehr93]. Recently, the game models of [AJM96, HO96, Nic94] have gained attention as they give rise to fully abstract models of Scott's PCF.

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Appendix A

Logics

A.1 Classical Logic

$$\frac{}{A \vdash A}\,Axiom$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta', \Delta}\,Cut$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,Weakening_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,Weakening_R$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,Contraction_L \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,Contraction_R$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\wedge_{L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,\wedge_{L2} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\,\wedge_R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\,1_L \qquad \frac{}{\vdash 1}\,1_R$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'}\,\Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\,\Rightarrow_R$$

Classical Logic - Disjunction

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee_L \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_{R1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_{R2}$$

$$\frac{}{0 \vdash} 0_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash 0, \Delta} 0_R$$

A.2 Intuitionistic Logic

$$\frac{}{A_1, ..., A_n \vdash A_q}\,Axiom$$

$$\frac{}{\Gamma \vdash 1}\,1_I$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge_I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\wedge_{E1} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\wedge_{E2}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\Rightarrow_I \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\,\Rightarrow_E$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash C}\,0_E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\vee_{I1} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\vee_{I2} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\,\vee_E$$

A.3 Classical Linear Logic

$$\frac{}{A \vdash A}\,Axiom$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta', \Delta}\,Cut$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\otimes_L \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\otimes_R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta}\,I_L \qquad \frac{}{\vdash I}\,I_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\parr_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\parr_R$$

$$\frac{}{\bot \vdash}\,\bot_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\,\bot_R$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta, \Delta'}\multimap_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\multimap_R$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,!_L \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\,!_R$$

$$\frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\,?_L \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}\,?_R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,Weakening_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\,Weakening_R$$

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,Contraction_L \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\,Contraction_R$$

Classical Linear Logic - the Additives

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ \&_{L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ \&_{L2} \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta}\ \&_R$$

$$\frac{}{\Gamma \vdash 1, \Delta}\ 1_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ \oplus_L \qquad\qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus_{R1} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus_{R2}$$

$$\frac{}{\Gamma, 0 \vdash \Delta}\ 0_L$$

A.4 Intuitionistic Linear Logic

$$\frac{}{A \vdash A}\ Axiom$$

$$\frac{}{\vdash I}\ I_I \qquad \frac{\Lambda \vdash I \quad \Gamma \vdash A}{\Gamma, \Lambda \vdash A}\ I_E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_I \qquad \frac{\Lambda \vdash A \otimes B \quad \Gamma, A, B \vdash C}{\Gamma, \Lambda \vdash C}\ \otimes_E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_I \qquad \frac{\Lambda \vdash A \multimap B \quad \Delta \vdash A}{\Lambda, \Delta \vdash B}\ \multimap_E$$

$$\frac{\Gamma_1 \vdash !A_1, ..., \Gamma_n \vdash !A_n \quad !A_1, ..., !A_n \vdash B}{\Gamma_1, ..., \Gamma_n \vdash !B}\ Prom. \qquad \frac{\Lambda \vdash !B}{\Lambda \vdash B}\ Dereliction$$

$$\frac{\Lambda \vdash !A \quad \Gamma, !A, !A \vdash B}{\Gamma, \Lambda \vdash B}\ Contraction$$

$$\frac{\Lambda \vdash !A \quad \Gamma \vdash B}{\Gamma, \Lambda \vdash B}\ Weakening$$

$$\frac{\Gamma_1 \vdash A_1, \ \ldots \ , \Gamma_n \vdash A_n}{\Gamma_1, ..., \Gamma_n \vdash 1}\ 1_I$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \&_I \qquad \frac{\Lambda \vdash A \& B}{\Lambda \vdash A}\ \&_{E1} \qquad \frac{\Lambda \vdash A \& B}{\Lambda \vdash B}\ \&_{E2}$$

$$\frac{\Gamma_1 \vdash A_1, \ \ldots \ , \Gamma_n \vdash A_n \quad \Lambda \vdash 0}{\Gamma_1, ..., \Gamma_n, \Lambda \vdash C}\ 0_E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \oplus_{I1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus_{I2} \qquad \frac{\Lambda \vdash A \oplus B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, \Lambda \vdash C}\ \oplus_E$$

Appendix B

Cut-Elimination for Classical Linear Logic

In this appendix we show that the multiplicative fragment of Classical Linear Logic satisfies the cut-elimination property. We actually give an algorithm transforming an arbitrary Classical Linear Logic proof (in the multiplicative fragment) into a cut-free proof with the same end-sequent. Our proof of cut-elimination is essentially taken from [BdP96]; it is along the lines of the proof of cut-elimination for Classical Logic given in [GLT89] where the linear context is taken into account. Towards the goal of proving this theorem we start with some preliminary results.

B.1 Some Preliminary Results

As usual with cut-elimination proofs, a cut can either be replaced by cuts involving simpler formulas, or it can be pushed upwards in the surrounding proof. The former situation amounts to the key-cases, and the latter situation is exemplified by the proof

$$\dfrac{\Gamma \vdash A, \Delta \qquad \dfrac{\Gamma_1, A \vdash \Delta_1}{\Gamma', A \vdash \Delta'}\,r'}{\Gamma', \Gamma \vdash \Delta', \Delta}$$

which is transformed into

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma_1, A \vdash \Delta_1}{\Gamma_1, \Gamma \vdash \Delta_1, \Delta}}{\Gamma', \Gamma \vdash \Delta', \Delta}\, r'$$

As is usual in cut-elimination proofs we need to define the degree of a formula.

Definition B.1.1 The degree $\partial(A)$ of a formula A is defined inductively as follows:

1. $\partial(A) = 1$ when A is atomic
2. $\partial(A \otimes B) = \partial(A \parr B) = \partial(A \multimap B) = max\{\partial(A), \partial(B)\} + 1$
3. $\partial(!A) = \partial(?A) = \partial(A) + 1$

The degree of a cut is defined to be the degree of the formula cut. The degree of a proof is defined as the supremum of the degrees of the cuts in the proof. The fundamental property of the key-cases is that a cut is replaced by cuts of strictly lower degree.

Now we deal with what we call the pseudo key-cases, which differ from the key-cases in that they do not decrease the degree of the cuts involved. There are two pseudo key-cases, namely $(!_R, C_L)$ and $(C_R, ?_L)$. In Lemma B.1.3 we show how to perform additional modifications when applying the pseudo key-cases such that the degree of the involved cuts *does* decrease. Let us examine the $(!_R, C_L)$ pseudo key-case

$$\dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\, !_R \qquad \dfrac{\Gamma', !A, !A \vdash \Delta'}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \quad \Gamma', !A, !A \vdash \Delta'}{\Gamma', !\Gamma, !A \vdash \Delta', ?\Delta}}{\Gamma', !\Gamma, !\Gamma \vdash \Delta', ?\Delta, ?\Delta}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

The $(!_R, C_L)$ pseudo key-case does not decrease the degree, as mentioned above. Thus, instead of just modifying the proof as prescribed, we have to "track" the cut formula $!A$ upwards in the right hand side immediate subproof which has C_L as the last rule. When we follow an antecedent $!A$ formula upwards in a proof, then a track ends due to one of the following reasons: The formula is introduced by an instance of the $!_L$ rule, the formula is absorbed by an instance of the W_L rule, or finally, we end up in an axiom. Note that on the way upwards the formula $!A$ might have proliferated into several copies while passing instances of the C_L rule. In each of the mentioned end-points there is an appropriate way of modifying the proof such that $!A$ is replaced by $!\Gamma$ and $?\Delta$ is added on the succedent in a way that decreases the degree. For example the following proof where none of the tracks are proliferated by passing instances of the C_L rule and they both both end up in an axiom

$$\dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\,!_R \qquad \dfrac{\dfrac{\dfrac{\overline{!A \vdash !A}}{\begin{array}{c}\vdots\\ \Gamma_1, !A \vdash \Delta_1\end{array}} \quad \dfrac{\overline{!A \vdash !A}}{\begin{array}{c}\vdots\\ \Gamma_2, !A \vdash \Delta_2\end{array}}}{\Gamma', !A, !A \vdash \Delta'}}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

is modified as follows

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\,!_R\\ \vdots\\ \Gamma_1, !\Gamma \vdash \Delta_1, ?\Delta\end{array} \qquad \begin{array}{c}\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\,!_R\\ \vdots\\ \Gamma_2, !\Gamma \vdash \Delta_2, ?\Delta\end{array}}{\Gamma', !\Gamma, !\Gamma \vdash \Delta', ?\Delta, ?\Delta}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

Note that a $(!_R, C_L)$ pseudo key-case was performed in the process of following the track(s) upwards. This is formalised in the next lemma which takes care of every situation where we have a cut such that the last rule of the left hand side immediate subproof is the $!_R$ rule. The lemma is proved by induction using the notion of height of a proof:

Definition B.1.2 The height $h(\tau)$ of a proof τ is defined inductively as follows:

1. if τ is an instance of a zero-premiss rule, then $h(\tau) = 1$,
2. if τ is obtained from the proof π by applying an instance of a one-premiss rule, then $h(\tau) = h(\pi) + 1$,
3. if τ is obtained from the proofs π and π' by applying an instance of a two-premiss rule, then $h(\tau) = max\{h(\pi), h(\pi')\} + 1$.

Now the promised lemma.

Lemma B.1.3 Assume that $!A^n$ denotes a list of n occurrences of the formula $!A$, and let a proof τ be given as follows

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi\\ !\Gamma \vdash A, ?\Delta\end{matrix}}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\begin{matrix}\vdots\,\pi'\\ \Gamma', !A^n \vdash \Delta'\end{matrix}}{\Gamma', !A \vdash \Delta'}}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}$$

where the subproofs π and π' have degrees strictly less than the degree of τ. Then we can construct a proof of $\Gamma', !\Gamma \vdash \Delta', ?\Delta$ which has degree strictly less than the degree of τ

Proof: The proof is by induction on $h(\pi')$. In what follows, the last rule used in π' is denoted by r' and we proceed on a case by case basis.

- r' is an instance of an axiom. We use the proof obtained by applying $!_R$ to π.
- One of the n occurrences of the formula $!A$ in the end-sequent of π' is introduced by r' which is an instance of $!_L$. We have two subcases. If $n = 1$ then we have a $(!_R, !_L)$ key-case which we change according to the table of key-cases. If $n > 1$ then we have the following proof

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma', !A^{n-1}, A \vdash \Delta'$}
\UnaryInfC{$\Gamma', !A^{n-1}, !A \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

which we transform into

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma', !A^{n-1}, A \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A, A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma, A \vdash \Delta', ?\Delta$}
\BinaryInfC{$\Gamma', !\Gamma, !\Gamma \vdash \Delta', ?\Delta, ?\Delta$}
\doubleLine
\UnaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

on which we apply the induction hypothesis on the appropriate subproof.

- One of the n occurrences of the formula $!A$ in the end-sequent of π' is introduced by r' which is an instance of W_L. We have two subcases. If $n = 1$ then we have a $(!_R, W_L)$ key-case which we change according to the table of key-cases. If $n > 1$ then we have the following proof

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma', !A^{n-1} \vdash \Delta'$}
\UnaryInfC{$\Gamma', !A^{n-1}, !A \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

which we transform into

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma', !A^{n-1} \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

on which we apply the induction hypothesis.

- One of the n occurrences of the formula $!A$ in the end-sequent of π' is introduced by r' which is an instance of C_L. We apply the induction hypothesis to the immediate subproof of π' where we consider the appropriate $n+1$ occurrences of $!A$.

- None of the n occurrences of the formula $!A$ in the end-sequent of π' are introduced by r'. We have two subcases. In the first subcase all of the n occurrences of the formula $!A$ in the end-sequent of π' are inherited from the same immediate subproof. If r' is a one-premiss rule, we have the following proof

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma_1, !A^n \vdash \Delta_1$}
\RightLabel{r'}
\UnaryInfC{$\Gamma', !A^n \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

which we transform into

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma_1, !A^n \vdash \Delta_1$}
\doubleLine
\UnaryInfC{$\Gamma_1, !A \vdash \Delta_1$}
\BinaryInfC{$\Gamma_1, !\Gamma \vdash \Delta_1, ?\Delta$}
\RightLabel{r'}
\UnaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

on which we apply the induction hypothesis on the appropriate subproof. The situation is analogous if r' is a two-premiss rule. In the second subcase the n occurrences of the formula $!A$ in the end-sequent of π' are not inherited from the same immediate subproof, so we have the following proof

$$
\begin{prooftree}
\AxiomC{$!\Gamma \vdash A, ?\Delta$}
\UnaryInfC{$!\Gamma \vdash !A, ?\Delta$}
\AxiomC{$\Gamma_1, !A^p \vdash \Delta_1$}
\AxiomC{$\Gamma_2, !A^q \vdash \Delta_2$}
\RightLabel{r'}
\BinaryInfC{$\Gamma', !A^{p+q} \vdash \Delta'$}
\doubleLine
\UnaryInfC{$\Gamma', !A \vdash \Delta'$}
\BinaryInfC{$\Gamma', !\Gamma \vdash \Delta', ?\Delta$}
\end{prooftree}
$$

which we transform into

$$
\dfrac{\dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\Gamma_1, !A^p \vdash \Delta_1}{\Gamma_1, !A \vdash \Delta_1}}{\Gamma_1, !\Gamma \vdash \Delta_1, ?\Delta} \qquad \dfrac{\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \qquad \dfrac{\Gamma_2, !A^q \vdash \Delta_2}{\Gamma_2, !A \vdash \Delta_2}}{\Gamma_2, !\Gamma \vdash \Delta_2, ?\Delta}}{\dfrac{\Gamma', !\Gamma, !\Gamma \vdash \Delta', ?\Delta, ?\Delta}{\Gamma', !\Gamma \vdash \Delta', ?\Delta}}\, r'
$$

on which we use the induction hypothesis on the appropriate two subproofs.

□

There is a dual to Lemma B.1.3; and it is the case that the proofs are also dual.

B.2 Putting the Proof Together

The following lemma is the engine of the cut-elimination proof.

Lemma B.2.1 Let a proof τ be given as follows

$$
\dfrac{\begin{array}{c} \vdots\ \pi \\ \Gamma \vdash A, \Delta \end{array} \qquad \begin{array}{c} \vdots\ \pi' \\ \Gamma', A \vdash \Delta' \end{array}}{\Gamma', \Gamma \vdash \Delta', \Delta}
$$

where the subproofs π and π' have degrees strictly less than the degree of τ. Then we can construct a proof of $\Gamma', \Gamma \vdash \Delta', \Delta$ which has a degree strictly less than the degree of τ.

Proof: Induction on $h(\pi) + h(\pi')$. In what follows, the last rules in π and π' are denoted by r and r', respectively, and the immediate subproofs of π and π' are denoted by π_i and π'_j, respectively. The formula A is denoted the principal formula. We proceed case by case.

- r is an instance of an axiom; we use π'.
- r' is an instance of an axiom; dual to the previous case.

We have now dealt with the cases where r or r' is an instance of an axiom.

- r is an instance of $!_R$ that introduces the principal formula; thus, $A = !A'$ for some A', and π looks as follows:

$$\frac{!\Gamma \vdash A', ?\Delta}{!\Gamma \vdash !A', ?\Delta}$$

  We use Lemma B.1.3.

- r' is an instance of $?_L$ that introduces the principal formula; dual to the previous case.

- r is an instance of $!_R$ that does not introduce the principal formula; thus, $A = ?A'$ for some A', and π looks as follows:

$$\frac{!\Gamma \vdash B, ?A', ?\Delta}{!\Gamma \vdash !B, ?A', ?\Delta}$$

  If the principal formula $?A'$ is not introduced by r', then we push π upwards in the right hand side subproof (note that r' cannot be an instance of $!_R$ or $?_L$) and use the induction hypothesis. If the principal formula $?A'$ is introduced by r', that is, π' looks as follows:

$$\frac{!\Gamma', A' \vdash ?\Delta'}{!\Gamma', ?A' \vdash ?\Delta'}$$

  then we push π' upwards in the left hand side subproof (note that each formula in $!\Gamma'$ has a ! at the top level and each formula in $?\Delta'$ has a ? at the top level) and use the induction hypothesis.

- r' is an instance of $?_L$ that does not introduce the principal formula; dual to the previous case.

- r is an instance of $?_L$; thus, $A = ?A'$ for some A', and π looks as follows:

$$\frac{!\Gamma, B \vdash ?A', ?\Delta}{!\Gamma, ?B \vdash ?A', ?\Delta}$$

If the principal formula $?A'$ is not introduced by r', then we push π upwards in the right hand side subproof (note that r' cannot be an instance of $!_R$ or $?_L$) and use the induction hypothesis. If the principal formula $?A'$ is introduced by r', that is, π' looks as follows:

$$\frac{!\Gamma', A' \vdash ?\Delta'}{!\Gamma', ?A' \vdash ?\Delta'}$$

then we push π' upwards in the left hand side subproof (note that each formula in $!\Gamma'$ has a ! at the top level and each formula in $?\Delta'$ has a ? at the top level) and use the induction hypothesis.

- r' is an instance of $!_R$; dual to the previous case.

We have now dealt with the cases where r or r' is an instance of $!_R$ or $?_L$.

- r does not introduce the principal formula; we push π' upwards in the left hand side subproof and use the induction hypothesis.
- r' does not introduce the principal formula; dual to the previous case.
- r introduces the principal formula on the right hand side and r' introduces the principal formula on the left hand side; we have a $(\otimes_R, \otimes_L)$, (I_R, I_L), $(\parr_R, \parr_L)$, $(\bot_R, \bot_L)$ or $(\multimap_R, \multimap_L)$ key-case. We change τ according to the table of key-cases.

$\square$

Lemma B.2.2 Given a proof τ of the sequent $\Gamma \vdash \Delta$ we can construct a proof of the same sequent with degree strictly lower than the degree of τ.

Proof: Induction on $h(\tau)$. If the last rule used in τ is not a cut with the same degree as the degree of τ, we are done by using the induction hypothesis on the immediate subproofs of τ. If the last rule in τ is a cut with the same degree as the degree of τ, we apply the induction hypothesis on the immediate subproofs of τ, and obtain a proof τ' which looks as follows

$$\dfrac{\begin{array}{c}\vdots\,\pi \\ \Gamma' \vdash A, \Delta'\end{array} \qquad \begin{array}{c}\vdots\,\pi' \\ \Gamma'', A \vdash \Delta''\end{array}}{\Gamma'', \Gamma' \vdash \Delta'', \Delta'}$$

where the proofs π and π' have degrees strictly lower than the degree of τ'. Note that Γ'', Γ' is equal to Γ and Δ'', Δ' is equal to Δ. We then use Lemma B.2.1. □

And now the Hauptsatz.

Theorem B.2.3 Given any proof of a sequent we can construct a cut-free proof of the same sequent.

Proof: Iteration of Lemma B.2.2. □

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